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On Robust Control System Design for Plants with Recycle

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Abstract: A direct procedure for deriving the component transfer functions in plants with recycle when mechanistic procedure is used in the mathematical modeling is demonstrated. Multi-loop feedback controllers are then designed for the plants with recycle incorporating recycle compensator and without recycle compensator using internal model control (IMC) parameterization. In general, it is found that controllers parameterized using the recycle compensated plant model result in closed-loop systems with better nominal and robust performance characteristics when implemented on the uncompensated plant model than when its controller is parameterized using the uncompensated plant model. Furthermore, recycle compensated plants result in overall best closed-loop systems when their feedback controllers are parameterized using the compensated plant model.

Keywords: Recycle plants, recycle compensator, multi-loop, robust performance, robust stability.

1. INTRODUCTION

Plants with recycle are widespread in the process industries. Under steady state operations, recycling is particularly appealing as it leads, for example, to efficient use of energy in integrated plants or greater overall conversion in a reactor. However, it results in poor dynamic characteristics (Denn and Lavie, 1982; Taiwo, 1986; Kapoor et al., 1986; Luyben, 1993; Scali and Ferrari, 1999; Morud and Skogestad, 1994; and Lakshminarayanan and Takeda, 2001). In order to keep the steady state advantage while preventing the dynamic penalty, Taiwo (1985, 1986, 1993 and 1996) proposed the recycle compensator whose efficacy for both nominal and robust performance has generally been corroborated by many investigators, see for example, Scali and Ferrari (1999), Lakshminarayanan and Takeda (2001), Tremblay et al. (2006), Meszaros et al. (2005) and Armbrust and Sharbaro (2011).

In order to compute the recycle compensator, the process transfer function must be decomposed into the forward path, recycle path and disturbance transfer functions. After specifying the recycle compensator in section 2, a direct procedure for this decomposition and controller design are undertaken in section 3 and the Appendix, for systems modelled from first principles using, for example, material, energy, momentum and force balances or combinations thereof. Section 4 contains a discussion and conclusions from the work.

2. THE GENERALIZED VIEW OF PROCESSES WITH RECYCLE AND THE SPECIFICATION OF A RECYCLE COMPENSATOR

Consider the block diagram involving a plant with recycle in Fig.1, where y, u', d', u, y_m are controlled , manipulated, disturbance, controller output and measured variables. C(s) is the feedback controller while F(s) is the recycle compensator. Each $G_i(s)$, with perhaps the exception of $G_2(s)$, is assumed to be a square matrix. First note that the (open-loop) recycle sub-plant output, z is given by

$$z(s) = (I - G_3)^{-1}G_1u'(s) + (I - G_3)^{-1}G_2d'(s) \quad (1a)$$

while the recycle plant output, y is given by

$$y(s) = G_4(I - G_3)^{-1}G_1u'(s) + G_4(I - G_3)^{-1}G_2d'(s) \quad (1b)$$

where $G_3(s)$ is the transfer function of the recycle process. The compensator F that totally cancels the detrimental effect of recycle is known as the perfect recycle compensator. Such a compensator restores the dynamically favourable transfer function of the original process without recycle, that is

$$z(s) = G_1u'(s) + G_2d'(s) \quad (2a)$$

$$y(s) = G_4G_1u'(s) + G_4G_2d'(s) \quad (2b)$$

In order to specify the recycle compensator, apply block diagram algebra to inner loop G in fig.1 to give (Taiwo, 1985, 1986, 1993, Taiwo and Krebs, 1996) eqn.(3).

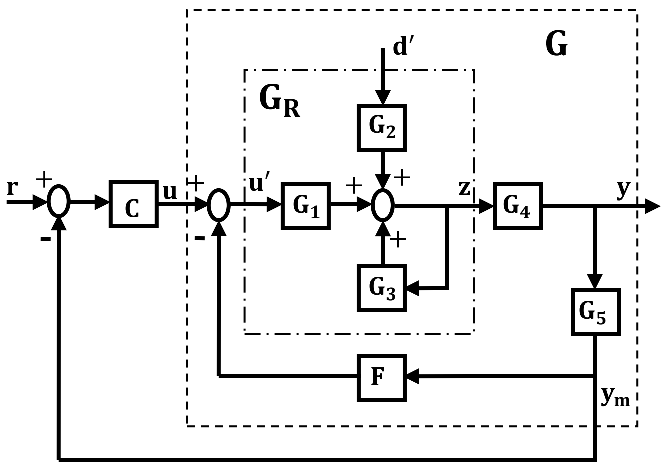

Fig.1: Block diagram of a system consisting of a plant with recycle, recycle compensator F and feedback controller C

$$F(s) = G_1{}^{-1}G_3G_4{}^{-1}G_5{}^{-1}(s) \tag{3}$$

$F(s)$ is realizable when eq.(3) is proper and causal. However, if one encounters realizability problems, a compensator which approximates the perfect recycle compensator may be used. Even the use of $F(0)$ alone should result in substantial improvement in process dynamics as this leads to the elimination of increased steady state sensitivity associated with plants with recycle. Note also that, when the arrangement of the component blocks in the plant is different from that given in Fig 1, block diagram algebra facilitates a transformation into this form, as illustrated in Example 1 below.

3. ILLUSTRATIVE DECOMPOSITION OF RECYCLE PLANTS AND FEEDBACK CONTROLLER DESIGN

3.1 Example 1

The block diagram (Fig.2a) for this SISO plant is taken from Del-Muro-Cuellar et al.(2005). Using block diagram algebra, it is clear that the enclosed block G_R in Fig 1 is equivalent to Fig 2b which is a transformation of Fig 2a.

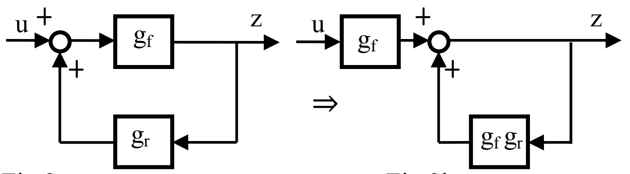

Fig.2a Fig.2b

Hence, the recycle compensator (3) is given by

$$F = g_r$$

Given that $g_f(s) = \frac{(s-3)e^{-1.2s}}{(s+1)^2}$ and $g_r(s) = \frac{e^{-0.8s}}{s+2}$, the enclosed transfer function G in Fig.1 which is the compensated system is given by $G(s) = \frac{y_m(s)}{u(s)} = g_f(s)$, since in this example $G_4(s)$=$G_5(s)$=1. Whereas, without recycle compensator,

$$G_R(s) = \frac{z(s)}{u'(s)} = \frac{(s+2)(s-3)e^{-1.2s}}{(s+1)^2(s+2)-(s-3)e^{-2s}}$$

Feedback Controller design for the compensated plant

This can be done straightforwardly using IMC parameterization. If it is desired to control the plant using a proportional plus integral (PI) controller, $g_f(s)$ may be simplified to a first order plus dead time model using, for example, Skogestad's half rule (Seborg et al. 2011), yielding $\frac{-3e^{-2.033s}}{(1.5s+1)}$. The final controller on adopting a tuning parameter $\tau_c = 2$ is $C(s) = -\frac{1}{8}\left(1+\frac{1}{1.5s}\right)$ the step responses of this process to reference step changes are good as shown in fig.5. No effort has been made here to design a controller satisfying a specified robust performance measure for space economy. However, an exposition of such methods are available (Taiwo and Krebs, 1996).

Feedback Controller design for the uncompensated plant

A simple analysis of the uncompensated plant $G_R(s)$ shows that the Nyquist plot marginally avoids the critical point (-1,0) because it crosses the negative real axis at the frequency $w_u = 0.69$ with $G_r(w_u) = -0.987$. It is therefore not surprising that the open loop response is highly oscillatory as shown in Fig 3. This is in sharp contrast to the well damped characteristics of the compensated plant (Fig.4). It was not straightforward designing a feedback controller giving a system with good performance. One method is to minimise a closed-loop performance measure such as the integral of the absolute error to a unit step change in reference input, IAE. When this was done, the parameters of the PID controller together with the metrics of its performance are given below. It is clear that the system with recycle compensator outperforms the system without recycle compensator especially with respect to the adequate damping of the closed loop response and its acceptable gain margin GM. Its phase margin, PM, is also adequate.

The performance metrics for the compensated system, are IAE=4.45, GM=3.0, PM=60.27^0. For the uncompensated plant, PID controller used is $C(s) = -0.25\left(1+\frac{1}{1.08s}+3.33s\right.$. Performance metrics are IAE=4.20, GM=1.135, PM=101.75^0.

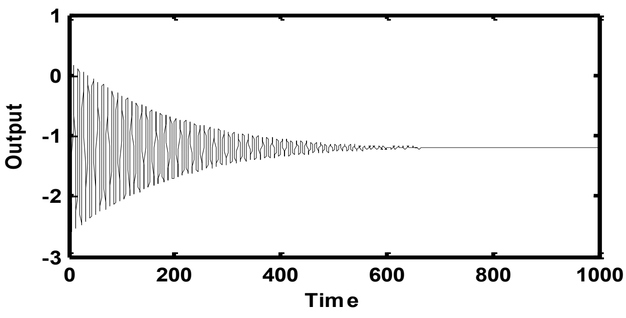

Fig.3: Open-loop step response for uncompensated plant

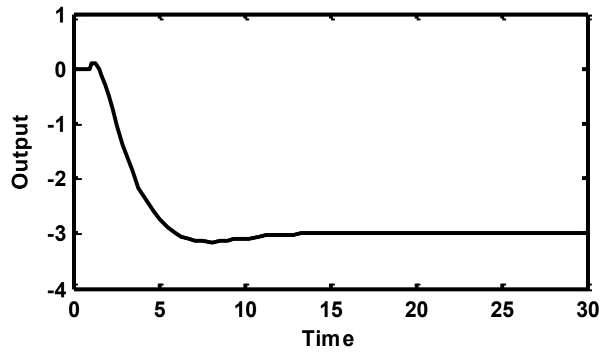

Fig.4:Open-loop step response for compensated plant

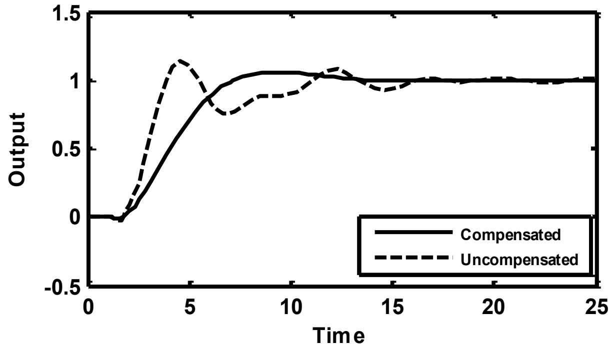

Fig.5: Closed-loop process step responses

3.2 Example 2: Two CSTR in series with recycle

This process consists of the continuously stirred tank reactors with recycle, see Fig 6 (Scali and Ferrari, 1999). Following the exposition in the Appendix, the state variable model for this process in deviation variables at the given operating point takes the form (note that for this plant $C = I_2$):

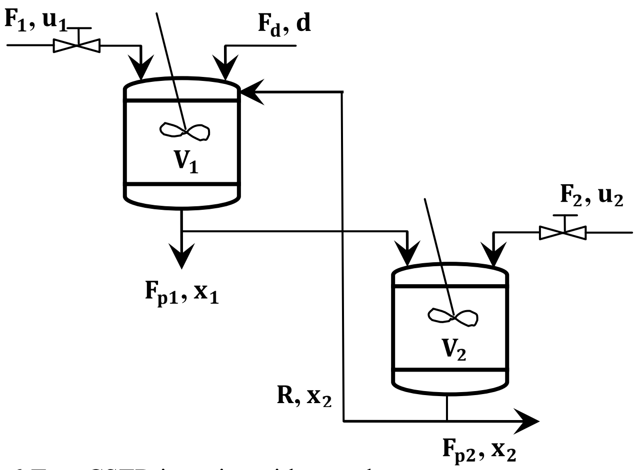

Fig.6:Two CSTR in series with recycle

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-12.5 & 0\\ 1.05 & 1.6\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0 & 10\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t-2)\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 0.05\end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix} + \begin{bmatrix}0.5\\ 0\end{bmatrix}d(t) \qquad (4)$$

Consequently the component transfer functions are

$$G_1(s) = \begin{bmatrix}\frac{1}{s+12.5} & 0\\ \frac{1.05}{(s+12.5)(s+1.6)} & \frac{0.05}{s+1.6}\end{bmatrix}, G_2(s) = \begin{bmatrix}\frac{0.5}{s+12.5}\\ \frac{0.525}{(s+12.5)(s+1.6)}\end{bmatrix}$$

$$G_3(s) = \begin{bmatrix}0 & \frac{10e^{-2s}}{s+12.5}\\ 0 & \frac{1.05e^{-2s}}{(s+12.5)(s+1.6)}\end{bmatrix}, G_4(s) = I_2 \text{ and}$$

$G_5(s) = \mathrm{diag}\,(e^{-s}, e^{-s})$

Hence, the recycle compensator is given by

$$F(s) = \begin{bmatrix}0 & 10e^{-s}\\ 0 & 0\end{bmatrix}$$

For the compensated plant, the model to use for feedback controller design is given by

$y(s) = G_5(s)G_1(s)u'(s) + G_5(s)G_2(s)d'(s)$

where

$$G_5(s)G_1(s) = \begin{bmatrix}\frac{0.08e^{-s}}{0.08s+1} & 0\\ \frac{1.05e^{-s}}{(s+12.5)(s+1.6)} & \frac{0.03125e^{-s}}{0.625s+1}\end{bmatrix}$$

And

$$G_5(s)G_2(s) = \begin{bmatrix}\frac{0.5e^{-s}}{s+12.5}\\ \frac{0.525e^{-s}}{(s+12.5)(s+1.6)}\end{bmatrix}$$

Whereas the recycle plant transfer function without the recycle compensator is given by

$$y_m(s) = G_5(I-G_3)^{-1}G_1u'(s) + G_5(I-G_3)^{-1}G_2d'(s)$$
$$= \frac{e^{-s}}{D(s)}\left\{\begin{bmatrix}(s+1.6) & 0.5e^{-2s}\\ 1.05 & 0.05(s+12.5)\end{bmatrix}u' + \begin{bmatrix}0.5(s+1.6)\\ 0.525\end{bmatrix}d'\right\}$$

$D(s) = (s+12.5)(s+1.6) - 10.5e^{-2s}$

Feedback controller design for the compensated system

Using the compensated plant model

$$G(s) = \begin{bmatrix}\frac{0.08e^{-s}}{5s+1} & 0\\ \frac{1.05e^{-s}}{(s+12.5)(s+1.6)} & \frac{0.03125e^{-s}}{0.625s+1}\end{bmatrix}, \text{RGA} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

This suggests a 1-1/2-2 pairing. Niederlinski Index is 1 affirming, by its non-negativity, that a multi-loop PI controller could be stable. With the uncertainty weight chosen as $Wu = 0.2\frac{0.2s+1}{0.02s+1}$ and the performance weight as $W_p(s) = \frac{s/2.5+0.1}{s}$ robust performance indices computed as summarized in Table I show that the system is robust.

Table I: Controller parameters and robustness indices for the compensated plant

	K_P	K_I	μ_{RP}	μ_{NP}	μ_{RS}
Loop1	1.5	6.25	0.95	0.828	0.2
Loop2	12.3	19.7			

Feedback controller design for the uncompensated system

For the uncompensated plant,

$$G_T(s) = \frac{1}{D(s)}\begin{bmatrix}(s+1.6)e^{-s} & 0.5e^{-3s}\\ 1.05e^{-s} & 0.05(s+12.5)e^{-s}\end{bmatrix}$$

Where $D(s) = (s+12.5)(s+1.6) - 10.5e^{-2s}$

$$\text{RGA} = \begin{bmatrix}2.11 & -1.11\\ -1.11 & 2.11\end{bmatrix}.$$

This suggests a 1-1/2-2 pairing. Niederlinski Index is 0.4755 affirming, by its non-negativity, that a multi-loop controller could be stable. Good controller settings obtained are summarized in table II. With the input weight chosen as $Wu = 0.2\frac{0.2s+1}{0.02s+1}$ and the performance weight as $W_p(s) = \frac{s/2+0.0075}{s}$ robust performance indices computed as summarized in Table II show that the system is robust.

On applying the controller tuned for the compensated plant on the uncompensated plant the system was unstable. This is not surprising as the uncompensated plant model has larger gains and a delay in its open loop characteristic quasi-polynomial. It also has a larger delay in its (1,2) element. Closed-loop instability may result whenever more realistic models are used. What is advisable in such a situation is to reduce the controller gains as has been done here where the closed loop system was stabilized by multiplying all the controller gains in Table I by 0.4. This was found to give a robust system also, as shown by the asterisked values in Table II.

Table II: Controller parameters and robustness indices for the uncompensated system

	K_P	K_I	μ_{RP}	μ_{NP}	μ_{RS}
Loop1(IIU)	3.695	1	0.8447	0.7381	0.2
Loop2(IIU)	9.546	2.26			
Loop1(*)	0.6	2.5	0.8838*	0.8029*	0.2*
Loop2(*)	4.92	7.88			

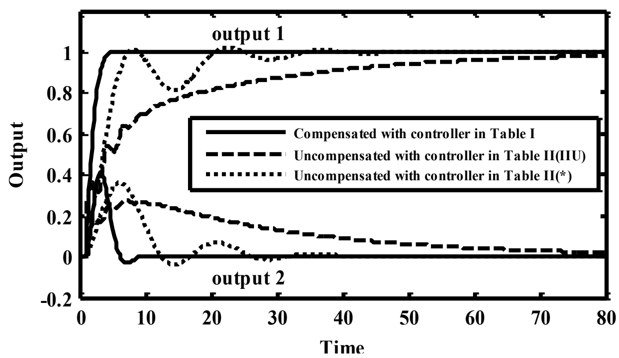

Fig.7:Closed-loop responses to a step change in reference 1

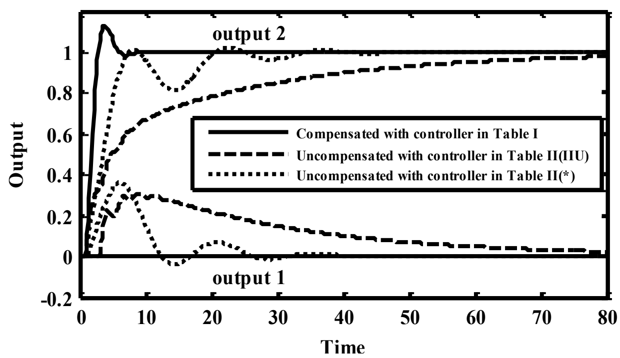

Fig.8:Closed-loop responses to a step change in reference 2

As commented in the last section, the controller parameterized using the compensated plant model results in a faster system when implemented on the uncompensated plant (see Figs 7 & 8). It is also robust. The relative sluggishness of the system incorporating the controller parameterized using the uncompensated plant model is a carryover from its open loop relatively larger time constants and other unfavourable characteristics. Utilizing the recycle compensator eliminates such undesirable characteristics thus making the recycle compensated system possess the most attractive characteristics as shown in Figs 7 & 8.

3.3 Example 3: Experimental three-tank system with recycle

Consider the three-tank system housed in our Process Systems Laboratory. The details about this equipment can be found in Amira, (2002) and Bamimore et al. (2009, 2012). We have modified the original system to incorporate a recycle stream from tank 2 to tank 1. The three tank system with recycle is as shown in Fig.9. It is a two-input, two-output process in which the controlled variables are the levels h_1 and h_2 inside tanks 1 and 2 and the manipulated variables are the volumetric flow-rates q_1 and q_2 respectively. The water level h_3 in tank 3 is observed but not controlled

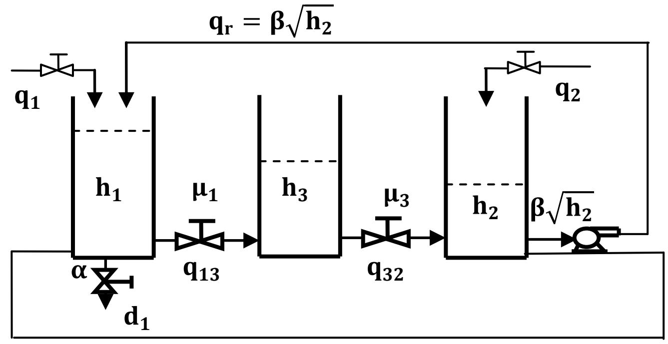

Fig.9: Experimental three tank system with recycle

The detailed mathematical model of this nonlinear process is available (Bamimore et al, 2011). Following (iii) in the Appendix, the linearized equations in deviation variables are:

$$\delta\dot{h} = A\delta h + Hz + B\delta q \qquad (5)$$

where $\delta h = \begin{bmatrix}\delta h_1\\ \delta h_2\\ \delta h_3\end{bmatrix}$, $z = \begin{bmatrix}\delta h_1\\ \delta h_2\end{bmatrix}$, $C = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$

Also, $A = \begin{bmatrix}-0.0153 & 0 & 0.0076\\ 0 & -0.0267 & 0.0099\\ 0.0076 & 0.0099 & -0.0175\end{bmatrix}$,

$B = \begin{bmatrix}0.0067 & 0\\ 0 & 0.0067\\ 0 & 0\end{bmatrix}$, $H = \begin{bmatrix}0 & 0.0335\\ 0 & 0\\ 0 & 0\end{bmatrix}$

$G_1 = C(sI - A)^{-1}B$, a third order model which easily simplifies to G_{1r} below using element by element simplification using power series expansion:

$$G_{1r}(s) = \begin{bmatrix}\frac{0.60}{120.4s+1} & \frac{0.122}{240.2s+1}\\ \frac{0.122}{240.2s+1} & \frac{0.344}{84.3s+1}\end{bmatrix}$$

$G_3 = C(sI - A)^{-1}H$, which simplifies to G_{3r} below using a similar procedure to the one above:

$$G_{3r}(s) = \begin{bmatrix} 0 & \dfrac{3}{120.2s+1} \\ 0 & \dfrac{0.611}{240.2s+1} \end{bmatrix}$$

From (i) in Appendix, $G_R = C(sI - (A+E))^{-1}B$, where E is 3 by 3 with zeros apart from e_{12} which is 0.0335, compute the transfer function matrix for the uncompensated plant model, which, by a similar procedure as above, simplifies to:

$$G_{Rr} = \begin{bmatrix} \dfrac{1.55}{498.3s+1} & \dfrac{2.78}{567.5s+1} \\ \dfrac{0.32}{618.2s+1} & \dfrac{0.88}{462.4s+1} \end{bmatrix}$$

The recycle compensator is calculated as:

$$F = G_1^{-1}G_3 = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}$$

Controller design for the compensated plant

Upon using the recycle compensator, the transfer function matrix for the compensated plant is given by

$$G_{1r}(s) = \begin{bmatrix} \dfrac{0.60}{120.4s+1} & \dfrac{0.122}{240.2s+1} \\ \dfrac{0.122}{240.2s+1} & \dfrac{0.344}{84.3s+1} \end{bmatrix}$$

It is easily verified that G(s) is diagonally dominant, hence designing a feedback controller using the diagonal elements would give a stable closed loop system involving G(s).

IMC-PI controllers thus obtained are

$$C(s) = \text{diag}(20.07 + 0.17/s, 30.63+0.36/s) \quad (6)$$

Feedback Controller design for the uncompensated plant

IMC-PI controllers were parameterized for the uncompensated plant using $G_{Rr}(s)$ obtained above.

The RGA for G_{Rr} is given by

$$RGA = \begin{bmatrix} 2.77 & -1.77 \\ -1.77 & 2.77 \end{bmatrix}$$

Consequently, a multi-loop IMC-PI controller

$$C(s) = \text{diag}(40.18+0.081/s, 65.33+0.14/s) \quad (7)$$

The controllers were implemented on a nonlinear SIMULINK model of the three tank system. The simulation results presented in Fig.10 show that the compensated system (thick line) displays a faster set-point tracking for both controlled levels h_1 and h_2 when compared to the uncompensated system (dashed line) being controlled using controller (7) which has been parameterised using the uncompensated plant model. On applying controllers in eqn (6) designed for the compensated plant model on the uncompensated plant (dotted line), a better setpoint tracking result is obtained than when using controller (7) parameterised by uncompensated plant model as shown in Fig 10. The recycle compensated plant using controller (6) however displays a faster set-point tracking for the controlled level h_1 than the uncompensated plant. It is noted that the controlled level h_2 is indistinguishable for both the compensated and uncompensated plants. This behaviour characterises simple models with adequate open-loop damping and without dead times. Otherwise, controllers designed for the compensated plant may lead to instability or oscillatory responses when implemented on the uncompensated plant, see, example 2, where controller gain reduction was mandatory.

Experimental results are currently being generated to verify the simulations.

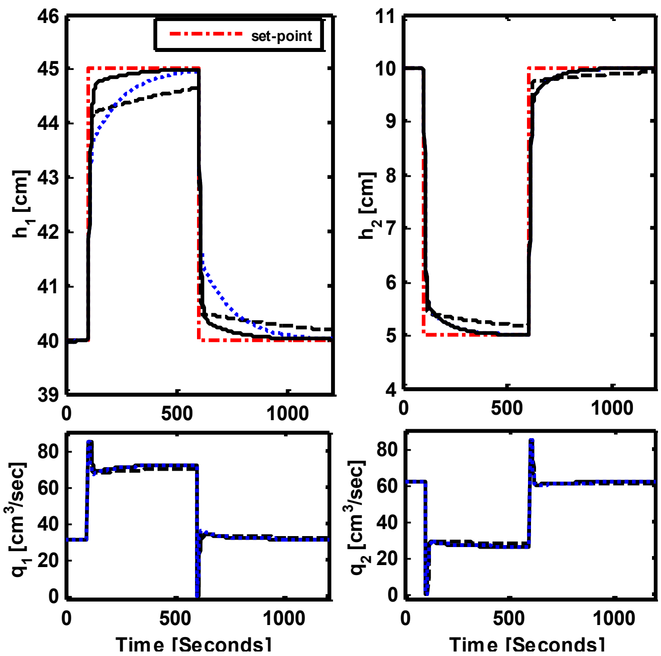

Fig.10:Closed-loop system outputs and inputs in response to step changes in references 1 and 2 . (Legend: thick, dashed and dotted lines respectively represent controller (6) on compensated plant, controller (7) on uncompensated plant and controller (6) on uncompensated plant.

4. DISCUSSION AND CONCLUSION

This work contains three examples which illustrate how the component transfer functions in plants with recycle can be simply derived irrespective of whether the output dimension is equal to the state dimension or not.

Example 1 illustrates that the recycle compensator can be used to restore adequate stability to open-loop marginally stable or unstable plants thus facilitating the design of robust closed-loop systems. It has been observed that plants with recycle exhibit much larger settling times when compared to the basic plants without recycle. Hence parameterizing the feedback controller for the uncompensated plant using the compensated plant model usually results in better closed-loop systems involving the uncompensated plants than when their controllers are parameterized using the uncompensated plant parameters. The only caveat is that such controllers may have to be modified, for example, by controller gain reduction, in order to ensure adequate stability of the closed-loop systems involving the uncompensated plant. This has been shown in example 2.

This work has used the IMC to parameterize PI controllers for both SISO and MIMO systems. It was only in the case of the marginally stable process in example 1 that PID controller parameters of the uncompensated system had to be determined using optimization. It has been found in general that the nominal and robust characteristics of the recycle compensated systems are better than those for the uncompensated systems.

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APPENDIX: A direct procedure for deriving the component transfer functions in plants with recycle

The linearized model of the recycle sub-plant, G_R (Fig 1) in deviation variables can be written as

$$\frac{dx}{dt} = Ax + Ex + Bu' + Dd' \quad \text{(i)}$$

$$z = Cx \quad \text{(ii)}$$

where x, z , u' and d' are the states, outputs, inputs and disturbances. Note that in (i) x has been split such that A operates on the states that have not been recycled to the process unit under consideration while E operates on recycled states for the same unit. For instance, in fig. 6, the first tank involves states x_1 and x_2. While x_1 is not a recycled state and thus would be operated on by matrix A, x_2 is a recycled state and would be operated on by E. Following fig.1 it is noted that inputs into the recycle process G_3 are z. It is therefore expedient to let E operate on z. This is achieved by re-expressing x on which E operates in terms of z utilizing (ii). That is, $x = C^+z$

where the subscript $+$ denotes the pseudoinversion of C (Maciejowski, 1989 and Strang, 1980). This is equivalent to ordinary matrix inversion if C is square and of full rank.

Consequently, (i) can be re-written

$$\frac{dx}{dt} = Ax + EC^+z + Bu' + Dd' \quad \text{(iii)}$$

Upon taking the Laplace transform of (iii) with zero initial condition and making x the subject of the formula, (iii) becomes

$$x(s) = (sI - A)^{-1}[Bu'(s) + EC^+z(s) + Dd'(s)] \quad \text{(iv)}$$

Hence,

$$z(s) = C(sI - A)^{-1}[Bu'(s) + Hz(s) + Dd'(s)] \quad \text{(v)}$$

where

$$H = EC^+ \quad \text{(vi)}$$

Finally on simplifying (v), we obtain

$$z(s) = G_1(s)u'(s) + G_3(s)z(s) + G_2(s)d'(s) \quad \text{(vii)}$$

where

$G_1(s) = C(sI - A)^{-1}B$, $G_3(s) = C(sI - A)^{-1}H$ and $G_2(s) = C(sI - A)^{-1}D$.

It is seen here that by separating the states that are recycled from those that are not, the direct identification of the component transfer functions in the recycle sub-plant G_R in fig.1 is directly facilitated.