Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

The nonlinear heat equation with state-dependent parameters and its connection to the Burgers' and the potential Burgers' equation ⋆

Christoph Josef Backi [*,1] **Jan Dimon Bendtsen** [**] **John Leth** [**]
Jan Tommy Gravdahl [*]

[*] *Department of Engineering Cybernetics, Norwegian University of Science and Technology, O.S. Bragstads plass 2D, 7034 Trondheim, Norway (e-mail: christoph.backi@itk.ntnu.no, jan.tommy.gravdahl@itk.ntnu.no)*
[**] *Department of Electronic Systems, Aalborg University, Fredrik Bajers vej 7C, 9220 Aalborg, Denmark (e-mail: dimon@es.aau.dk, jjl@es.aau.dk)*

Abstract: In this work the stability properties of a nonlinear partial differential equation (PDE) with state-dependent parameters is investigated. Among other things, the PDE describes freezing of foodstuff, and is closely related to the (Potential) Burgers' Equation. We show that for certain forms of coefficient functions, the PDE converges to a stationary solution given by (fixed) boundary conditions that make physical sense. We illustrate the results with numerical simulations.

Keywords: Parabolic PDE, nonlinear heat equation, Burgers' equation, state-dependent parameters

1. INTRODUCTION

Freezing is an essential part of shelf life extension of foodstuff. Especially for rapidly spoiling food, like e.g. fish, reliable and gentle freezing is essential in order to guarantee a safe and good product. Thus the physical process of freezing gets more and more attention in the scientific community and various mathematical tools describing heat transfer phenomena get applied. These tools can also be applied to other applications than freezing fish; in fact they can be applied to a whole range of physical processes where phase change occurs.

As available computational power grows, the possibilities of simulating complex heat exchange processes modeled by PDEs are enhanced as well. Even if finding an explicit analytical solution to the PDE is hard or impossible, simulations can provide qualitative and quantitative results. The process of freezing foodstuff is described in a whole range of publications, where Pham (2006b) and Pham (2006a) give an overview over how to model heat and mass transfer in frozen foods. Woinet et al. (1998) compare experimental and theoretical results of freezing with a simplified heat equation model. Costa et al. (1997) present numerical results for a latent heat thermal energy storage system modeled by a diffusion equation. Cleland et al. (1987) give experimental data for freezing and thawing of multi-dimensional objects modeled by finite element techniques.

An example of a PDE model describing freezing of a specific material (fish species), taking the phenomenon of *thermal arrest* caused by *latent heat of fusion* into account, was introduced in Backi and Gravdahl (2013). The parameters have to be state- (i.e. temperature)-dependent because their values change significantly not just above and below, but also around the freezing point according to the latent heat of fusion principle. In this case, the method used to model the latent heat of fusion is the so-called *apparent heat capacity method*, as introduced e.g. in Muhieddine et al. (2008). In this paper, we wish to study the stability properties of this model.

The model we consider is closely related to Burgers' equation,

$$u_t(t,x) = \varepsilon u_{xx}(t,x) + u(t,x)u_x(t,x)$$

where subscript refers to partial derivative wrt. the argument, e.g., $u_t(t,x) \equiv \partial u(t,x)/\partial t$.

This PDE is commonly used to describe turbulent flows and is closely related to the Navier-Stokes equations. The Burgers' equation is one of the very few nonlinear partial differential equations that can be solved exactly (for a restricted set of initial functions only, and for constant parameter ε). In the context of gas dynamics, Hopf (1950) and Cole (1951) independently showed that this equation can be transformed into the linear diffusion equation and solved exactly for arbitrary initial conditions (but again for constant ε). The study of the general properties of the Burgers' equation has motivated considerable attention due to its applications in fields as diverse as number theory, gas dynamics (Korshunova and Rozanova, 2009), heat conduction (Hills, 2006), elasticity (Sugimoto and Kakutani, 1985), etc.

The stability properties of the Burgers' equation with constant and time-varying parameters have been studied previously. Krstic (1999) presents stability results for both viscous and inviscid Burgers' equation by defining control laws satisfying a Lyapunov analysis in the L^2-norm. Balogh and Krstic (2000) introduce H^1-stability for the Burgers' equation with nonlinear boundary feedback. Krstic et al. (2008) show results in nonlinear stabilization of shock-like unstable equilibria in the viscous Burgers' equation, whereas Krstic et al. (2009) go a step further

⋆ This work was realized during the main author's research visit at the Department of Electronic Systems at Aalborg University, Denmark from August to December 2013
[1] corresponding author

and present results for the same problem in trajectory generation, tracking and observer design.

However, as explained above, the PDE considered here represents a freezing case with phase transition. This change in phase is the reason for introducing state-dependent parameters due to the fact that the physical properties of the material to be frozen change significantly after crossing the freezing point.

This means that the parameter ε depends on the state variable itself, namely $\varepsilon = \varepsilon(u(t,x))$, which is a more challenging case than the situations outlined above. This problem was initially introduced in Backi and Gravdahl (2013) for an application that describes the freezing of fish in a vertical plate freezer. Inspired by this problem, the present paper attempts to investigate the stability properties of the Burgers' equation with specific functional forms of ε and its derivatives.

As this paper deals with nonconstant, state-dependent parameters, the earlier described transformations from Burgers' equation to linear heat equation cannot be applied. Furthermore, the system considered in Backi and Gravdahl (2013) is quite limited in actuation, meaning that only limited Dirichlet and Neumann boundary control can be applied. Strictly speaking the Dirichlet boundary condition is equal to the temperature of the cooling medium, whereas the Neumann boundary condition represents heat flux through the boundary which is proportional to the difference between the temperature at the boundary and the temperature of the cooling medium. The main contribution of the paper is to generalize Backi and Gravdahl (2013) and show that under certain assumptions, the version of Burgers' equation we consider converges to a stationary solution determined by (constant) boundary conditions.

The rest of the paper is organized as follows. First, Section 2 provides a brief overview of previous results. Section 3 describes the problem setting of Backi and Gravdahl (2013). Section 4 provides the main stability results, whereupon Section 5 shows a few numerical examples that highlight the results in the previous section. Finally, Section 6 provides some concluding remarks.

Notation: We write f_x for the (partial) derivative of the function f with respect to x. Moreover, let $L^2([0,1])$ denote the space of real-valued, square integrable functions f defined on $[0,1]$ with finite L^2-norm; $\|f\|^2 = \int_0^1 f(x)^2 dx < \infty$. The space $H^1([0,1])$ is the subspace of $L^2([0,1])$ consisting of functions g with finite H^1-norm; $\|g\|_{H^1} = \|g\|^2 + \|g_x\|^2 < \infty$. In this paper we deal with functions $w = w(t,x)$ of time t and space x (the spatial variable). To ease notation we frequently leave out the dependency on t and/or x, e.g., $\|w(t)\|$ is the L^2-norm of the function $x \mapsto w(t)(x) = w(t,x)$.

2. PRELIMINARIES

First of all the diffusion equation for a constant parameter ε is introduced. Let $u = u(t,x)$ be a function of two variables, time $t \in \mathbb{R}_+$ and space x. For simplicity and without loss of generality, only one spatial dimension $x \in [0,L] \subset \mathbb{R}$ is considered.

The function u must satisfy a partial differential equation of the form

$$u_t = [\varepsilon u_x]_x, \quad u(0,x) = U(x), \tag{1}$$

where ε is a parameter and $U(x)$ denotes an initial condition. For the problem to be well posed, u must also satisfy various relevant boundary conditions. With constant ε, the diffusion equation (1) can be rewritten as the linear heat equation

$$u_t = \varepsilon u_{xx}. \tag{2}$$

The stability properties of (1) and thus of (2) have been studied extensively in many publications, see for example Krstic and Smyshlyaev (2008). In particular, it is known that the heat equation is stable in the sense that $u(t,x) \to \bar{u}(x)$ for *any* $\varepsilon > 0$, where $\bar{u}(x)$ describes the steady-state solution.

If the parameter ε depends on the spatial variable x, i.e. $\varepsilon = \varepsilon(x)$, we obtain the slightly more complicated expression for (1)

$$u_t = \varepsilon_x u_x + \varepsilon u_{xx}. \tag{3}$$

This may be handled by means of so-called *gauge transformations*, which eliminate the spatial dependency of $\varepsilon(x)$, see e.g., Smyshlyaev and Krstic (2010). In addition, the spatial derivative of the parameter, ε_x, vanishes after transforming the system; hence system (3) can be transformed into system (2) for which the stability properties mentioned above are known to hold.

Similar techniques can be applied to Burgers' equation. Heredero et al. (1999) showed that the standard Burgers' equation with parameters set to 1,

$$u_t = u_{xx} + 2uu_x \tag{4}$$

can be transformed into the Potential Burgers' equation by using the transformation $u = v_x$, resulting in

$$v_{tx} = v_{xxx} + 2v_x v_{xx}.$$

After integrating this expression we obtain

$$v_t = v_{xx} + v_x^2$$

which represents the potential form of the Burgers' equation. After introducing the transformation $w = e^v$ the Potential Burgers' equation then boils down to the linear heat equation, as outlined above.

Finally, if the coefficient ε is a known function of time, Burgers' equation becomes

$$u_t = \varepsilon(t)u_{xx} + uu_x \tag{5}$$

For this case, it was shown in Sophocleous (2004) that a *time*-dependent gauge transformation exists that transforms the nonlinear PDE into a linear one.

In the sequel, we shall refer to the following two well-known lemmas taken from Krstic and Smyshlyaev (2008).

Lemma 1. (Poincaré's Inequality) For any continuously differentiable function $\omega = \omega(z)$, the following inequalities hold:

$$\|\omega\|^2 \le 2\omega(0)^2 + 4\|\omega_z\|^2,$$
$$\|\omega\|^2 \le 2\omega(1)^2 + 4\|\omega_z\|^2.$$

Lemma 2. (Agmon's Inequality) For any function $\omega = \omega(t,x)$ with $\omega(t) \in H^1([0,1])$, the following inequalities hold:

$$\max_{x\in[0,1]} |\omega(t,x)|^2 \le \omega(0)^2 + 2\|\omega(t)\|\|\omega_x(t)\|,$$
$$\max_{x\in[0,1]} |\omega(t,x)|^2 \le \omega(1)^2 + 2\|\omega(t)\|\|\omega_x(t)\|.$$

3. PROBLEM FORMULATION

In the case considered in the present paper and Backi and Gravdahl (2013), a parabolic PDE is formulated in the state variable T representing temperature, as follows:

$$\rho(T)\, c(T)\, T_t = [\lambda(T)\, T_x]_x$$

subject to Dirichlet boundary conditions

$$T(t,0) = T(t,L) = \bar{T} \tag{6}$$

where $\rho(T)$ denotes the density, $c(T)$ indicates the specific heat capacity at constant pressure and $\lambda(T)$ describes the thermal conductivity of the medium to be frozen. Note that $\rho(T)$, $c(T)$ and $\lambda(T)$ all depend on the temperature T. The boundary condition $\bar{T}$ is given by the refrigerant temperature at $x = 0$ and $x = L$. Since λ depends on T, differentiation yields

$$\lambda_x(T) = \lambda_T(T)\, T_x$$

and thus

$$\rho(T)\, c(T)\, T_t = \lambda_T(T)\, T_x^2 + \lambda(T)\, T_{xx}. \tag{7}$$

Figure 1 displays a qualitative sketch of parameter variations in $\lambda(T)$ and $c(T)$ over T. Note that the variation in $\rho(T)$ over T is of minor consequence and therefore negligible, i.e. $\rho(T) = const$.

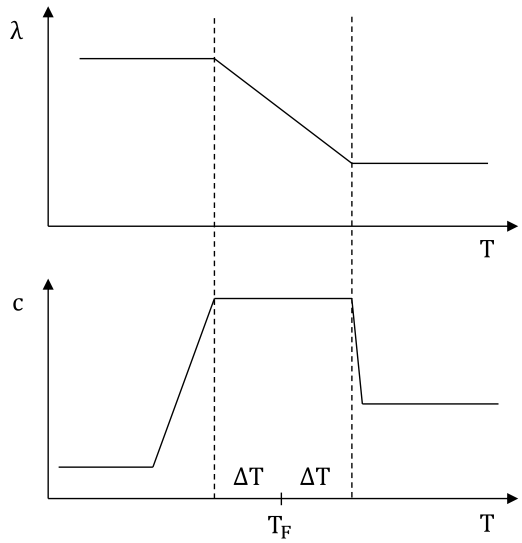

Fig. 1. Parameter variations in λ and c over T

To keep notation simple, two new parameters can be introduced as

$$k(T) = \frac{\lambda(T)}{\rho(T)\, c(T)} \tag{8}$$

and

$$\kappa(T) = \frac{\lambda_T(T)}{\rho(T)\, c(T)}. \tag{9}$$

This leads to a rewritten form of (7):

$$T_t = \kappa(T)\, T_x^2 + k(T)\, T_{xx}, \tag{10}$$

which is still subject to the boundary conditions defined in (6).

Here we remark that differentiating (10) with respect to x yields

$$\begin{aligned} T_{tx} &= \left[\kappa T_x^2 + k T_{xx}\right]_x \\ &= \kappa_T T_x^3 + 2\kappa T_x T_{xx} + k_T T_x T_{xx} + k\, T_{xxx} \\ &= \kappa_T T_x^3 + 2 T_x T_{xx}\left(\kappa + \frac{k_T}{2}\right) + k\, T_{xxx}. \end{aligned} \tag{11}$$

After introducing a change of variable $\varphi = T_x$ the following equation is obtained:

$$\varphi_t = \kappa_T \varphi^3 + 2\varphi\varphi_x\left(\kappa + \frac{k_T}{2}\right) + k\, \varphi_{xx} \tag{12}$$

which, apart from the additional term $\kappa_T \varphi^3$, is similar to the standard Burgers' equation in (4). As (10) is closely related to the Potential Burgers' equation, it will thus be referred to as a Potential Burgers'-like equation.

As mentioned, the stability properties of the Burgers' equation have been subject to many publications. However, these results are not applicable to (12) due to the fact that an additional term φ^3 is present and also that the effect of the non-constant parameters κ_T, κ, k_T and k on the stability properties of the overall PDE are unknown. This motivates the subsequent investigation.

4. STABILITY ANALYSIS

The PDE (10) in the previous section is specific for the freezing application. In this section, however, we choose to take a more general view of the problem and to indicate this, we follow the general notation from the Introduction and change the state variable from T to u.

In general, the function u is the sum of a transient part $w(t,x)$ and a stationary part $\bar{u}(x)$, i.e. $u(x,t) = w(x,t) + \bar{u}(x)$. In the present case we have $\bar{u}(x) = const$ due to the symmetric boundary conditions (6); for the general case, refer to Kreiss and Kreiss (1986). We normalize the spatial coordinate to belong to $[0,1]$ and, with slight abuse of notation, let $k = k(w+\bar{u})$ and $\kappa = \kappa(w+\bar{u})$. With these conventions, we study the following equivalent form of (10):

$$w_t = \frac{\kappa}{L^2} w_x^2 + \frac{k}{L^2} w_{xx} \tag{13a}$$

with boundary conditions

$$w(t,0) = w(t,1) = 0. \tag{13b}$$

Moreover, we will only be interested in continuously differentiable solutions with finite H^1-norm. Although the question of existence of such solutions is an important issue from a stringent mathematical point of view, we will not address that here. However, we note that our application studies indicate that at least some solutions of this form exist. With this remark we proceed to state the following result.

Lemma 3. Let w satisfy (13). Suppose that there exists constants $\beta > \alpha > 0$ such that $\alpha \le k \le \beta$. If

$$(\kappa + k_u)^2 < 2\left(\kappa k_u - k_{uu} k + k_u^2\right) \tag{14a}$$

$$k_{uu} k < k_u^2 + \kappa k_u \tag{14b}$$

then $\|w(t)\| \to 0$ as $t \to \infty$.

Proof. Define the Lyapunov candidate V by

$$V = \int_0^1 \frac{1}{k} w^2\, dx \tag{15}$$

and note that

$$V \ge \frac{1}{\beta}\|w(t)\|^2 \tag{16}$$

since $k \le \beta$ by assumption.

Differentiating (15) with respect to time leads to

$$
\begin{aligned}
\dot V &= \int_0^1 \frac{2}{k} w w_t - \frac{k_u}{k^2} w_t w^2 \, dx \\
&= \frac{1}{L^2} \int_0^1 \left[\frac{2\kappa}{k} \, w w_x^2 + 2 \, w w_{xx} - \frac{\kappa k_u}{k^2} w^2 w_x^2 - \frac{k_u}{k} w^2 w_{xx} \right] \, dx.
\end{aligned} \tag{17}
$$

Integrating the term ww_{xx} by parts yields

$$
\int_0^1 w w_{xx} \, dx = [w w_x]_0^1 - \int_0^1 w_x^2 \, dx \tag{18}
$$

with $[ww_x]_0^1 = 0$ due to (13b). Furthermore, by integrating the term $\frac{k_u}{k} w^2 w_{xx}$ by parts as well, we arrive at

$$
\begin{aligned}
\int_0^1 \frac{k_u}{k} w^2 w_{xx} \, dx = \left[\frac{k_u}{k} w^2 w_x \right]_0^1 - 2 \int_0^1 \frac{k_u}{k} w w_x^2 \, dx \\
- \int_0^1 \frac{k_{uu} k - k_u^2}{k^2} w^2 w_x^2 \, dx.
\end{aligned} \tag{19}
$$

Then, after inserting (18) and (19) into (17) and collecting terms the following expression is obtained

$$
\dot V = \frac{1}{L^2} \int_0^1 - \frac{w_x^2}{k} \left[a w^2 + b w + c \right] \, dx \tag{20}
$$

where we have used the shorthand

$$
a = \frac{\kappa k_u - k_{uu} k + k_u^2}{k} \tag{21a}
$$

$$
b = -2(\kappa + k) \tag{21b}
$$

$$
c = 2k. \tag{21c}
$$

Now, if (14) hold, we see that $a > 0$ and $b^2 - 4ac < 0$; hence there exists a lower bound $0 < \underline{K}$ for $aw^2 + bw + c$, and therefore

$$
\dot V \le - \frac{\underline{K}}{L^2} \int_0^1 \frac{1}{k} w_x^2 \, dx. \tag{22}
$$

Moreover, using that $\alpha \le k$ followed by *Poincaré's Inequality* (Lemma 1), we obtain

$$
\dot V \le - \frac{\underline{K}}{L^2 \alpha} \int_0^1 w_x^2 \, dx \le - \frac{\underline{K}}{4 L^2 \alpha} \int_0^1 w^2 \, dx = - \frac{\underline{K}}{4 L^2 \alpha} \| w(t) \|^2 \tag{23}
$$

which, together with (16) and (Henry, 1981, Theorem 4.1.4), proves the lemma.

Remark: Stability for Neumann boundary conditions can be investigated in much the same fashion as above. Indeed, the main difference boils down to the term $[ww_x]_0^1$ in (18), which is now not guaranteed to hold. In order for Lemma 3 to hold for Neumann boundary conditions as well, we would thus need the extra condition

$$
[w(t,1) \, w_x(t,1) - w(t,0) \, w_x(t,0)] \overset{!}{\le} 0. \tag{24}
$$

We now extend Lemma 3 to the H^1-case.

Lemma 4. Suppose that the assumptions of Lemma 3 hold true. If moreover

$$
2k\kappa_u - k_u \kappa \le 0 \tag{25a}
$$

$$
4\kappa + k_u \le 0 \tag{25b}
$$

$$
w_x(t,1) w_{xx}(t,1) - w_x(t,0) w_{xx}(t,0) \le 0 \tag{25c}
$$

then $\|w(t)\|_{H^1} \to 0$ as $t \to \infty$.

Proof. Define the Lyapunov candidate Λ by

$$
\Lambda = V_1 + V = \int_0^1 \frac{1}{k} w_x^2 \, dx + V \tag{26}
$$

where V denotes the Lyapunov function defined by (15).

Note that

$$
\Lambda \ge \frac{1}{\beta} \| w(t) \|_{H^1} \tag{27}
$$

since $k \le \beta$ by assumption.

The time derivative of V_1 is

$$
\dot V_1 = \int_0^1 \frac{2}{k} w_x w_{tx} \, dx - \frac{k_u}{k^2} w_t w_x^2. \tag{28}
$$

To obtain an expression for w_{tx} in terms of spatial derivatives of w only, the derivative of (13) with respect to x is calculated and one obtains

$$
w_{tx} = \frac{1}{L^2} \Big(\kappa_u w_x^3 + 2\kappa w_x w_{xx} + k_u w_x w_{xx} + k w_{xxx} \Big). \tag{29}
$$

Combining (29) and (28) gives

$$
\begin{aligned}
\dot V_1 = \frac{1}{L^2} \int_0^1 \Big(\frac{2\kappa_u}{k} \, w_x^4 + \frac{4\kappa + 2k_u}{k} w_x^2 w_{xx} + 2 w_x w_{xxx} \\
- \frac{\kappa k_u}{k^2} w_x^4 - \frac{k_u}{k} w_x^2 w_{xx} \Big) dx.
\end{aligned} \tag{30}
$$

Integrating the term $2 w_x w_{xxx}$ by parts yields

$$
\int_0^1 2 \, w_x w_{xxx} \, dx = [2 \, w_x w_{xx}]_0^1 - 2 \int_0^1 w_{xx}^2 \, dx
$$

and putting this into (30) one obtains

$$
\begin{aligned}
\dot V_1 &= \frac{1}{L^2} \int_0^1 \left[\frac{2k\kappa_u - k_u \kappa}{k^2} \, w_x^4 - 2 w_{xx}^2 + \frac{4\kappa + k_u}{k} \, w_x^2 w_{xx} \right] \, dx \\
&\quad + \frac{1}{L^2} \left[2 \, w_x w_{xx} \right]_0^1 \\
&= \frac{1}{L^2} \int_0^1 \left[\frac{2k\kappa_u - k_u \kappa}{k^2} \, w_x^4 - 2 w_{xx}^2 \right] dx \\
&\quad + \frac{1}{L^2} \int_0^1 \frac{4\kappa + k_u}{k} \, w_x^2 \, dw_x + \frac{1}{L^2} \left[2 \, w_x w_{xx} \right]_0^1.
\end{aligned}
$$

Using (25) and *Poincaré's Inequality* one receives

$$
\dot V_1 \le - \frac{2}{L^2} \int_0^1 w_{xx}^2 \, dx \le - \frac{2}{L^2} \int_0^1 w_x^2 \, dx.
$$

Hence by (23) and with $C = \min\{ \frac{\underline{K}}{4L^2 \alpha}, \frac{2}{L^2} \} > 0$ it can be concluded that

$$
\dot \Lambda \le -C \| w(t) \|_{H^1},
$$

which, together with (27) and (Henry, 1981, Theorem 4.1.4), proves the lemma.

Together with *Agmon's Inequality*, Lemma 4 now immediately implies the following main result of the paper.

Theorem 5. Let w satisfy (13). Suppose that the assumptions of Lemma 4 hold true. Then $w(t,x) \to 0$ as $t \to \infty$, hence $u(t,x) \to \bar{u}(x) = const$ as $t \to \infty$.

5. SIMULATION EXAMPLES

We now return to the original freezing application discussed in Section 3. Simulations were conducted for different boundary conditions (BCs) and initial conditions (ICs) for the purpose of illustrating the theoretical developments in Section 4.

Strictly speaking, the assumptions (25) are not satisfied by the freezing application in the minute region from $T_F + \Delta T$ to where c becomes constant in Figure 1. However, the assumptions (25) are conservative and can be relaxed by adding some technical adjustments to the proof of Lemma 4. These adjustments would be sufficient to prove formally that the freezing application is indeed stable, but would make the presentation and arguments in the previous section much more cumbersome and are therefore left out.

The simulation parameters for an actual physical freezing process have been used and can be found in Backi and Gravdahl (2013). The PDE has been discretized in the spatial domain only, using second order central differences and first order backward differences. This approach resulted in a set of coupled ODEs representing a spatial resolution of $1 \cdot 10^{-3}$ m ($N = 100$ discretization steps).

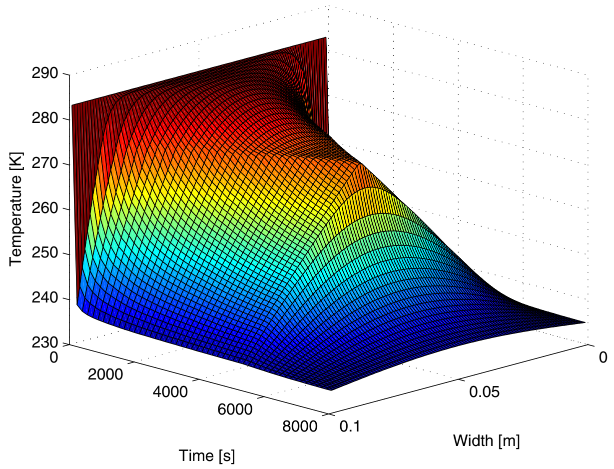

Fig. 2. Symmetric BCs and evenly distributed IC

In Figure 2 a case with BCs $T(t,0) = T(t,0.1) = 235$ K and an evenly distributed IC $T(0,x)) = 283$ K is shown. The figure shows how a ‘flattened triangular’ profile is prevalent in the temperature distribution until about 4000 s due to the removal of latent heat around the freezing point. Subsequently, the temperature profile converges smoothly towards a constant level of 235 K, the stationary solution given in Theorem 5.

To illustrate the conservativeness of Theorem 5, we now consider cases not covered by it, but where the PDE exhibits exactly the same stable behavior.

Figure 3 shows a more general case with asymmetric BCs $T(t,0) = 250$ K and $T(t,0.1) = 235$ K and evenly distributed IC $T(0,x) = 283$ K. In comparison to the first case it takes longer to observe the convergence trend towards the steady state solution but the solution is clearly stable in accordance

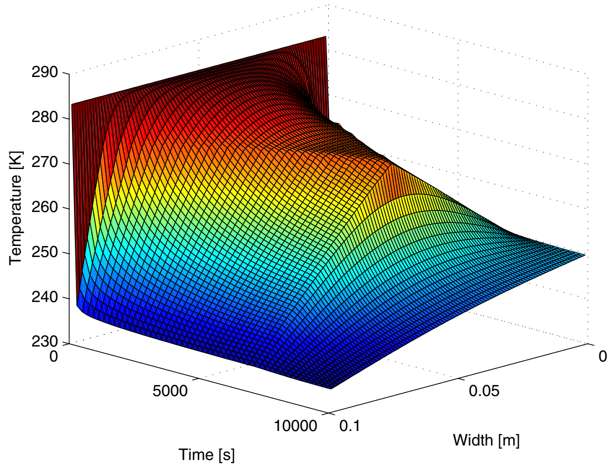

Fig. 3. Asymmetric BCs and evenly distributed IC

with Theorem 5. The reason for the slower convergence lies in the fact that $T(t,0)$ is larger than in the first case, resulting in a smaller temperature difference between the initial and the boundary temperature at $x = 0$ m.

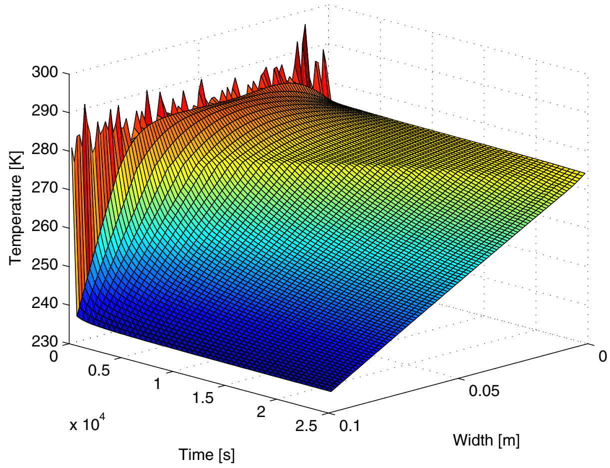

Fig. 4. Asymmetric BCs and noisy IC

In Figure 4 a simulation with asymmetric BCs $T(t,0) = 275$ K and $T(t,0.1) = 235$ K as well as noisy IC can be seen. Note that the BCs at $x = 0$ m and at $x = 0.1$ m are above and below the freezing point ($T_F = 272$ K), respectively. The noisy IC is represented by a sum of sinusoidal functions with different frequencies and added white gaussian noise. Converging to the steady state solution takes excessively longer than for the case demonstrated in Figure 3, due to the much larger boundary temperature $T(t,0)$ and the yet smaller difference between the initial and the boundary temperature at $x = 0$ m.

Figure 5 shows the behaviour in the first 50 seconds for the case shown in Figure 4. This shall illustrate the converging character even for noisy IC. As can be seen, the high frequency peaks caused by gaussian white noise get flattened out quite quickly, whereas the lower frequency sinusoidals need more time to be flattened out.

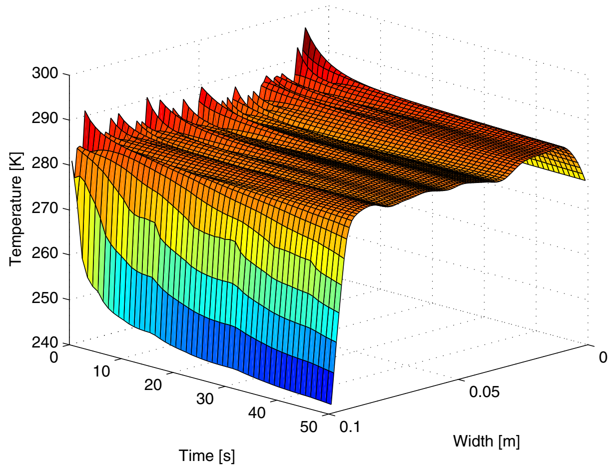

Fig. 5. Asymmetric BCs and noisy IC - 0 to 50 s

6. CONCLUSION

In this paper a stability investigation of a partial differential equation derived from the diffusion equation has been performed. The PDE has similarities to the potential form of the Burgers' equation and thus to the Burgers' equation itself. However, gauge transformations like the ones used in previous literature could not immediately be applied due to the presence of state-dependent coefficient functions. Therefore, we first showed stability in the sense of convergence in norm, and then in terms of absolute value of the transient part of the solution. This came at the price of some conservativeness, as we were forced to impose restrictions on derivatives and signs of the coefficient functions, see (25).

The simulation examples presented in Section 5 show that the final temperature distribution converges to the steady state solution in all cases, further indicating conservativeness of the analysis.

REFERENCES

Backi, C.J. and Gravdahl, J.T. (2013). Optimal boundary control for the heat equation with application to freezing with phase change. In *Proceedings of the 3rd Australian Control Conference*. Perth, Australia.

Balogh, A. and Krstic, M. (2000). Burgers' equation with nonlinear boundary feedback: H1 stability, well–posedness and simulation. *Mathematical Problems in Engineering*, 6, 189–200.

Cleland, D., Cleland, A., Earle, R., and Byrne, S. (1987). Experimental data for freezing and thawing of multi-dimensional objects. *International Journal of Refrigeration*, 10, 22–31.

Cole, J. (1951). On a quasi–linear parabolic equation occuring in aerodynamics. *Quarterly of Applied Mathematics*, 9, 225–236.

Costa, M., Buddhi, D., and Oliva, A. (1997). Numerical simulation of a latent heat thermal energy storage system with enhanced heat conduction. *Energy Conversion and Management*, 39(3/4), 319–330.

Henry, D. (1981). *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin.

Heredero, R.H., Levi, D., and Winternitz, P. (1999). Symmetries of the discrete burgers equation. *Journal of Physics A: Mathematical and General*, 32(14), 2685–2695.

Hills, R.G. (2006). Model validation: Model parameter and measurement uncertainty. *Journal of Heat Transfer*, 128(4), 339–351.

Hopf, E. (1950). The partial differential equation ut + uux = μuxx. *Communications on Pure and Applied Mathematics*, 3(3), 201–230.

Korshunova, A. and Rozanova, O. (2009). The riemann problem for the stochastically perturbed non-viscous burgers equation and the pressureless gas dynamics model. In *Proceedings of the International Conference Days on Diffraction, DD 2009*, 108–113. St. Petersburg, Russia.

Kreiss, G. and Kreiss, H.O. (1986). Convergence to steady state of solutions of burgers' equation. *Applied Numerical Mathematics*, 2(3–5), 161–179.

Krstic, M. (1999). On global stabilization of burgers' equation by boundary control. *Systems & Control Letters*, (37), 123–141.

Krstic, M., Magnis, L., and Vazquez, R. (2008). Nonlinear stabilization of shock-like unstable equlibria in the viscous burgers pde. *IEEE Transactions on Automatic Control*, 53(7), 1678–1683.

Krstic, M., Magnis, L., and Vazquez, R. (2009). Nonlinear control of the viscous burgers equation: Trajectory generation, tracking and observer design. *Journal of Dynamic Systems, Measurement and Control*, 131.

Krstic, M. and Smyshlyaev, A. (2008). *Boundary Control of PDEs - A Course on Backstepping*. Advances in Design and Control. Society for Industrial and Applied Mathematics.

Muhieddine, M., Canot, É., and March, R. (2008). Various approaches for solving problems in heat conduction with phase change. *International Journal of Finite Volume Method*, 6(1).

Pham, Q.T. (2006a). Mathematical modeling of freezing processes. In *Handbook of Frozen Food Processing and Packaging*, chapter 7. Taylor & Francis Group, LLC.

Pham, Q.T. (2006b). Modelling heat and mass transfer in frozen foods: a review. *International Journal of Refrigeration*, 29, 876–888.

Smyshlyaev, A. and Krstic, M. (2010). *Adaptive Control of Parabolic PDEs*. Princeton University Press, Princeton, New Jersey, USA.

Sophocleous, C. (2004). Transformation properties of a variable-coefficient burgers equation. *Chaos, Solitons and Fractals*, 20(5), 1047–1057.

Sugimoto, N. and Kakutani, T. (1985). 'generalized burgers' equation' for nonlinear viscoelastic waves. *Wave Motion*, 7(5), 447–458.

Woinet, B., Andrieu, J., and Laurent, M. (1998). Experimental and theoretical study of model food freezing. part 1. heat transfer modelling. *Journal of Food Engineering*, 35, 381–393.