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Toward a Rational Matrix Approximation of the Parameter-Dependent Riccati Equation Solution

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Abstract: This paper considers the problem of solving parameter-dependent Riccati equations. In this paper, a tractable iterative scheme involving mainly additions and multiplications is developed for finding solutions to arbitrary accuracy. It is first presented in the parameter-independent case and then extended to the parametric case. It hinges upon two results: (i) a palindromic quadratic polynomial matrix characterization of the matrix sign and square root functions. (ii) a particular representation of parameter-dependent matrices with negative and positive power series with respect to parameters. Several numerical examples are given throughout the paper to prove the validity of the proposed results.

Keywords: Algebraic Riccati Equation, Matrix Sign Function, Matrix Square Root, Parameter-dependent systems.

1. INTRODUCTION

Parameter-dependent algebraic Riccati equation (ARE) often arises in the design of controllers such as, for instance, gain scheduling control (Apkarian *et al.*, 1995), (Stilwell *et al.*, 1999), trade-off dependent control (Dinh *et al.,* 2003), (Dinh *et al.,* 2005). The main goal of this paper is to propose a computationally tractable algorithm for finding solutions to such problems, to arbitrary accuracy, based on the matrix sign or matrix square root functions framework (see for instance (Roberts, 1980), (Kenney *et al.*, 1995), (Higham, 1997) and references therein).

It is worthwhile to note that algorithms based on the matrix sign function have proven to be particularly attractive for the solution of large-scale ARE problems in the parameter-independent case. Several results, Supporting this statement, are available in the literature for linear continuous-time (Kenney *et al.*, 1989), (Kenney *et al.*, 1992), (Kenney *et al.*, 1995), (Quintana-Orti *et al.*, 1998), (Higham, 2008) and discrete-time (Fabbender *et al.*, 1999) invariant systems. Furthermore, some recent works try to extend these results to the parameter dependent case (Rice *et al.*, 2010), (Guerra *et al.*, 2012). Specifically, the method developed in (Rice *et al.*, 2010), consists in using the main iteration for the matrix sign function that is Newton's method associated to a linear fractional transformation (LFT) parametric dependence. However, an LFT order reduction is needed at each step. If it can be performed efficiently in the single parameter case using standard linear time-invariant state-space model order reduction, it is known to be a very hard problem in the multi-parametric case. In (Guerra *et al.*, 2012) an alternative method, seeking for exact solution of the ARE problem, is proposed based on the matrix sign function integral definition. Unfortunately, it seems to be time consuming in the multi-parametric case.

The proposed approach, hereinafter, does not rely on the Newton recursion for the matrix sign function. It proposes instead a "multiplication rich" (i.e. with only one inverse) scheme that converges to a rational matrix approximation of the ARE solution. On the one hand it gives a new insight of using the matrix sign (or matrix square root) for solving ARE's in the parameter-independent case, and on the other hand it is easily generalized to the multi-parametric case.

First, the application of the matrix sign to the solution of the standard ARE's and also the link between the matrix square root and a particular ARE (Incertis Carro *et al.*, 1977) (*i.e.* when the system has only real poles and under some restrictions on the output matrix) are outlined. Next, relying on a quite recent palindromic quadratic polynomial matrix characterization of the matrix sign and square root functions (see (Iannazzo *et al.*, 2011)), a new iterative scheme is proposed by use of Laurent series expansion of LTI systems. Then an extension to ARE problems with coefficient matrices in a class of parameter-dependent matrices with negative and positive power series with respect to parameters is described.

The remainder of this paper is organized as follows. Section II recalls some preliminary results on the matrix sign and square root functions and their applications to the solution of some ARE problems. Section III is devoted to the main results of this paper. These results will be presented in the parameter-independent case before a generalization to the multi-parametric case in section IV. All through the paper, some examples are presented in order to demonstrate the efficiency of the proposed approach.

Notations: Hereafter $\otimes$ denotes the Kronecker product of matrices. $Z(\theta)$ denotes matrices of monomials (with negative and positive power series) in θ. $\rho(Z)$ denotes the spectral radius of matrix Z. The matrix I_n is the identity matrix of dimension $n \times n$. $\|Z\|$ is any subordinate matrix norm. $\mathbb{C}^*$ denotes the union of the open right-half ($\mathbb{C}_+$) and the open left-half ($\mathbb{C}_-$) complex plane. Finally, the form $\binom{p}{k}$ with $p \in \mathbb{N},\ k \in \mathbb{N}$, is the binomial coefficient.

2. PRELIMINARIES

In this section, we introduce some preliminary definitions and results that are used extensively in the sequel.

2.1 The matrix sign and square root functions

Let us introduce the matrix sign function developed in (Roberts, 1980). It is well-known that the sign function is defined for $z \in \mathbb{C}^*$ by:

$$sign(z) = \begin{cases} 1, & \text{if } \mathrm{Re}(z) > 0 \\ -1, & \text{if } \mathrm{Re}(z) < 0 \end{cases} \tag{1}$$

If $\mathrm{Re}(z) = 0$ then $sign(z)$ is undefined. By analogy, the matrix sign function is restricted to square matrices with no eigenvalues on the imaginary axis. Let the matrix $Z \in \mathbb{R}^{n\times n}$ have a Jordan canonical form that is: $Z \triangleq T\ J\ T^{-1}$ with $J = diag(J_1, J_2)$ where the square matrices J_1 and J_2 have eigenvalues respectively in $\mathbb{C}^-$ and $\mathbb{C}^+$. Then the sign of the matrix $Z \in \mathbb{R}^{n\times n}$ can be defined as:

$$sign(Z) \triangleq T \begin{bmatrix} -I_{J_1} & 0 \\ 0 & I_{J_2} \end{bmatrix} T^{-1} \tag{2}$$

where I_{J_1} and I_{J_2} are identity matrices of the same dimensions as J_1 and J_2 respectively. This definition leads to some easily verified properties of $S = sign(Z)$:

(P1) $S^2 = I_n$.
(P2) S is diagonalizable with eigenvalues ± 1.
(P3) $SZ = ZS$.
(P4) if Z is real, S is real.
(P5) eigenvalues of SZ belong to $\mathbb{C}^{*+}$.

Based on property (P1) stated above we are led to an iterative method to calculate the matrix sign called Newton iterations (Roberts, 1980), (Kenney *et al.*, 1989):

$$\begin{aligned} &Z_0 = Z \\ &Z_{k+1} = \frac{1}{2}\left(Z_k + Z_k^{-1}\right),\ k = 1,2,3,\ldots \\ &sign(Z) = \lim_{k\to\infty} Z_k \end{aligned} \tag{3}$$

Another closed method is the Newton-Shultz (N-S) iterations scheme that is multiplication rich and is presented as follows:

$$\begin{aligned} &Z_0 = Z \\ &Z_{k+1} = \frac{1}{2} Z_k \left(3I_n - Z_k^2\right),\ k = 1,2,3,\ldots \\ &sign(Z) = \lim_{k\to\infty} Z_k \end{aligned} \tag{4}$$

Theorem 1: (Convergence of N-S iterations)
Let $Z \in \mathbb{C}^{n\times n}$ have no pure eigenvalues on the imaginary axis.

$$\text{If } \left\| I - Z^2 \right\| < 1 \text{ then } Z_k \to sign(Z) \text{ as } k \to \infty \tag{5}$$

and $\left\| I - Z_k^2 \right\| < \left\| I - Z^2 \right\|^{2^k}$.

Proof: see (Kenney *et al.*, 1992).

Remark 1: Theorem 1 describes a local convergence of the N-S iteration.

Another concise definition of the sign of Z is given by (Kenney *et al.*, 1989) and recalled hereafter.

Lemma 1:
Let $Z \in \mathbb{C}^{n\times n}$ have no eigenvalues on the imaginary axis.

$$sign(Z) = Z\left(Z^2\right)^{-1/2} \tag{6}$$

Proof: Suppose that $Z = SN$ with $S = sign(Z)$. Property (P1) leads to $N = S^{-1}Z = SZ$. As Z commutes with S according to property (P3), $N^2 = SZS^{-1}Z = ZSS^{-1}Z = Z^2$. Finally, using (P5) the principal square root of $N = \left(Z^2\right)^{1/2}$ exists and (6) is verified.

As for the matrix sign function, let us recall the matrix square root Newton-Shulz (SRN-S) iterations (see (Higham, 1997)). For this aim, consider $Z \in \mathbb{C}^{n\times n}$ a matrix with no nonpositive real eigenvalues. Note that:

$$sign\left(\begin{bmatrix} 0 & Z \\ I_n & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & Z^{\frac{1}{2}} \\ Z^{-\frac{1}{2}} & 0 \end{bmatrix},\ Z \in \mathbb{C}_+^{n\times n} \tag{7}$$

Thus, the SRN-S scheme is given by:

$$\begin{cases} Y_0 = Z \text{ and } W_0 = I_n \\ Y_{k+1} = \dfrac{1}{2} Y_k \left(3I_n - W_k Y_k\right) \ ,\ k = 1,2,3,\ldots \\ W_{k+1} = \dfrac{1}{2}\left(3I_n - W_k Y_k\right) W_k \end{cases} \tag{8}$$

with $\lim_{k\to\infty} Y_k = Z^{\frac{1}{2}}$ and $\lim_{k\to\infty} W_k = Z^{-\frac{1}{2}}$.

Theorem 2: (Convergence of N-S iterations)
Let $Z \in \mathbb{C}^{n\times n}$ a matrix with no nonpositive real eigenvalues.

$$\text{If } \left\| \begin{bmatrix} I_n - Z & 0 \\ 0 & I_n - Z \end{bmatrix} \right\| < 1 \text{ then } \begin{bmatrix} 0 & Y_k \\ W_k & 0 \end{bmatrix} \to \begin{bmatrix} 0 & Z^{\frac{1}{2}} \\ Z^{-\frac{1}{2}} & 0 \end{bmatrix} \text{ as } k \to \infty \tag{9}$$

Proof: Obvious from Theorem 1.

2.2 Application to some AREs

Two different AREs are alternately considered in this paper. The first one is a standard ARE of the form

$$A^T X + XA - XBR^{-1}B^T X + C^T C = 0 \tag{10}$$

with the coefficient matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $R \in \mathbb{R}^{m\times m}$. We assume that:

(A1) R is positive definite,
(A2) (A, B) is stabilizable
(A3) (A, C) is detectable.

Let us also define, as usual, the associated Hamiltonian matrix:

$$H = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \in \mathbb{R}^{2n\times 2n} \quad (11)$$

According to the assumptions (A2)-(A3), H has no imaginary axis eigenvalues. The following result, due to (Roberts, 1980), shows the application of the matrix sign to the solution of ARE (10).

Theorem 3: (A matrix sign based solution to the ARE)

Let $sign(H) = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$. Then X, the unique symmetric positive semi-definite stabilizing solution of ARE (10) is a solution of

$$\begin{bmatrix} S_{12} \\ S_{22}+I_n \end{bmatrix} X = -\begin{bmatrix} S_{11}+I_n \\ S_{21} \end{bmatrix} \quad (12)$$

Proof: Can be found for instance in (Roberts, 1980).

Moreover, a subclass of the ARE problem given by (10) will also be considered in this paper. This problem, first introduced in (Jones, 1976) and extended in (Incertis Carro *et al.*, 1977), concerns systems with real poles under the following constraints on the output matrix C:

(A4) C is invertible

(A5) $(C^TC).A$ is a symmetric matrix

and will be denoted here (CARE). According to the results in (Incertis Carro *et al.*, 1977), the solution to the CARE problem is given by

$$X = C^T\left(\left(CBR^{-1}B^TC^T + CA^2C^{-1}\right)^{1/2} - CAC^{-1}\right)^{-1} C \quad (13)$$

which obviously links the solution of this constrained problem to the positive definite square root matrix of $\left(CBR^{-1}B^TC^T + CA^2C^{-1}\right)$. Interested readers can refer to (Incertis Carro *et al.*, 1977) for more details.

Remark 2: The solution of the standard ARE problem implies the computation of the matrix sign of a $2n\times 2n$ matrix. While for the CARE problem the solution is linked to the square matrix of only a $n\times n$ matrix.

3. THE PARAMETER-INDEPENDENT ARE AND CARE PROBLEMS

In this section, a new method is proposed to solve the ARE and CARE problems with parameter-independent coefficients matrices. For this aim, let us first define a palindromic Laurent polynomial matrix as:

$$L(z,P,Q) \triangleq Pz^{-1} + Q + Pz \quad (14)$$

with $z\in\mathbb{C}$, $P\in\mathbb{R}^{n\times n}$ and $Q\in\mathbb{R}^{n\times n}$.

Lemma 2: (Iannazzo *et al.*, 2011)

The Laurent matrix polynomial (14) is invertible in an open annulus containing the unit circle if and only if the matrix $M = Q^{-1}P$ does not have real eigenvalues of modulus greater than or equal to ½. Moreover, this polynomial is invertible for any $z\in\mathbb{C}$ such that $r<|z|<1/r$, with:

$$r = \rho\left(-2M\left(I+\left(I-4M^2\right)^{1/2}\right)^{-1}\right) \quad (15)$$

and the inverse $L(z,P,Q)^{-1} \triangleq \sum_{i=-\infty}^{i=+\infty} F_i z^i$ is such that F_0 is given by:

$$F_0 = \left(I-4M^2\right)^{-\frac{1}{2}} Q^{-1} \quad (16)$$

Proof: see Lemma 7 in (Iannazzo *et al.*, 2011).

3.1 New computation method for the matrix sign

One of the contributions of this paper is to present a rational matrix approximation of the matrix sign by use of a new method of computing the constant coefficient matrix F_0 given by (16).

Theorem 4: (A new matrix sign characterization)

Let $Z\in\mathbb{R}^{n\times n}$ be a matrix with no imaginary axis eigenvalues such that $r = \rho\left(2\left(I_n+Z^2\right)^{-1} - I_n\right) < 1$, then the matrix sign of Z is given by:

$$sign(Z) = 2\left(I_n+Z^2\right)^{-1} Z \cdot \left(I_n + \sum_{l=1}^{\infty}\binom{2l}{l}\frac{1}{2^{2l}}\left(2\left(I_n+Z^2\right)^{-1} - I_n\right)^{2l}\right) \quad (17)$$

Proof: Let us first consider $L\left(z, \frac{1}{4}\left(Z^{-1}-Z\right), \frac{1}{2}\left(Z^{-1}+Z\right)\right)$ and the following change of variable:

$$\tilde{z}^{-1} = z + \frac{1}{z},\ z\in\mathbb{C}, z\neq 0, r<|z|<1/r \quad (18)$$

Note that if $r<1$ then $\tilde{z}$ exist. Hence, according to (14)

$$L\left(\tilde{z}, \frac{1}{4}\left(Z^{-1}-Z\right), \frac{1}{2}\left(Z^{-1}+Z\right)\right) = \frac{1}{4}\left(Z^{-1}-Z\right)\tilde{z}^{-1} + \frac{1}{2}\left(Z^{-1}+Z\right) \quad (19)$$

Thus, the inverse of the Laurent polynomial matrix $L(\tilde{z}) \triangleq L\left(\tilde{z}, \frac{1}{4}\left(Z^{-1}-Z\right), \frac{1}{2}\left(Z^{-1}+Z\right)\right)$ can be expressed as a proper transfer function with a state-space representation of the form

$$\left(L(\tilde{z})\right)^{-1} = D_L + C_L\left(\tilde{z}I_n - A_L\right)^{-1} B_L \quad (20)$$

Using the well-known Laurent series expansion of (20) leads to

$$\left(L(\tilde{z})\right)^{-1} = \tilde{F}_0 + \sum_{k=1}^{\infty}\tilde{F}_k\tilde{z}^{-k} = \tilde{F}_0 + \sum_{k=1}^{\infty}\tilde{F}_k\left(z+\frac{1}{z}\right)^{-k} \quad (21)$$

where $\tilde{F}_0 = D_L$ and $\tilde{F}_k = C_L A_L{}^k B_L, k=0,1,\ldots$ are the Markov parameters. Besides, according to Theorem 8 in (Iannazzo *et al.*, 2011), for the following particular choice of the pair of matrices (P,Q): $P = \frac{1}{4}\left(Z^{-1}-Z\right)$ and $Q = \frac{1}{2}\left(Z^{-1}+Z\right)$, the constant coefficient matrix F_0 given by (16) is the matrix sign of Z that is

$$sign(Z) = F_0 = \tilde{F}_0 + \sum_{l=1}^{\infty} C_l^{2l} \tilde{F}_{2l} \tag{22}$$

Finally, noting that $A_L = B_L = -\left(I_n + Z^2\right)^{-1} + (1/2) I_n$ and $C_L = D_L = \tilde{F}_0 = 2\left(I_n + Z^2\right)^{-1} Z$, it is easy to see that (17) holds.

In the sequel the following matrix:

$$Z_q = 2\left(I_n + Z^2\right)^{-1} Z.\left(I_n + \sum_{l=1}^{q} \binom{2l}{l} \frac{1}{2^{2l}} \left(2\left(I + Z^2\right)^{-1} - I_n \right)^{2l} \right) \tag{23}$$

will denote the q^{th} rational matrix approximation of $sign(Z)$.

Remark 3: Note that (17) involves only one matrix inverse $\left(I + Z^2\right)^{-1}$.

Remark 4: Note also that a scheme using a rational approximation in the initialization stages of the iteration and then switching to a N-S procedure in the final iterations will be particularly pertinent when computing the matrix sign of a parameter dependant matrix.

Hereafter, a new algorithm for matrix sign computation is proposed. This algorithm needs the computation of only one matrix inverse $\left(I + Z^2\right)^{-1}$ at the initialization step.

Algorithm 1: Given a matrix $Z \in \mathbb{R}^{n \times n}$, with no imaginary axis eigenvalues, such that $\rho\left(2\left(I_n + Z^2\right)^{-1} - I_n\right) < 1$ and a tolerance ε for testing convergence.

1 $X_0 = Z$, $k = 0$, $0 < \varepsilon << 1$.
2 while $\left\| I_n - X_k^2 \right\| \geq 1$,
3 $\quad k = k+1$, $X_k = Z_k$, (Z_k is given by (23))
4 end
5 while $\left(\left\| X_k - X_{k-1} \right\| / \left\| X_k \right\| \right) > \varepsilon$
6 $\quad X_{k+1} = \frac{1}{2} X_k \left(3 I_n - X_k^2\right)$, $k = k+1$
7 end
8 $sign(Z) = X_k$.

Remark 5: Note that a scaling step can be added to Algorithm 1 and is omitted here for brevity. Moreover, a convergence test more suitable for N-S iterations can be used instead of the test given in line 6.

Furthermore, it is important to note that a rational matrix approximation (23) with a small number q is needed so that the condition $\left\| I_n - X_k^2 \right\| < 1$ holds which allows switching to N-S iterations. This was observed on a large number of examples such as those described in Example 1 and Example 2.

Example 1: This example is borrowed from (Koç *et al.*, 1994) and uses some matrices proposed, for instance, in (Higham, 1991). Let $Z = UDU^T$, where $D = randsvd\left(n, \kappa, 3, kl, ku\right)$, (see (Higham, 2002) for details on "randsvd" function), is a $\left(kl, ku\right)$ banded random matrix, with geometrically distributed singular values. U is an arbitrary unitary matrix. We choose a condition number $\kappa = 100$ and equal lower and upper bandwidth of 2, 4 and 8 for matrix dimension $n = 4$, 8 and 16, respectively. A tolerance $\varepsilon = 10^{-10}$ is considered. Algorithm 1 is tested in comparison to Newton's method and Table 1 summarizes the obtained results.

Table 1. Results for Example 1

n	**Algorithm 1** (§3)		Newton alg.
	Number q of the rational approximation	Number of N-S steps	Number of Newton steps
4	1	5	6
8	2	5	7
16	2	6	9

Example 2: This example is from (Kenney *et al.*, 1992a). Let $Z = diag\left(\lambda_1, \ldots, \lambda_n\right) + T \in \mathbb{R}^{n \times n}$ with λ_i randomly and uniformly distributed in $[-10, 10]$ and T is strictly upper triangular with entries uniformly distributed in $[-1, 1]$. Algorithm 1 is tested again in comparison to Newton's method with $\varepsilon = 10^{-10}$. Table 2 summarizes the obtained results.

Table 2. Results for Example 2

n	**Algorithm 1** (§3)		Newton alg.
	Number q of the rational approximation	Number of N-S steps	Number of Newton steps
4	1	5	7
8	1	6	8
16	2	6	9

All the numerical example tested in Example 1 and Example 2 verify the condition $\rho\left(2\left(I_n + Z^2\right)^{-1} - I_n\right) < 1$.

These examples show that Algorithm 1 lead to a solution, with an arbitrary accuracy, after approximately the same number of iterations as the Newton's method using only one matrix inverse.

Example 3: Consider the ARE problem of the form (10) with the following data:

$$A = \begin{bmatrix} 3 & 1 & 4 \\ -1 & 2 & 5 \\ -1 & 3 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}, \; R = 1$$

The associated Hamiltonian matrix H has no eigenvalues on the imaginary axis and verifies $\rho\left(2\left(I_6 + H^2\right)^{-1} - I_6\right) < 1$.

Applying Algorithm 1, the first rational approximation (given by (24) with $q = 1$) of $sign(H)$ is found to be

$$Z_1 = \begin{bmatrix} 0.720 & -0.479 & 0.135 & -0.017 & -0.037 & 0.022 \\ 0.101 & 0.162 & 0.578 & -0.037 & -0.150 & 0.053 \\ 0.083 & 0.406 & -0.193 & 0.022 & 0.053 & -0.029 \\ -0.450 & 0.065 & -0.493 & -0.720 & -0.101 & -0.083 \\ 0.065 & -0.010 & 0.089 & 0.479 & -0.162 & -0.406 \\ -0.493 & 0.089 & -0.766 & -0.135 & -0.578 & 0.193 \end{bmatrix}$$

and verifies $\left\| I_6 - Z_1^2 \right\|_2 = 0.989 < 1$. 8 iterations of the N-S steps (line 5 of Algorithm 1) lead to

$$sign(H) = \begin{bmatrix} 0.996 & -0.313 & 0.566 & -0.032 & -0.053 & 0.038 \\ -0.008 & 0.390 & 1.083 & -0.053 & -0.148 & 0.049 \\ -0.026 & 0.725 & -0.370 & 0.038 & 0.049 & -0.131 \\ -0.532 & -0.210 & -0.402 & -0.996 & 0.008 & 0.026 \\ -0.210 & -0.342 & -0.013 & 0.313 & -0.390 & -0.725 \\ -0.402 & -0.013 & -0.836 & -0.566 & -1.083 & 0.370 \end{bmatrix}$$

with the convergence tolerance $\varepsilon = 10^{-10}$. Finally, the approximate positive solution of the ARE (10) is found to be

$$\tilde{X} = \begin{bmatrix} 207.31 & -63.151 & 36.043 \\ -63.151 & 31.969 & -0.817 \\ 36.043 & -0.817 & 14.857 \end{bmatrix}.$$

3.2 A new computation method for the square root matrix

Following the lines of §III.A, a new rational approximation of the matrix square root of a given matrix Z is presented hereafter.

Theorem 5: (A new matrix square root characterization)
Let $Z \in \mathbb{R}^{n\times n}$ be a matrix with no nonpositive real eigenvalue such that $\rho\left(2Z(I_n+Z)^{-1} - I_n\right) < 1$, then a matrix square root of Z is given by:

$$Z^{1/2} = 2\left(I_n + \sum_{l=1}^{q} \binom{2l}{l} \frac{1}{2^{2l}} \left(2Z\left(I_n+Z\right)^{-1} - I_n\right)^{2l}\right).Z\left(I_n+Z\right)^{-1} \quad (24)$$

Proof: Similarity to the proof of Theorem 3 and using the Laurent polynomial $L\left(\tilde{z}, \frac{1}{4}\left(I_n - Z^{-1}\right), \frac{1}{2}\left(I_n + Z^{-1}\right)\right)$ and the following matrices $A_L = C_L = \left(I_n + Z^{-1}\right)^{-1} - (1/2)I_n$ and $B_L = D_L = 2\left(I_n + Z^{-1}\right)^{-1}$.

Furthermore, the following matrix will denotes, in the sequel, the q^{th} rational approximation of $Z^{1/2}$

$$Z_q^{sr} = 2\left(I_n + \sum_{l=1}^{q} \binom{2l}{l} \frac{1}{2^{2l}} \left(2Z\left(I_n+Z\right)^{-1} - I_n\right)^{2l}\right).Z\left(I_n+Z\right)^{-1} \quad (25)$$

Algorithm 2: Given a matrix $Z \in \mathbb{R}^{n\times n}$ with no nonpositive real eigenvalue such that $\rho\left(2Z\left(I_n+Z\right)^{-1} - I_n\right) < 1$ and a tolerance ε for testing convergence.

$X_0 = Z$, $W_0 = I_n$, $k = 0$, $0 < \varepsilon << 1$.

while $\left\| \begin{bmatrix} X_k - I_n & 0 \\ 0 & X_k - I_n \end{bmatrix} \right\| \geq 1$

  $k = k+1$, $X_k = Z_k^{sr}$, (Z_k^{sr} is given by (25))

end

while $\left(\left\| X_k - X_{k-1} \right\| / \left\| X_k \right\| \right) > \varepsilon$

$$\begin{cases} X_{k+1} = \frac{1}{2} X_k \left(3I_n - W_k X_k\right) \\ W_{k+1} = \frac{1}{2}\left(3I_n - W_k X_k\right) W_k \end{cases}, W_k = X_k X_0^{-1}, \quad k = k+1$$

end

$Z^{1/2} = X_k$.

Example 4: Consider the CARE problem with the following data

$$A = \begin{bmatrix} 2 & -1 & 5 \\ 1 & 5 & -3 \\ 2 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, R = 1 \quad (26)$$

It is easy to see that this example verifies the assumptions (A3)-(A5). Applying Algorithm 2, the first rational approximation (given by (24) with $q = 1$) of $\left(CBR^{-1}B^TC^T + CA^2C^{-1}\right)^{1/2}$ is found to be

$$Z_1^{sr} = \begin{bmatrix} 2.484 & 0.131 & 0.247 \\ 0.131 & 2.587 & -0.348 \\ 0.247 & -0.348 & 2.262 \end{bmatrix}$$

with $\left\| Z_1^{sr} - I_3 \right\|_2 = 0.773 < 1$. Thence, the condition to switch to the N-S iterations is verified. After 8 N-S steps we obtain:

$$X_8 = \begin{bmatrix} 3.654 & 0.261 & 0.763 \\ 0.261 & 4.837 & -1.594 \\ 0.763 & -1.594 & 3.447 \end{bmatrix}$$ with the following convergence tolerance $\varepsilon = 10^{-10}$. Then, we found the approximate solution of the CARE problem given by:

$$\tilde{X} = C^T\left(X_8 - CAC^{-1}\right)^{-1} C = \begin{bmatrix} 1850.5 & 3686.2 & -119.57 \\ 3686.2 & 7378.9 & -258.72 \\ -119.57 & -258.72 & 19.937 \end{bmatrix}.$$

4. THE PARAMETER-DEPENDENT ARE AND CARE PROBLEMS

In this section, the coefficient matrices of the ARE and the CARE problems are supposed to belong to the following class of parameter dependent matrices:

$$Z(\theta) \triangleq \bar{Z}\left(\bar{\theta} \otimes I_n\right) \in \mathbb{R}^{n\times n} \quad (27)$$

with $\bar{Z} \in \mathbb{R}^{n\times nw^\gamma}$, $\bar{\theta} = \left[\bar{\theta}_1 \otimes \cdots \otimes \bar{\theta}_\gamma\right]^T \in \mathbb{R}^{w^\gamma}$ and $\bar{\theta}_k = \left[\theta_k^{-g} \ \cdots \ \theta_k^{-1} \ 1 \ \theta_k^{1} \ \cdots \ \theta_k^{h}\right]^T \in \mathbb{R}^w$, $\forall k \in [1, \gamma]$.

In the sequel, $Z(\theta)$ is supposed to be non singular. Basic arithmetic over the class of parameter-dependent matrices introduced here follows directly according to the proposed definition (27) and is summarized in the following lemma.

Lemma 3: Given $Z(\theta) = \bar{Z}\left(\bar{\theta} \otimes I_n\right) \in \mathbb{R}^{n\times n}$, $V(\theta) = \bar{V}\left(\bar{\theta} \otimes I_n\right) \in \mathbb{R}^{n\times n}$, a partition of $\bar{Z}$ given by $\bar{Z} = \left[\bar{Z}_1 \ \cdots \ \bar{Z}_w\right]$, $\bar{Z}_{i\in[1,w]} \in \mathbb{R}^{n\times n}$ and $\bar{\theta} = \left[\bar{\theta}_1 \otimes \cdots \otimes \bar{\theta}_\gamma\right]^T$

then the sum $Z(\theta)+V(\theta)$, the product $Z(\theta)V(\theta)$ and the power of $Z(\theta)$ can be represented by:

$$Z(\theta)+V(\theta)=(\bar{Z}+\bar{V})(\bar{\theta}\otimes I_n) \tag{28}$$

$$Z(\theta)\cdot V(\theta)=\bar{Z}(I_{m^\gamma}\otimes\bar{V})((\bar{\theta}\otimes\bar{\theta})\otimes I_n) \tag{29}$$

$$Z(\theta)^l=\bar{Z}\left(\prod_{j=1}^{l-1}(I_{w^j}\otimes\bar{Z})\right)(\bar{\theta}_l\otimes I_n) \tag{30}$$

with $\bar{\theta}_l \triangleq \underbrace{\bar{\theta}\otimes\bar{\theta}\otimes\ldots\otimes\bar{\theta}}_{l}$.

Remark 6: Equations (28), (29) and (30) lead to simple computations on some constant matrices.

However, the inverse of $Z(\theta)$, that is $Z(\theta)^{-1}=\dfrac{1}{\det(Z(\theta))}Z_{adj}(\theta)$, can be written under the form (27) at the cost of the introduction of an additional parameter representing $1/\det(Z(\theta))$ since the adjoint matrix $Z_{adj}(\theta)$ is of the form (27) and can easily be obtained.

The last Remark implies the use of a multiplication rich iteration scheme for finding the matrix sign (or matrix square root) of $Z(\theta)$. Indeed, such scheme can avoid the introduction of additional parameters as suggested for the inverse matrix $Z(\theta)^{-1}$.

Moreover, for standard ARE problem, the assumptions (A1)-(A3) and the condition $\rho\left(2\left(I_n+H(\theta)^2\right)^{-1}-I_n\right)<1$, where $H(\theta)$ is the Hamiltonian matrix, are supposed to hold in a l_2 norm ball of radius δ centred on θ_0.

Similarly, for the CARE problem the assumptions (A1)-(A5) and the condition $\rho\left(2Z(\theta)(I_n+Z(\theta))^{-1}-I_n\right)<1$, where $Z(\theta)=\left(C(\theta)B(\theta)R(\theta)^{-1}B(\theta)^T C(\theta)^T+C(\theta)A(\theta)^2C(\theta)^{-1}\right)$ are supposed to hold in a l_2 norm ball of radius δ centred on θ_0.

Example 5: Consider the ARE problem of the form (10) with the following parameter-dependent coefficient matrices

$$A(\theta)=\begin{bmatrix}\theta-3 & 1-2\theta^2\\ -1 & -3\theta\end{bmatrix},\ B=\begin{bmatrix}0\\1\end{bmatrix},\ C(\theta)=[\theta\ \ -1],\ R=1$$

with $\theta_0=0$ and $\delta=1.5$.

Hence, the associated Hamiltonian matrix is given by

$$H(\theta)=\begin{bmatrix}\theta-3 & 1-2\theta^2 & 0 & 0\\ -1 & -3\theta & 0 & -1\\ -\theta^2 & \theta & 3-\theta & 1\\ \theta & -1 & 2\theta^2-1 & 3\theta\end{bmatrix}$$

We then apply Algorithm 1, using the arithmetic described in Lemma 3 with fixed, *a priori*, number $q=1$ and number of N-S steps $N=6$. Fig. 1 shows that the condition $\left\|I_4-(H_1(\theta))^2\right\|_2<1$ holds for all $\theta\in[-\delta,\delta]$ where $H_1(\theta)$ is the first rational approximation given by (23).

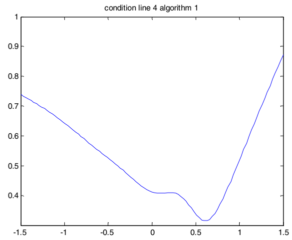

Fig. 1. The condition "line 4 of Algorithm 1" for the first rational matrix approximation of $sign(H(\theta))$.

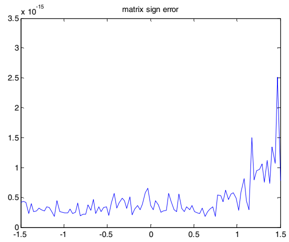

Fig. 2. The error $\left\|\tilde{S}(\theta)-S_N(\theta)\right\|_2$.

Let $\tilde{S}(\theta)$ be the approximate matrix sign of $H(\theta)$ obtained by the method presented above, while $S_N(\theta_i)$ is the matrix sign found by the Newton iterations (3) for a given gridding of θ in the interval $[-\delta,\delta]$. Fig. 2 shows the error $\left\|\tilde{S}(\theta_i)-S_N(\theta_i)\right\|_2$, $\theta_i\in[-\delta,\delta]$.

Finally, we found an approximate $\tilde{X}(\theta)$ that satisfies the ARE (10) with a residual error norm of only

$$\max_{\theta\in[-\delta,\delta]}\Big\|A(\theta)^T\tilde{X}(\theta)+\tilde{X}(\theta)A(\theta)$$
$$-\tilde{X}(\theta)BR^{-1}B^T\tilde{X}(\theta)+C(\theta)^TC(\theta)\Big\|\approx\ 5.69.10^{-14}$$

Example 6: Consider the CARE problem with the following parameter-dependent coefficient matrices

$$A(\theta)=\begin{bmatrix}4\theta+3 & 1-\theta & 3\theta+1\\ 2 & \theta^2+1 & \theta+3\\ -4\theta-2 & \theta+1 & -7\theta-2\end{bmatrix},\ B=\begin{bmatrix}0\\0\\1\end{bmatrix},C(\theta)=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix},\ R=1$$

with $\theta_0=0$ and $\delta=1.5$.

We then apply Algorithm 2 to find the square root matrix of

$$Z(\theta)=C(\theta)BR^{-1}B^TC(\theta)^T+C(\theta)A(\theta)^2C(\theta)^{-1}=$$

$$=\begin{bmatrix}(4\theta+2)^2+6 & -2\theta^2-6\theta+2 & 12\theta^2+4\theta+1\\ -2\theta^2-6\theta+2 & (\theta+1)^2+(\theta^2+1)^2+4 & \theta^3-2\theta^2-10\theta-3\\ 12\theta^2+4\theta+1 & \theta^3-2\theta^2-10\theta-3 & 26\theta^2+18\theta+6\end{bmatrix} \tag{31}$$

Fig. 3 and Fig. 4 summarize the results obtained using $q=1$ and $N=8$ N-S steps. In particular, Fig. 3 shows that $\left\|Z_1^{sr}(\theta)-I_3\right\|_2<1$ holds for $\theta\in[-\delta,\delta]$ where $Z_1^{sr}(\theta)$ is the first rational matrix approximation given by (25).

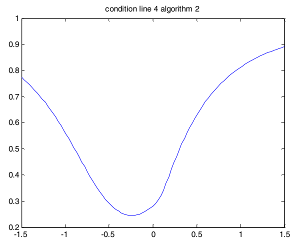

Fig. 3. The condition "line 4 of Algorithm 2" for the first rational matrix approximation of the square root matrix.

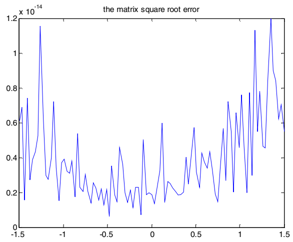

Fig. 4. The error $\left\|S^r(\theta)-S_N^r(\theta)\right\|_2$.

Following the lines of Example 5, Fig. 4 shows the error $\left\|\tilde{S}^{sr}(\theta_i)-S_N^{sr}(\theta_i)\right\|_2$, $\theta_i\in[-\delta,\delta]$ where $\tilde{S}^{sr}(\theta)$ is the approximate matrix square root of (31) obtained by Algorithm 2 and $S_N^{sr}(\theta_i)$ is the matrix square root found by a Newton-type scheme for a given gridding of θ in the interval $[-\delta,\delta]$. Finally, we found an approximate $\tilde{X}(\theta)$ that satisfies the CARE with a residual error norm of only

$$\max_{\theta\in[-\delta,\delta]}\Big\|A(\theta)^T\tilde{X}(\theta)+\tilde{X}(\theta)A(\theta)$$
$$-\tilde{X}(\theta)BR^{-1}B^T\tilde{X}(\theta)+C(\theta)^TC(\theta)\Big\|\approx 2.5.10^{-7}$$

6. CONCLUSION

In this paper, two iterative algorithms, that have the advantage of involving mainly additions and multiplications, were described for the computation of the matrix sign and square root functions. A new rational matrix approximation of these matrix functions, based on palindromic quadratic polynomial matrices and Laurent series-expansion of LTI systems, is presented. The proposed algorithms use the proposed rational matrix approximation in the initialization stages and a Newton-Shultz procedure in the final iterations. These algorithms are tailored for computing the matrix sign or square root functions for a class of parameter-dependent matrices with negative and positive power series with respect to parameters. Finally, some promising results were obtained when applying these algorithms to solve the parameter-dependent standard and constrained ARE problems.

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