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Time-Varying Gain Second Order Sliding Mode Differentiator ⋆,⋆⋆

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Abstract: The problem of first order differentiation, when an estimation of an upper bound of the second derivative is available on-line, is studied in this paper. The proposed method include two approaches: Second Order Sliding Mode and High-Gain algorithms, and consider the differentiator gain as a continuous time varying function. This scheme provides chattering attenuation, increasing at the same time the overall performance. Experimental results, realized over an Industrial Hydraulic System, confirm the efficiency of the methodology.

1. INTRODUCTION

In the last years, High Order Sliding Mode Differentiator, has shown very good performance even in the presence of noise. Basically, the work of Levant, see Levant [1998], introduced a robust exact differentiator using a Second Order Sliding Mode technique, known as Super-Twisting algorithm. This first order differentiator has become focus of intensive research. An observer for mechanical systems was proposed in Davila et al. [2005]. In order to estimate the convergence time, in Polyakov and Poznyak [2011] and Moreno and Osorio [2012] the design of strong Lyapunov functions was introduced. Uniformly convergent algorithms were designed in Cruz-zavala et al. [2011] and Angulo et al. [2013]. Recently, some interesting remarks about the convergence time and disturbance rejection were presented in Utkin [2013]. In order to attenuate the characteristic chattering phenomenon, some adaptive schemes have been developed in Utkin and Poznyak [2013] and Shtessel et al. [2012]. In Gonzalez et al. [2013], a Variable-Gain approach was proposed, achieving chattering attenuation. An adaptive Super-Twisting algorithm for actuator oscillatory failure case reconstruction was proposed in Alwi and Edwards [2013].
The extension to arbitrary order has been developed in Levant [2003], assuming that the $(n+1)$th-order derivative is bounded by a known constant, L. However, if a global constant bound is chosen for the whole practical operation region, the constant would be excessively large that results in increasing the differentiator errors. Recently, a differentiator of signals with unbounded higher derivatives was developed in Levant and Livne [2012], removing the requirement of L, the differentiator gain, to be a constant, and assuming that the $(n+1)$th order derivative has a variable upper bound available in real time. Main system features are often determined by a few variables available in real time (pressures in cylinders in the case of study) allowing a variable upper bound for $(n+1)$th-order derivative. In this note we want to use this idea for an Industrial Hydraulic System, where measurements of the pressures in the cylinders are available; this information can be used to obtain an upper bound of the second derivative of the cylinders position. Then, the time varying gain, $L(t)$ is designed using this time varying upper bound.
Three approaches are explored in this paper: High-Gain, Super-Twisting and Asymptotic Second Order Sliding Mode algorithms with time-varying gain. Besides, it is designed a new time-varying Second-Order Sliding Mode algorithm, which allows the gain to be dependent of a time-varying upper bound, available in real time. This algorithm combine two approaches: High-Gain and Sliding Mode algorithms. A Lyapunov based analysis is presented to demonstrate its properties, including the convergence rate and the ultimate boundennes of differentiation error. Experiment over an Industrial Hydraulic System have been carried out, obtaining very good results. The comparison of the presented methods, including an off-line estimation of the velocity, verify the efficacy of the methodology.
The remainder of this paper is organized as follows. First, a brief description of the first order differentiator and Super-Twisting algorithm is presented in section II. Section III presents the Time-Varying High-Gain Observer. Section IV, introduces the design of the Time-Varying Gain Second Order Sliding Mode differentiator. The model description of mechanical and hydraulic systems is presented in Section V. Experimental results are presented in Section V. Finally in Section VI, the Conclusions are drawn for this study.

2. FIRST ORDER DIFFERENTIATOR

In Mechanical systems, the first order differentiator should estimate the first derivative of a position signal $x(t)$. In this note we consider the next assumptions:

(i) The second derivative of the position signal $x(t)$ is uniformly bounded by a signal $L(t)$: $|\ddot{x}(t)| \leq L(t)$.
(ii) The signal $L(t)$ is available in real time and $\underline{L} \leq L(t) \leq \overline{L}$.
(iii) The derivative of $L(t)$ is bounded: $|\dot{L}(t)| \leq \delta_0$, δ_0 a known constant.

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Defining $x_1 := x$ and $x_2 := \dot{x}$, the problem can be settled as the design of an observer for the system below:

$$\dot{x}_1 = x_2,\ \dot{x}_2 = \ddot{x}, \tag{1}$$

with output $y = x$, and $\ddot{x}$ is described by Newton's Law like equation. We will study two approaches in order to solve this problem: Second Order Sliding Modes and Second Order High-Gain Observer. The design is motivated by Levant and Livne [2012], where the inclusion of a time varying gain which depend of the signal $L(t)$, available on-line, is the key idea.

Remark 1. If some noise μ is present in the measurement, $y = x + \mu$, and this noise is uniformly bounded by δ, a constant; then, no first order exact differentiator can provide for accuracy better than $L^{\frac{1}{2}}||\delta||^{\frac{1}{2}}$, see Levant [2003] and Vasiljevic and Khalil [2008].

2.1 Constant-Gain Super-Twisting Differentiator

First, the Super-Twisting, ST, algorithm with constant gains will be presented:

$$\begin{aligned} \dot{\hat{x}}_1 &= -\kappa_1|\hat{x}_1 - x(t)|^{\frac{1}{2}}\operatorname{sign}(\hat{x}_1 - x(t)) + \hat{x}_2, \\ \dot{\hat{x}}_2 &= -\kappa_2 \operatorname{sign}(\hat{x}_1 - x(t)), \end{aligned} \tag{2}$$

where κ_1 and κ_2 are positive constants to be designed, and $x(t)$ is the position measurement, such that $|\ddot{x}(t)| < L$, L a known constant. Defining $e_1 = \hat{x}_1 - x(t)$ and $e_2 = \hat{x}_2 - \dot{x}(t)$:

$$\begin{aligned} \dot{e}_1 &= e_2 - \kappa_1|e_1|^{\frac{1}{2}}\operatorname{sign}(e_1), \\ \dot{e}_2 &= -\kappa_2\operatorname{sign}(e_1) - \ddot{x}(t). \end{aligned} \tag{3}$$

Solutions of (2) and (3) are understood in Filippov sense, see Filippov [1988]. A necessary condition of convergence is $\kappa_2 > L$, if in addition, we select the gain κ_1 sufficiently large, the appearance of a Second Order Sliding Mode is guaranteed after a finite time transient, i.e. $e_1 = e_2 = 0$ in system (3). A very crude condition is $2(\kappa_2 + L)^2/(\kappa_1^2(\kappa_2 - L)) < 1$, see [Shtessel et al., 2014, Theorem 4.6, p. 159.]. In Moreno and Osorio [2012], a Lyapunov function is introduced that permits the design of κ_2 and κ_1 providing the estimation of convergence time. Recently, in Utkin [2013], it was demonstrated that for any $\kappa > 0$, $\kappa = \kappa_2 - L$, and $\delta > 0$, there exists λ^* such that e_1 is reduced to zero in finite time less than $(e_1(0)/\alpha) + \delta$, if $\kappa_1 > \lambda^*$. The convergence time cannot be less than $e_1(0)/\kappa$. However, if the maximum acceleration, $\ddot{x}$, is excessively large, it will imply the selection of high gain parameters κ_1 and κ_2, resulting in the amplification of differentiation errors. This will be overcome in the next sections with the introduction of time varying parameters.

3. TIME-VARYING HIGH-GAIN DIFFERENTIATOR

High Gain Observers have been demonstrated to achieve similar steady-state errors in presence of noises, see Vasiljevic and Khalil [2008]. In this section we implemented the same idea, over a first order High Gain Observer. We consider the differentiator:

$$\begin{aligned} \dot{\hat{x}}_1 &= -\tfrac{\kappa_1}{\varepsilon}L_h^{\frac{1}{2}}(t)(\hat{x}_1 - x(t)) + \hat{x}_2, \\ \dot{\hat{x}}_2 &= -\tfrac{\kappa_2}{\varepsilon^2}L_h(t)(\hat{x}_1 - x(t)), \end{aligned} \tag{4}$$

where parameters κ_1, κ_2 and ε are positive constants, and $L_h(t)$ is a time varying definite positive continuous function, to be designed. Defining the errors functions $e_1 = (\hat{x}_1 - x(t))\varepsilon^{-1}$, $e_2 = \hat{x}_2 - \dot{x}(t)$ and $e = [e_1, e_2]^T$, we obtain:

$$\dot{e} = \varepsilon^{-1}A(t)e + g(t), \tag{5}$$

with:

$$A(t) = \begin{bmatrix} -\kappa_1 L_h^{\frac{1}{2}} & 1 \\ -\kappa_2 L_h & 0 \end{bmatrix},\ g(t) = \begin{bmatrix} 0 \\ -\ddot{x}(t) \end{bmatrix}.$$

Consider the Lyapunov function $V = e^T P(t) e$, with:

$$P(t) = \begin{bmatrix} \delta_1 L_h^{\frac{1}{2}} & -1 \\ -1 & \frac{\alpha}{L_h^{\frac{1}{2}}} \end{bmatrix}, \tag{6}$$

where $\delta_1 = \kappa_2\alpha - \kappa_1$ and $\alpha > \kappa_1/\kappa_2$. Consider $\underline{L}_h \le L_h(t) \le \overline{L}_h$. If $\underline{L}_h/\overline{L}_h > 1/(\delta_1\delta_3)^2$ we have $P(t) > 0$. Moreover, this matrix $P(t)$ satisfy, $P_0 < P(t) < P_1$, where:

$$P_0 = \begin{bmatrix} \delta_1 \underline{L}_h^{\frac{1}{2}} & -1 \\ -1 & \frac{\alpha}{\overline{L}_h^{\frac{1}{2}}} \end{bmatrix},\ P_1 = \begin{bmatrix} \delta_1 \overline{L}_h^{\frac{1}{2}} & -1 \\ -1 & \frac{\alpha}{\underline{L}_h^{\frac{1}{2}}} \end{bmatrix}.$$

Taking the derivative of V we obtain:

$$\dot{V} = -2e^T Q(t) e + 2e^T P(t) g(t),$$

$$Q(t) = \begin{bmatrix} \delta_2 L_h - \frac{\delta_1\varepsilon\dot{L}_h}{4L_h^{\frac{1}{2}}} & 0 \\ 0 & 1 + \frac{\alpha\varepsilon\dot{L}_h}{4L_h^{1.5}} \end{bmatrix},$$

with $\delta_2 = \kappa_2 - \kappa_1\delta_1$. If the condition below is satisfied:

$$|\dot{L}_h| < \min\{\frac{4\delta_2}{\varepsilon\delta_1}\underline{L}_h^{\frac{3}{2}}, \frac{4}{\varepsilon\alpha}\underline{L}_h^{\frac{3}{2}}\} = \delta_3,$$

then, it imply $Q(t) > 0$. Furthermore,

$$Q(t) > Q_0 = \begin{bmatrix} \delta_2 \underline{L}_h - \frac{\delta_1\varepsilon\delta_3}{4\underline{L}_h^{\frac{1}{2}}} & 0 \\ 0 & 1 - \frac{\alpha\varepsilon\delta_3}{4\underline{L}_h^{1.5}} \end{bmatrix}, \tag{7}$$

taking into account $|\ddot{x}| \le L(t)$, it implies,

$$\dot{V} \le -2\varepsilon^{-1}\frac{\lambda_{\min}[Q_0]}{\lambda_{\max}[P_1]}V + 2\sqrt{2}\frac{L\max\{1, \alpha/L_h^{\frac{1}{2}}\}}{\lambda_{\min}^{\frac{1}{2}}[P_0]}V^{\frac{1}{2}}.$$

Setting $W^2 = \varepsilon^2 V$, we obtain: $\dot{W} \le -\varepsilon^{-1}\beta_0 W + \varepsilon\beta_1$, where $\beta_0 = \frac{\lambda_{\min}[Q_0]}{\lambda_{\max}[P_1]}$ and $\beta_1 = \frac{L\max\{1, \delta_3/L_h^{\frac{1}{2}}\}}{\lambda_{\min}^{\frac{1}{2}}[P_0]}$. ensuring ultimate boundedness. Note that, if $\varepsilon \to 0$, the ultimate bound of the error also tends to zero.

4. SECOND ORDER SLIDING MODE DIFFERENTIATOR

With the addition of a discontinuous term, the time-varying high-gain algorithm (4) has the next form:

$$\begin{aligned} \dot{\hat{x}}_1 &= -\kappa_1 L_h^{\frac{1}{2}}(t)e_1 + \hat{x}_2, \\ \dot{\hat{x}}_2 &= -\tfrac{\kappa_2}{\varepsilon}L_h(t)e_1 - \underbrace{\kappa_3 L(t)\operatorname{sign}(e_1)}_{\text{SM-term}}, \end{aligned} \tag{8}$$

where κ_3 is a positive constant and $L(t)$ is a positive continuous function. As before, we defined $e_1 = (\hat{x}_1 - x)\varepsilon^{-1}$ and $e_2 = \hat{x}_2 - \dot{x}$, obtaining the error dynamics:

$$\begin{aligned} \dot{e}_1 &= \varepsilon^{-1}(-\kappa_1 L_h^{\frac{1}{2}}e_1 + e_2), \\ \dot{e}_2 &= -\varepsilon^{-1}\kappa_2 L_h e_1 - \kappa_3 L(t)\operatorname{sign}(e_1) - \ddot{x}, \end{aligned} \tag{9}$$

with $|\ddot{x}| \le L(t)$ and $\underline{L} \le L(t) \le \overline{L}$. A necessary condition of convergence is $\kappa_3 > 1$, and the sliding set is given by $e_1 = e_2 = 0$, however, instead of finite time convergence, in this case the convergence is asymptotic, see [Fridman and Levant, 2002, p. 65]. In order to prove the ultimate boundedness of the error, consider the next Lyapunov function $V = e^T P(t) e + \gamma|e_1|$, with $P(t)$ as in (6), but considering $\alpha = 1/(\kappa_3\alpha_0)$,

α_0 a positive constant. By other hand $\gamma = 2\varepsilon L/(\alpha_0\sqrt{L_h})$. Considering the conditions below:

$$\alpha_0 < \min\left\{\frac{\kappa_2}{\kappa_1\kappa_3}, \frac{\kappa_1}{2(\kappa_3+1)}\right\}, \quad \frac{L}{\overline{L}} > \frac{\kappa_3^2\alpha_0^2}{\kappa_2 - \kappa_1\kappa_3\alpha_0},$$

function V is positive definite and decrescent. Besides, function V is differentiable almost everywhere, taking its derivative, it is obtained:

$$\dot{V} \le -\frac{2}{\varepsilon}\min\{\lambda_{\min}[Q_0], \varepsilon c\}(||e||^2 + |e_1|) + 2\sqrt{2}\frac{L}{\alpha_0\kappa_3 L_h^{\frac{1}{2}}}||e||.$$

where Q_0 is defined as in (7), and:

$$c = \frac{\kappa_1 - \alpha_0(\kappa_3+1)}{\alpha_0}L - \varepsilon\frac{2L_h|\dot{L}| + L|\dot{L}_h|}{2\alpha_0 L_h^{1.5}} > 0$$

with $|\dot{L}| \le \delta_0$, $\dot{L}_h \le \delta_3$ and $\kappa_1 > \alpha_0(\kappa_3+1) + \varepsilon L^{-1}L_h^{-3/2}(2L_h\delta_0 + L\delta_3)$. Considering $W^2 = \varepsilon^2 V$ and $L_h = \phi L$, it is obtained: $\dot{W} \le -\varepsilon^{-1}\beta_0 W + \varepsilon\beta_1$, where $\beta_0 = \frac{\min\{\lambda_{\min}[Q_0], \varepsilon c\}}{\max\{\lambda_{\max}[P_1], \gamma\}}$ and $\beta_1 = \frac{\sqrt{2L}}{\alpha_0\phi\kappa_3\lambda_{\min}^{\frac{1}{2}}[P_0]}$, ensuring ultimate boundedness.

4.1 Time-Varying Gain Super-Twisting Differentiator

In this section we introduce the time varying version of ST algorithm. This is a particular case of the general version of the algorithm presented in Levant and Livne [2012]. Again, considering the measurement of position $x(t)$, the estimation of its velocity is obtained by the next differentiator:

$$\begin{aligned}\dot{\hat{x}}_1 &= -\kappa_1 L^{\frac{1}{2}}(t)|\hat{x}_1 - x(t)|^{\frac{1}{2}}\operatorname{sign}(\hat{x}_1 - x(t)) + \hat{x}_2,\\ \dot{\hat{x}}_2 &= -\kappa_2 L(t)\operatorname{sign}(\hat{x}_1 - x(t)).\end{aligned} \tag{10}$$

where κ_1 and κ_2 are positive constants and the parameter $L(t)$ is a time varying definite positive continuous function. The recommendation given in Levant and Livne [2012] is $\kappa_1 = 1.5$ and $\kappa_2 = 1.1$. Defining the errors functions $e_1 = \hat{x}_1 - x(t)$ and $e_2 = \hat{x}_2 - \dot{x}(t)$, we obtain:

$$\begin{aligned}\dot{e}_1 &= e_2 - \kappa_1 L^{\frac{1}{2}}|e_1|^{\frac{1}{2}}\operatorname{sign}(e_1),\\ \dot{e}_2 &= -\kappa_2 L\operatorname{sign}(e_1) - \ddot{x}(t).\end{aligned} \tag{11}$$

With $\kappa_2 > 1$, equalities $e_1 = 0$ and $e_2 = 0$ define a formal Filippov solution. Following Levant and Livne [2012], it is assumed the propositions below:

- $|e_1(0)| \le \delta_0 L(t)$ for some constant δ_0.
- $|\dot{L}/L| \le \delta_1$, for some constant δ_1.

Then, differentiator (10) gives the exact derivative for any $t \ge t_0 + T(t_0)$. Consider any arbitrary moment $t_0 > 0$, and define some $L(t)$ in the interval $L_0(1-\gamma) \le L(t) \le L_0(1+\gamma)$, where γ and L_0 are positive constants. Since the logarithmic derivative of $L(t)$, $\dot{L}/L$, is bounded, it follows $T(t_0) \le T_0 = \ln(1+\gamma)/\delta_1$. Choosing $L(t)$ sufficiently large, the finite time convergence of (11) is ensured. In fact, the constants δ_0 and T_0 depend only on δ_1. See the details in [Levant and Livne, 2012, Theorems 1 and 2].

5. INDUSTRIAL HYDRAULIC SYSTEM

The experimental setup under the study is a laboratory prototype of an Industrial Hydraulic Forestry Crane. Such industrial equipment is widely used in forestry and is a subject of many researches aimed to automation of these systems, see Papadopoulos et al. [2003]. One important issue of the automation is the online velocity estimation problem. In this section we show how this problem can be successfully solved by the proposed differentiators. We solve this problem for a telescopic link of the crane, however similar results can be easily obtained for other joints. Some physical parameters of the link are given in the table I.

Table I: Physical parameters of the link

A_a, m^2	A_b, m^2	V_{a0}, m^3	V_{b0}, m^3
$1.26 \cdot 10^{-3}$	$0.76 \cdot 10^{-3}$	$0.012 \cdot 10^{-3}$	$1.19 \cdot 10^{-3}$
P_t, Pa	P_s, Pa	$\bar{q}$, l/min	β, Pa
$5 \cdot 10^5$	$180 \cdot 10^5$	90	$17 \cdot 10^8$
m, kg	$\bar{f}_h$, N	f_c, N	f_v, N·s/m
200	8000	750	6500

5.1 Mechanical and Hydraulic systems

The telescopic link of the crane consists of a double-acting single-side hydraulic cylinder and a solid load which is attached to a piston of the cylinder. Position of the link x varies from 0 to 1.55m; positive velocity $\dot{x}$ corresponds to extraction of the cylinder. This link can be described as a 1-DOF mechanical system actuated by a hydraulic force, and the equation of the motion is

$$m\ddot{x} = f_h - f_{grav} - f_{fric},$$

where m is the mass, f_h is the force generated by the hydraulics, f_{grav} is the gravity force and f_{fric} is the friction force. The force generated by the hydraulics is given by:

$$f_h = P_a A_a - P_b A_b. \tag{12}$$

The piston areas A_a and A_b are known geometric parameters, P_a and P_b are the measured pressures in chambers A and B of the cylinder. The friction is modeled as a Columbus friction plus a viscous friction: $f_{fric} = f_c\operatorname{sign}(\dot{x}) + f_v\,\dot{x}$. Hence

$$\begin{aligned}\ddot{x} &= \frac{f_h}{m} - \frac{1}{m}\left(f_{grav} + f_c\operatorname{sign}(\dot{x})\right) - \frac{f_v}{m}\dot{x}\\ &= \frac{f_h}{m} - \mu_0 - \mu_1\,\dot{x}.\end{aligned} \tag{13}$$

The dynamics of the pressures is given, see [Merrit, 1967, Sec. 3.8], by

$$\dot{P}_a = \tfrac{\beta}{V_a(x)}\left(-\dot{x}\,A_a + q_a\right), \; \dot{P}_b = \tfrac{\beta}{V_b(x)}\left(\dot{x}\,A_b - q_b\right), \tag{14}$$

where $V_a(x) = V_{a0} + x\,A_a$ and $V_b(x) = V_{b0} - x\,A_b$ are volumes of the chambers A and B at the given piston position x, V_{a0} and V_{b0} are known geometric constants, β is a known bulk modulus, q_a and q_b are flows to the chamber A and from the chamber B. Differentiating (12) and substituting (14) leads to

$$\begin{aligned}\dot{f}_h = &\frac{\beta}{V_a(x)\,V_b(x)}\left(A_a\,V_b(x)\,q_a + A_b\,V_a(x)\,q_b\right)\\ &- \frac{\beta}{V_a(x)\,V_b(x)}\left(A_a^2\,V_b(x) + A_b^2\,V_a(x)\right)\dot{x}.\end{aligned}$$

It leads to $\dot{x} = \eta_0(x, q_a, q_b) - \eta_1(x)\,\dot{f}_h$, where:

$$\begin{aligned}\eta_0(x, q_a, q_b) &= \frac{A_a\,V_b(x)\,q_a + A_b\,V_a(x)\,q_b}{A_a^2\,V_b(x) + A_b^2\,V_a(x)},\\ \eta_1(x) &= \frac{V_a(x)\,V_b(x)\,\beta^{-1}}{A_a^2\,V_b(x) + A_b^2\,V_a(x)}.\end{aligned} \tag{15}$$

From (13) it follows:

$$\begin{aligned} \ddot{x} &= \frac{1}{m} f_h - \mu_0 - \mu_1\, \eta_0(x, q_a, q_b) + \mu_1\, \eta_1(x)\, \dot{f}_h \\ &= \frac{f_h}{m} - c_0(x, q_a, q_b) + c_1(x)\, \dot{f}_h. \end{aligned} \qquad (16)$$

5.2 Bounds of the variables

Both pressures P_a and P_b are bounded by the tank pressure P_t and the supply pressure P_s. However it is not a realistic practical situation when both pressures have extreme contrary values simultaneously. Due to internal restrictions the practical bound is $|f_h| \leq \bar{f}_h$. The bound for the gravity force $\bar{f}_g$ is defined by the given mass. Hence for equation (13) we can define the upper bound $|\mu_0| \leq \bar{\mu}_0$ with $\bar{\mu}_0 = (f_{grav} + f_c)/m$. The parameter $\mu_1 > 0$ is constant.

Both flows q_a and q_b are bounded by a factory-set level of a maximum flow through a valve, $|q_{a,b}| \leq \bar{q}$, see Aranovskiy [2013].. Moreover, the flows cannot go in the same direction simultaneously, i.e. they always are of the same sign. Hence upper bounds $|\eta_0(x, q_a, q_b)| \leq \bar{\eta_0}$ and $|\eta_1(x)| \leq \bar{\eta_1}$ can be defined for equation (15), and functions $c_0(x, q_a, q_b)$ and $c_1(x)$ in equation (16) are bounded by $|c_0(x, q_a, q_b)| \leq \bar{c}_0$ and $|c_1(x)| \leq \bar{c}_1$, where $\bar{c}_0 = \bar{\mu}_0 + \mu_1\, \bar{\eta}_0$ and $\bar{c}_1 = \mu_1\, \bar{\eta}_1$.

A practical bound of the velocity is $|\dot{x}| \leq \bar{x}^{(1)}$ with $\bar{x}^{(1)} = 1.1\mathrm{m/s}$. It follows from (13) that the acceleration $\ddot{x}$ is bounded by $|\ddot{x}| \leq \bar{x}^{(2)}$, where

$$\bar{x}^{(2)} = \frac{1}{m}\, \bar{f}_h + \bar{\mu}_0 + \mu_1\, \bar{x}^{(1)}.$$

As the flows q_a and q_b are bounded it follows from (14) that both time derivatives $\dot{P}_a$ and $\dot{P}_b$ are bounded $|\dot{P}_i| \leq (\beta/V_{i0})(A_i \bar{x}^{(2)} + \bar{q})$, $i = a, b$. This imply $|\dot{f}_h| \leq |\dot{P}_a| A_a + |\dot{P}_b| A_b \leq c_2$.

5.3 Measured and estimated signals

The experimental tests are carried out with a real-time platform dSpace 1401 at a sampling interval 1ms using forward Euler integration method. The pressures are measured with installed pressure transducers that allows to estimate the force (12). The position of the telescopic link is measured with a wire-actuated encoder. The encoder provides 2381 counts for the range from 0 to 1.55m; the quantization interval is $Q = 0.651\mathrm{mm}$. The measured signal x can be seen as the position signal with an additive uniform noise with a variance $Q^2/12$. Such quantization interval makes it hard to use a direct difference of the position for velocity estimation as resulting velocity quantization interval is inappropriately high.

The differentiators considered in this paper are online ones. However, it is obvious that a better velocity estimation can be achieved with an offline method when both previous and future values of the position are used. Based on this idea we suppose to post-process the measured position with the offline velocity estimation method to obtain an estimation $\hat{x}_{2,off}$. Further we evaluate the designed online differentiators in comparison with this offline estimation.

To obtain the offline estimation we use splines. First the measured signal $x(t)$ is fitted with a smoothing spline $x_{spl}(t)$. Next $\hat{x}_{2,off}$ is obtained as an analytical differentiation of the spline $x_{spl}(t)$. The smoothing spline $x_{spl}(t)$ is found as a cubic spline which minimizes the following expression, see e.g. [Biagiotti and Melchiorri, 2008, p. 194]:

$$\rho \sum_{i=1}^{N} \left(x(t_i) - x_{spl}(t_i)\right)^2 + (1-\rho) \int_{t_1}^{t_N} \ddot{x}_{spl}^2\, dt,$$

where N is a number of measured points, $0 \leq \rho \leq 1$ is a smoothing parameter. The smoothing parameter determines a trade-off between fitting of the measured data and smoothing. The value $\rho = 0$ leads to maximum smoothing, i.e. linear approximation; and $\rho = 1$ leads to a classic cubic spline with exact fitting and without any smoothing. For our purpose we tune the smoothing parameter in order to obtain the smoothest possible estimation, i.e. the smallest ρ, keeping the fitting error within the quantization interval Q.

6. EXPERIMENTAL STUDIES

6.1 Design of $L(t)$

The problem is to obtain the velocity of the spool of the cylinder from the measured position. In order to obtain the estimation of the velocity, $\dot{x}$, differentiator (10) is proposed. For this purpose we need to design some appropriated time varying gain $L(t)$, such that $|\ddot{x}| \leq L(t)$ and $|\dot{L}| \leq \delta_1$. Besides, from equation (16), it is obtained:

$$|\ddot{x}| \leq \bar{c}_0 + \frac{1}{m}|f_h| + \bar{c}_1 |\dot{f}_h|, \qquad (17)$$

When the cylinder is moving with constant velocity, we have $\ddot{x} \approx 0$ which means that a small constant gain L can be selected. Besides, if the cylinder is moving with varying velocity, then the acceleration $\ddot{x}(t)$ is not close to zero anymore and it implies that the gain L should increase proportionally with rate of variation of the cylinder velocity. In order to include both cases, constant and time-varying profiles of velocity, and motivated by expressions (16) and (17), we proposed the next time varying gain:

$$L(t) = \gamma_0 + \gamma_1 |f_h| + \gamma_2 \zeta(f_h), \qquad (18)$$

where the parameters γ_0, γ_1 and γ_2 are positive constants; the rate of variation of f_h, is given by $\zeta(f_h)$, which is a positive function that depends of the available pressure measurements. One option is:

$$\zeta(f_h) = \frac{|f_h(t - \tau_1) - f_h(t - \tau_2)|}{\tau_2 - \tau_1},$$

with $\tau_2 > \tau_1 > 0$. For constant profiles of velocity $\zeta(f_h) \approx 0$. With this selection, the upper bound of the derivative, $\dot{L}$ is given by: $|\dot{L}| \leq \gamma_1\, c_2 + \frac{2\, \gamma_2\, c_2}{\tau_2 - \tau_1}$.

Four velocity estimation algorithms are tested:

- Super-Twisting with constant gain: ST, (2).
- Super-Twisting with time-varying gain: STV, (10).
- High-Gain with time-varying gain: HGV, (4).
- High-Gain + Sliding Mode (time-varing): HGV+SM, (8).

The STV differentiator, (10), is implemented, considering the parameters $\kappa_1 = 1.5$, $\kappa_2 = 1.1$ and the time varying gain (18). For the gain $L(t)$ it is taking: $\gamma_0 = 0.5$, $\gamma_1 = 0.0005$, $\gamma_2 = 0.0004$, $\tau_1 = 0.004$ and $\tau_2 = 0.01$. The sign function is approximated by $\mathrm{sign}(x) = \frac{x}{|x| + \epsilon}$, with $\epsilon = 0.001$. For the HGV differentiator, algorithm (4), we consider the parameters $\kappa_1 = \kappa_2 = 2$, $\varepsilon = 0.015$, and the time varying gain (18) for the design of L_h, with: $\gamma_0 = 0.5$, $\gamma_1 = 0.00002$, $\gamma_2 =$

0.00001, $\tau_1 = 0.004$ and $\tau_2 = 0.01$. For the combined scheme, HGV+SM algorithm (8), we consider the same gain L_h as in the case of HGV, $\kappa_3 = 1.1$ and $L(t)$ as in the case of STV. For comparison purposes we also test the ST algorithm with constant gain. Two values were considerd: $L = 5$ and $L = 15$; increasing of this value resulted in increasing of chattering. For evaluation purposes we compare all the algorithms with the velocity estimation obtained using post-processing of the measured data, i.e. off-line velocity estimation, see Sec. 5.3. In the experiment we consider different input profiles, and fig. 1 shows the measured cylinder position, x. In this experiment the cylinder was in motion with constant and varying velocity.

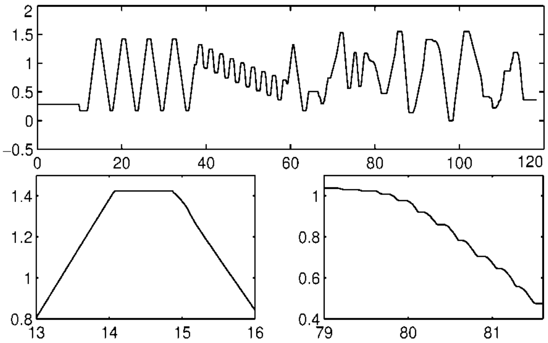

Fig. 1. Measured position, x, (m) vs time (seconds).

6.2 *Super-Twisting: Time-Varying vs Constant Gain*

In this section we compared the ST algorithm, using constant and time-varying gains. Fig. 2 shows the velocity estimation in the interval of time $(13, 16)$ and the corresponding differentiator gains, constant and time-varying. In addition, we computed the estimation of the acceleration, $\ddot{x}$, using the presented off-line method, see section 5.3. It is clear that with a constant gain we can not compensate the acceleration in the whole interval. By other hand, with the variable gain we can cover the acceleration in the whole region without the increasing of chattering.

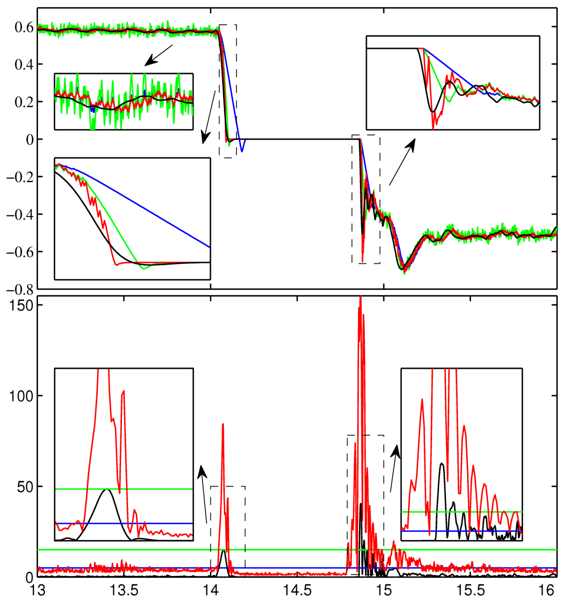

Fig. 2. Top: (Velocity (m/s) vs time (seconds)): ST(L=5,–), ST(L=15,–), STV($L(t)$,–), Off-line(–).
Bottom: Gains vs $|\hat{\ddot{x}}|$-(Off-line).

6.3 *Comparison of time-varying algorithms*

In this section we present the obtained result with the proposed time-varying algorithms. Fig. 3 shows the velocity estimation in the interval of time $(13, 16)$.

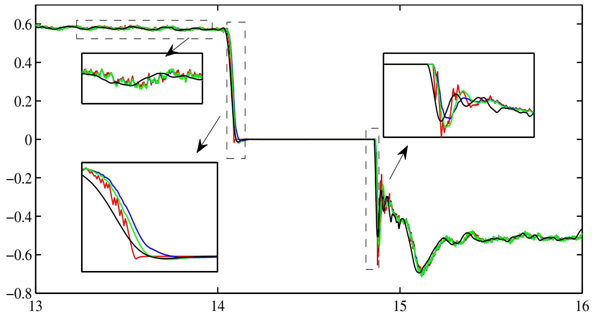

Fig. 3. Velocity (m/s) vs time (seconds): Off-line(–), STV(–), HGV(–), HGV+SM(–).

Fig. 4 shows the velocity estimation in the interval of time $(79, 81.6)$ and the corresponding time varying gain, $L(t)$. In this interval a sequence of acceleration and deceleration inputs was included in the experiment, producing abrupt changes in velocity. In this case we can not use a constant gain, since the gain to select should be $L = 100$, increasing at the same time the chattering effect.

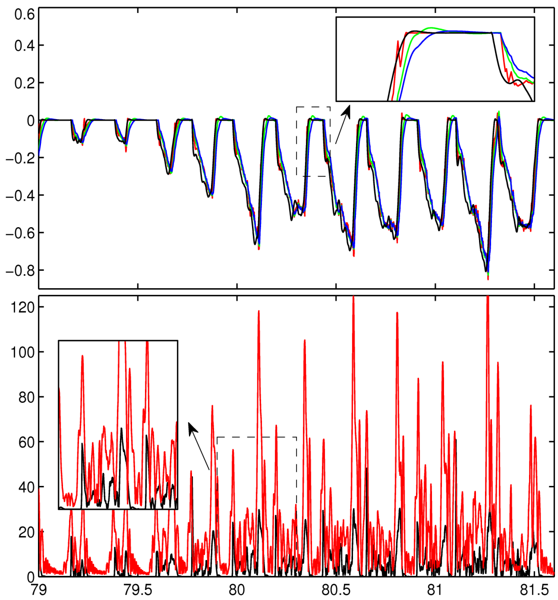

Fig. 4. Top (Velocity (m/s) vs time (seconds)): Off-line(–), STV($L(t)$,–), HGV(–), HGV+SM(–).
Bottom: $L(t)$ vs $|\hat{\ddot{x}}|$-(Off-line).

6.4 *Computation of errors*

Considering the off-line estimation as the true value of velocity, $\dot{x}$, in this section, the second and first norm of the error are computed for the considered intervals of time. For this purpose we define $e_i = \hat{\dot{x}}_i - \dot{x}$, where $\dot{x}$ is the true velocity (off-line estimation) and $\hat{\dot{x}}_i$ is the on-line estimation for i = ST, STV, HGV and HGV+SM. Table II shows the normalized error, $||e_i||/||e_{\text{HGV+SM}}||$, during intervals of time that correspond with fig. 2, fig. 3 and fig. 4, with $||e_i|| = \sqrt{\frac{1}{t_f-t_0}\int_{t_0}^{t_f}|e_i(\tau)|^2 d\tau}$. By other hand, table III shows the normalized error $||e_i||_1/||e_{\text{HGV+SM}}||_1$, where $||e_i||_1 = \max\limits_{t_0\leq t\leq t_f}|e_i|$. In general, the HGV+SM algorithm gives a very good performance and this is the reason to choose $e_{\text{HGV+SM}}$ for the normalization. It means that if the value in the table is below 1, then the corresponding algorithm performs *better*, then HGV+SM. And vice versa. In the interval of almost constant velocity, $(13, 14)$, the difference in performance is not considerable, except in the case of $L = 15$ where the increasing of chattering is evident. The reason for this, is because during this intervals the acceleration $\ddot{x}$ decreases to the smallest values, and the amplitude of acceleration can be covered with a relatively small constant gain $L = 5$.

Table II: $||e_i||/||e_{\text{HGV+SM}}||$

i	$(13,14)$ Const. Vel.	$(14,14.2)$ Abrupt Acce.	$(79,81.6)$ Var. Vel.
ST (L=5)	1.14	3.32	2.79
ST (L=15)	2.52	1.17	1.4
STV	1.11	0.93	0.79
HGV	0.96	1.3	1.21

Besides, if abrupt changes in acceleration present, we can not cover the acceleration amplitude choosing a constant gain. As it can be seen from the columns $(14, 14.2)$ and $(79, 81.6)$ of the tables II and III, the ST algorithm with the constant gain $L = 5$ significantly degrades in performance. Thus, the use of the algorithm with constant gain for the whole operation region results in increase of differentiation errors either for the constant velocity range, or for the abrupt acceleration range. However, time varying algorithms efficiently perform for the both profiles of acceleration and provide small errors for the whole operation region.

Table III: $||e_i||_1/||e_{\text{HGV+SM}}||_1$

i	$(13,14)$ Const. Vel.	$(14,14.2)$ Abrupt Acce.	$(79,81.6)$ Var. Vel.
ST (L=5)	1.34	2.34	2.33
ST (L=15)	3.34	1.1	1.5
STV	1.27	0.97	1.19
HGV	0.94	1.23	1.16

7. CONCLUSION

The problem of first order differentiation was studied in this paper. We verify that using a time varying gain is a better option, instead of use a global constant bound for the whole operation region. The design of ST and HG differentiators with time varying gains is presented. Besides, a new differentiator formed by the HG and SM algorithms with time-varying gains, is proposed in this work. A Lyapunov based analysis was provided, in order to demonstrate the stability and convergence properties for both algorithms. We tested and validated the proposed scheme in an Industrial Platform, obtaining very good results. The proposed methodologies showed an increase in performance with respect to the constant gain algorithm, including chattering attenuation. The HGV+SM differentiator proposes a good trade-off between HGV and SM, compromising a transient performance and chattering effect. Extensions of this algorithm applied to a more general class of mechanical systems are considered for future work.

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