Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

High Order Sliding Mode Differentiator for Dynamical Inversion of Non-Involutive Systems

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Abstract: - This paper deals with the left inversion problem for a class of nonlinear systems. It is well-known that such problem is a challenging one because of the presence of non-smooth unknown inputs and non-involutivity of the forced fields' distribution. To overcome these difficulties, we first propose a novel left invertible form. Then, using this form, we design an observer in order to reconstruct the state and the control inputs. Numerical results are given in order to validate the proposed approach.

Keywords: Left invertibility, High order sliding mode observers, Non-involutivity distributions, Non-smooth unknown inputs.

1. INTRODUCTION

Since the early nineties, the left inversion problem (i.e recovering inputs from outputs and their derivatives knowledge) has been treated in both the geometric and algebraic contexts Respondek [1990], Fliess [1986]. Additionally, the concept of inverse dynamics is used to overcome some constraints in the dynamical inversion problems Daoutidis and Kravaris [1991]. In the majority of the work on the left invertibility and dynamical inversion, it is assumed that the inputs are either available or analytical or at least sufficiently smooth. In this paper, we relax these assumptions - as in our earlier work Barbot et al. [2009], Boutat et al. [2013] on the topic - and consider the more generic case whereby the inputs can be non-smooth but bounded; such as in the case of a square signal for example. One restriction for left inversion is due to the fact that some nonlinear systems do not have certain integrability properties. Another restriction arises from the non-involutivity of the forced terms or fields. To overcome these limitations, we first propose a new left inversion form, building on our earlier work Barbot et al. [2009], Boutat et al. [2013] on the subject. Note that the integrability problem and non-smoothness of the inputs can be addressed using the algebraic formulation together with the left invertible form as given in Respondek [1990] and Fliess [1986]. However, such a form requires some specific differentiators for recovering the unknown input. For technical reasons, finite time differentiators seem to constitute natural candidates to design a 'reconstructor' for the proposed left invertible form. One can find more details about of finite time differentiators in the following references, Levant [1998], Davila et al. [2005], Liu et al. [2011], Floquet and Barbot [2007], to name a few. Nevertheless, due to some constrains such as the observability matching condition, Perruquetti and Barbot [2002], only few of them are deemed suitable. Recently, high order sliding mode differentiators Barbot et al. [1996], Fridman et al. [2008] are being employed for this purpose. However, they must meet some specific order imposed by the observability matching condition. This type of observer was used in the context of classical dynamical inversion in Fridman et al. [2008].

The rest of the paper is organized as follow. Section 2 present a formal description of the problem. Section 3 outlines some previous results on the topic. In Section 4, we introduce the proposed left invertibility form and state the conditions under which the form can be obtained. In Section 5, we first recall some well-known results on high order sliding mode and then propose high order sliding mode differentiator. In Section 6, the efficiency of the proposed algorithm is verified through numerical results and some concluding remarks are drawn in Section 7.

2. PROBLEM STATEMENT

Throughout the paper, the following MIMO dynamical system is considered

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i \quad (1)$$

$$y = h(x) \quad (2)$$

where $x \in U \subseteq \Re^n$ represents the state with an open set U that contains $x = 0$, $y = (y_1, ..., y_m)^T \in \Re^m$ represents the output and $u = (u_1, ..., u_m)^T \in \Re^m$ is the vector of unknown inputs. The unknown inputs are considered bounded and PieceWise Smooth (PWS). The vectors fields f, g_i and h are assumed to be sufficiently smooth.

Assumption 1: There exists ρ_i such that $\sum_{i=1}^{m} \rho_i = n$

$$Rank \begin{pmatrix} dh_1 \\ \vdots \\ dl_f^{\rho_1 - 1} h_1 \\ \vdots \\ dh_m \\ \vdots \\ dl_f^{\rho_m - 1} h_m \end{pmatrix}_{|x \in U} = n$$

Thus, the pair (h, f) is locally weakly observable Hermann and Krener [1977]. This assumption is made in order to avoid the singularity problem that might occur in the diffeomorphism construction. The properties for recovering unknown inputs from outputs can be defined from two point of view: The first one is the left invertibility in the sense of Hirshorn Hirschorn [1979], Respondek [1990]. The second one is the invertible equivalent to the previous one but expressed in term of its internal dynamic Daoutidis and Kravaris [1991]. Our definition of left invertibility is more restrictive than that of Hirshorn. Hence, the invertibility conditions are more stringent.

Definition 2.1. The dynamical system (1)-(2) is said to be invertible at x_0, if two distinct inputs $u^1 \neq u^2$ yield two distinct outputs $y(t; u^1, x_0) \neq y(t; u^2, x_0)$.

Definition 2.2. The dynamical system (1)-(2) is said to be left invertible at x_0, if a realization of its inverse dynamic can be obtained from the following equation

$$\dot{\eta} = \varphi(\eta, y, \dot{y}, ...) \quad (3)$$

$$u = \omega(\eta, y, \dot{y}, ...) \quad (4)$$

where η is a substate of the state x and η is globally strongly input-to-state stable (GSISS).

The property of GSISS is more restrictive than the well-known input-to-state stability property; for more details please refer to Sontag [1989].

Definition 2.3. The system $\dot{\eta} = f(\eta, w)$ is GSISS if for all bounded input $w(t)$ and all bounded initial conditions $\eta_1(0)$ and $\eta_2(0)$:

- $\forall t \geq 0 \; ||\eta(t, w, \eta_1(0))||$ and $||\eta(t, w, \eta_1(0))||$ are bounded
- $lim_{t \to +\infty} ||\eta(t, w, \eta_1(0)) - \eta(t, w, \eta_2(0))|| = 0$

Note that the form (3)-(4) is one of the equivalent conditions in Theorem 5.1 in Respondek [1990]. In Respondek [1990], the system is left invertible in the sense of Hirshorn. Here, the stability condition on η is simply added. Moreover, in Respondek [1990] the comparison is also done with an algebraic approach and with a flat system Fliess et al. [1995]. More precisely, if the dimension of η is zero, then the system (1)-(2) is flat and the output is also flat. Nevertheless, in both geometric and algebraic approaches, the unknown inputs are considered analytic. If a distribution associated to the system is non-involutive and the unknown inputs are non-smooth, these approaches cannot be used. Next, we show the difficulty that arises when inputs are non-smooth through this example.

$$\begin{aligned} \dot{x}_1 &= x_3 + u_1 \\ \dot{x}_2 &= x_1 + u_1 \\ \dot{x}_3 &= u_2 \\ y_1 &= x_1 \\ y_2 &= x_2 \end{aligned} \quad (5)$$

where u_1, u_2 are PWS. It is easy to see that u_2 appears on the derivative on the second derivative of y_1 (i.e $\ddot{y}_1 = u_2 + \dot{u}_1$) but as u_1 is not differentiable, then $\ddot{y}_1$ does not exist. Nevertheless, it is possible to reconstruct the inputs from the outputs with any input derivative. The solutions are $u_1 = \dot{y}_2 - y_1$ and $u_2 = \dot{\bar{y}} - \dot{y}_1$ with $\bar{y} = \dot{y}_1 - \dot{y}_2$, where $\bar{y}$ is called a dummy output. In this example, it is important to note that $\dot{\bar{y}}$ exists even if $\ddot{y}_1$ and $\ddot{y}_2$ do not exist. Note that the derivative and addition operators do not commute in this case [1]. Now consider the following non-involutive example:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_2 - x_1 - x_5 + (x_5^2 + 1)u_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= x_2 - x_3^3 - x_4 + u_2 \\ \dot{x}_5 &= -x_5^3 + (x_2^2 + 1)u_2 \\ y_1 &= x_1 \\ y_2 &= x_3 \end{aligned} \quad (6)$$

Arranging the system (6) in the form (1)-(2), we get

$$g_1 = \begin{pmatrix} 0 \\ x_5^2 + 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ x_2^2 + 1 \end{pmatrix} \quad (7)$$

And the Lie bracket $[g_1, g_2]$ is

$$[g_1, g_2] := L_{g_1} g_2 - L_{g_2} g_1 = \begin{pmatrix} 0 \\ -2x_5(x_2^2 + 1) \\ 0 \\ 0 \\ 2x_2(x_5^2 + 1) \end{pmatrix} \quad (8)$$

It can be noticed from the above that $[g_1, g_2]$ is not in the $span\{g_1, g_2\}$, thus the distribution is not involutive. Therefore, the integrability problem occurs and it is not possible to transform the system in the form (3)-(4) Isidori [1995]. We will later show in Section 4 that the above system can be transformed into the left invertibility form.

[1] More precisely $\ddot{\widehat{y_1 - y_2}}$ exists and $\ddot{y}_1 + \ddot{y}_2$ doesn't exist

For the linear system (5), it is always the input u_1 which arrives first generates difficulties for the output derivatives. However, for nonlinear system even if the system is involutive another difficulty arises, which we demonstrate through this example.

$$\begin{aligned}\dot{x}_1 &= x_3 + \beta_1(X)u_1\\ \dot{x}_2 &= x_1 + \beta_2(X)u_1\\ \dot{x}_3 &= u_2\\ y_1 &= x_1\\ y_2 &= x_2\end{aligned}\tag{9}$$

with u_1 and u_2 PWS and bounded. If $\beta_1(X)$ and $\beta_2(X)$ are not function of x_3 and are generically different from zero, it is possible to reconstruct the inputs from the outputs. In this case, the unknown input are given by $u_1 = \frac{\dot{y}_2 - y_1}{\beta_2(X)}$ and $u_2 = \frac{\beta_2(x)\ddot{y}+\beta_1(x)\beta_2\dot{y}_1-\dot{\beta}_2(x)(\bar{y}-\beta_1(x)x_1)-\dot{\beta}_1\beta_2 y_1}{\beta_2(X)^2}$. Obviously, the system (9) is left invertible in the meaning of Respondek [1990] and Fliess [1986]. But if $\beta_2(X) = 0$ with either $\beta_1(X)$ or $\beta_2(X)$ is function of x_3, then the system (9) is not left invertible as per Definition 2. This is due to the fact that u is non-smooth.

3. DYNAMIC INVERSION ALGORITHM

This section is based on the algorithm introduced in Barbot et al. [2009], and consistent with the context of dynamical inversion given in Boutat et al. [2013]. Consider the system (1)-(2) and compute its characteristic matrix as follows

$$\Gamma_0 = \begin{pmatrix} L_{g_1}L_f^{(r_1-1)}h_1 & \dots & L_{g_m}L_f^{(r_1-1)}h_1\\ \dots & \dots & \dots\\ \dots & \dots & \dots\\ L_{g_1}L_f^{(r_m-1)}h_m & \dots & L_{g_m}L_f^{(r_m-1)}h_m \end{pmatrix}$$

Defining $\pounds$ be the commutative algebra of the measured outputs and their successive Lie derivatives up to order $r_i - 1$ the relative degree of the i_{th} output [2] is given as $\pounds_0 = \text{span}\{h_1, ..., L_f^{r_1-1}h_1, ..., h_m, ..., L_f^{r_m-1}h_m\}$. The above definitions are used to initialize the following algorithm.

Algorithm

0) Algorithm initialization set $i = 0$

(1) Set $\pounds = \pounds_i$ and $\Gamma = \Gamma_i$ and compute rank(Γ_i),
  (a) if rank$(\Gamma) = m$ then go to step 4), else
  (b) compute $(1 \times m)$ row vector $K(x) = (k_1(x), ..., k_m(x)) \neq 0$ such that:
$$K(x)\Gamma(x) = 0 \text{ for all } x \in U \tag{10}$$
  (c) if there is no $K(x) \in \pounds$ then go to 5, else go to Step 2).

(2) From $K(x)$, the function is defined
$$h_K(x) = \sum_{i=1}^{m} k_i(x)L_f^{r_i}h_i(x) = \sum_{i=1}^{m} k_i(x)y_i^{(r_i)}$$
  (a) if all $h_K(x) \in \pounds$ then go to 5, else
  (b) compute the relative degree of $h_K(x)$, if there is no $h_K(x)$ with finite relative degree then go to Step 5), else go to Step 3).

(3) Set $y = (y^T, h_K(x))^T$, increment i and compute the corresponding Γ_i and $\pounds_i$, and go back to Step 1).
(4) End algorithm there is a solution.
(5) End algorithm there is no solution.

Remark 1. The $1-c$) guarantees that the new output is only function of available information at the current step. Moreover, step 2 defines a new dummy output which is independent of u and which is derivable.

If after γ iterations, the algorithm stops at Step 4, then it supplies enough outputs h_i for $i \in \{m+1, ..m + l_\gamma\}$ [3] to obtain a matrix

$$\Gamma_\gamma = \begin{pmatrix} L_{g_1}L_f^{(r_1-1)}h_1 & . & L_{g_m}L_f^{(r_1-1)}h_1\\ . & . & .\\ . & . & .\\ L_{g_1}L_f^{(r_m-1)}h_m & . & L_{g_m}L_f^{(r_m-1)}h_p\\ L_{g_1}L_f^{(r_{m+1}-1)}h_{m+1} & . & L_{g_m}L_f^{(r_{m+1}-1)}h_{p+1}\\ . & . & .\\ L_{g_1}L_f^{\left(r_{m+l_\gamma}-1\right)}h_{m+l_\gamma} & . & L_{g_m}L_f^{\left(r_{m+l_\gamma}-1\right)}h_{m+l_\gamma} \end{pmatrix}$$

of rank equal to m. This verifies Assumption 1 and there exists a diffeomorphism $\phi(x) = (\xi^T, \eta^T)^T$ such that the system can be transformed in the form:

$$\dot{\xi}_{i,j} = \xi_{i,j+1} \tag{11}$$
$$\text{for } i \in \{1, .., m + l_\gamma\} \text{ and } 1 \leq j \leq \theta_i - 1$$
$$\dot{\xi}_{i,\theta_i} = b_i(\xi,\eta) + \sum_{j=1}^{m} a_{i,j}(\xi,\eta)u_j \tag{12}$$
$$\text{for } 1 \leq i \leq m + l_\gamma$$
$$\dot{\eta} = \overline{f}(\xi,\eta,u) \tag{13}$$

where $\eta = (\eta_1, ..., \eta_\rho)^T$ is independent from ξ with $\rho \geq n - \sum_{i=1}^{m+l_\gamma}\theta_i$ with l_γ is the number of dummy outputs added by the algorithm and $\theta_i \leq r_i$ with respect to the chosen basis. Moreover, $b_i(\xi,\eta) = L_f^{\theta_i}h_i$ for $i = 1 : m + l_\gamma$ and $a_{i,j} = L_{g_j}L_f^{\theta_i - 1}h_i$ for $j = 1 : m + l_\gamma$ and the $a_{i,j}$ are the $\Gamma_\gamma(x)$ coefficients if $\theta_i = r_i$ and are equal to zero if $\theta_i < r_i$. From (11)-(13), it is possible to deduce a form similar to that in (3)-(4). However, u appears in the dynamic of η as follows

$$\left\{\begin{array}{l} \dot{\eta} = \overline{f}(\xi,\eta,u)\\ u = (P\Gamma_\gamma)^{-1}P\left(\begin{pmatrix}\dot{\xi}_{1,\theta_1}\\ \dots\\ \dot{\xi}_{m,\theta_m}\\ \dots\\ \dot{\xi}_{m+l_\gamma,\theta_{m+l_\gamma}}\end{pmatrix} - \begin{pmatrix} b_1(\xi,\eta)\\ \dots\\ b_m(\xi,\eta)\\ \dots\\ b_{m+l_\gamma}(\xi,\eta)\end{pmatrix}\right)\end{array}\right. \tag{14}$$

where $P \in \Re^{m \times m+l_\gamma}$ is a linear projection matrix such that rank$(P\Gamma_\gamma) = $ rank$(\Gamma_\gamma) = m$. Moreover, the choice of P is made with respect to ξ, η basis choice (i.e. the selected rows are $\theta_i = r_i$).

In (14), the dynamic of η is function of the unknown inputs, thus it is not possible to estimate η directly.If the distribution $span\{g_i, g_j\}$ is involutive, there exists a diffeomorphism ϕ such that the η can be segregated from ξ and the η dynamic, and hence, the dynamics of η will be free from the unknown input u Isidori [1995]. For the involutive case, a constructive method based on the

[2] r_i is the smallest number such that $\frac{\partial^{r_i} y_i}{\partial t^{r_i}}$ depends explicitly of u.

[3] l_γ is the number of dummy outputs

dedicated projector idea was given in Boutat et al. [2013] to estimate η as follows

$$\begin{cases} \dot{\eta} = \overline{f}(\xi,\eta) \\ u = (P\Gamma_\gamma)^{-1}P\left(\begin{pmatrix} \dot{\xi}_{1,r_1} \\ \dots \\ \dot{\xi}_{m,r_m} \\ \dots \\ \dot{\xi}_{m+l_\gamma,r_{m+l_\gamma}} \end{pmatrix} - \begin{pmatrix} b_1(\xi,\eta) \\ \dots \\ b_m(\xi,\eta) \\ \dots \\ b_{m+l_\gamma}(\xi,\eta) \end{pmatrix}\right) \end{cases} \tag{15}$$

Moreover, if $\dot{\eta} = \overline{f}(\xi,\eta)$ is globally strongly input state stable (GSISS), then the system is left invertible. However, such a method is not applicable when the distribution $span\{g_1,...,g_m\}$ is not involutive and the inputs are not smooth. In the next section, we have proposed a novel left invertible approach that allows to deal with non-involutive systems with unknown non-smooth inputs.

4. LEFT INVERTIBLE FORM

Using high high order sliding mode observer theory, it is possible to observe the system (11)-(13) in finite time for all $\xi_{i,j}$, $\dot{\xi}_{i,\theta_i}$, $i \in \{1,..,m+l_\gamma\}$ and $j \in \{1, \theta_i - 1\}$. To achieve this, we substitute u in (14) to obtain the closed-loop system

$$\dot{\eta} = \overline{\overline{f}}(\xi,\eta,\dot{\xi}_{1,\theta_1},...,\dot{\xi}_{m+l_\gamma,\theta_{m+l_\gamma}}) \tag{16}$$

Using (16), we propose a new left invertible form as follows

$$\dot{\xi}_{i,j} = \xi_{i,j+1} \tag{17}$$
$$\text{for } i \in \{1,..,m+l_\gamma\} \text{ and } 1 \le j \le \theta_i - 1$$
$$\dot{\xi}_{i,\theta_i} = b_i(\xi,\eta) + \sum_{j=1}^{m} a_{i,j}(\xi,\eta)u_j \tag{18}$$
$$\text{for } 1 \le i \le m + l_\gamma$$
$$\dot{\eta} = \overline{\overline{f}}(\xi,\eta,\dot{\xi}_{1,\theta_1},...,\dot{\xi}_{m+l_\gamma,\theta_{m+l_\gamma}}) \tag{19}$$

Proposition 4.1. Under Assumption 1, there exists a diffeomorphism $\phi(x) = (\xi^T, \eta^T)^T$ which transform system (1)-(2) in the left invertible form (17)-(19) if there exists $\gamma \ge 0$ such that $Rank\{\Gamma_\gamma\} = m$.

Then, it is now possible to give the following theorem:

Theorem 4.2. If there exists a diffeomorphism ϕ which transform the system (1)-(2) into the form (17)-(19) with $\dot{\eta} = \overline{\overline{f}}(\xi,\eta,\dot{\xi}_{1,\theta_1},...,\dot{\xi}_{m+l_\gamma,\theta_{m+l_\gamma}})$ globally strongly input state stable, then the system is left invertible.

Proof 4.3. The proof is given in the next section.

5. HIGH ORDER SLIDING MODE OBSERVER

In Fridman et al. [2008], an observer structure was proposed for state estimation and input reconstruction if $rank\{\Gamma_0\} = m$ and the distribution $span\{g_1,...,g_m\}$ is involutive. In this paper, the system does not satisfy at least one of these assumptions. To address this, we propose a novel finite time observer for the system (17)-(19) as follows

$$\begin{aligned}
\dot{\hat{\xi}}_{i,1} &= \nu_{i,1} \\
\nu_{i,1} &= \hat{\xi}_{i,2} - \lambda_{i,1}|\hat{\xi}_{i,1} - \xi_{i,1}|^{\frac{r_i}{r_i+1}} sign(\hat{\xi}_{i,1} - \xi_{i,1}) \\
\dot{\hat{\xi}}_{i,2} &= \nu_{i,2} \\
\nu_{i,2} &= \hat{\xi}_{i,3} - \lambda_{i,2}|\hat{\xi}_{i,2} - \nu_{i,1}|^{\frac{r_i-1}{r_i}} sign(\hat{\xi}_{i,2} - \nu_{i,1}) \\
\vdots &= \vdots \\
\dot{\hat{\xi}}_{i,\theta_i-1} &= \nu_{i,\theta_i-1} \\
\nu_{i,\theta_i-1} &= \hat{\xi}_{i,\theta_i} - \lambda_{i,\theta_i-1}|\hat{\xi}_{i,\theta_i-1} - \nu_{i,\theta_i-2}|^{\frac{1}{2}} \\
&\quad sign(\hat{\xi}_{i,\theta_i-1} - \nu_{i,\theta_i-2}) \\
\dot{\hat{\xi}}_{i,\theta_i} &= -\lambda_{i,\theta_i} sign(\hat{\xi}_{i,\theta_i} - \nu_{i,\theta_i-1})
\end{aligned} \tag{20}$$

It is shown in Levant [2005], Levant [2007], that observer (20) converges in finite time. More precisely, there exists t_0 such that for all $i \in \{1,...,m+l_\gamma\}$ and $j \in \{1,...,\theta_i\}$, $\hat{\xi}_{i,j}$ is equal to $\xi_{i,j}$. Thus, defining $e_{i,\theta_i} = \xi_{i,\theta_i} - \hat{\xi}_{i,\theta_i}$, $\forall t \ge t_0$ and using the finite-time relation, we get $\dot{e}_{i,\theta_i} = 0$, which implies

$$\begin{aligned}
\lambda_{i,\theta_i} sign(\hat{\xi}_{i,\theta_i} - \nu_{i,\theta_i-1}) &= b_i(\xi,\eta) + \sum_{j=1}^{m} a_{i,j}(\xi,\eta)u_j \\
\lambda_{i,\theta_i} sign(\hat{\xi}_{i,\theta_i} - \nu_{i,\theta_i-1}) &= \dot{\xi}_{i,\theta_i}
\end{aligned} \tag{21}$$

As known, the higher order sliding mode observer suffers from chattering problems. To address this, $\lambda_{i,\theta_i} sign(\hat{\xi}_{i,\theta_i} - \nu_{i,\theta_i-1})$ is filtered with a low pass filter with a cutoff frequency between the highest frequency of the system and the chattering frequency. The filtered state is denoted as $\dot{\xi}^f_{i,\theta_i}$. Similarly, the dummy outputs are also filtered. Using (21) and (14), we estimate the unknown input $\hat{u}$ for all $t \ge t_0$ as follow

$$\hat{u} = (P\hat{\Gamma}_\gamma)^{-1}P\left(\begin{pmatrix} \dot{\xi}^f_{1,\theta_1} \\ \dots \\ \dot{\xi}^f_{m,\theta_m} \\ \dots \\ \dot{\xi}^f_{m+l_\gamma,\theta_{m+l_\gamma}} \end{pmatrix} - \begin{pmatrix} b_1(\xi,\hat{\eta}) \\ \dots \\ b_m(\xi,\hat{\eta}) \\ \dots \\ b_{m+l_\gamma}(\xi,\hat{\eta}) \end{pmatrix}\right) \tag{22}$$

where $\hat{\Gamma}_\gamma$ estimates Γ_γ. Substituting (22) into (19), we obtain the following estimation for η

$$\dot{\hat{\eta}} = E_1\overline{\overline{f}}(\xi,\hat{\eta},\dot{\xi}^f_{1,\theta_1},...,\dot{\xi}^f_{m+l_\gamma,\theta_{m+l_\gamma}}) \tag{23}$$

where $E_1 = 1$ when $t > t_0$, t_0 is the time when the observation error $e := \xi - \hat{\xi}$ and $\lambda_{i,\theta_i} sign(\hat{\xi}_{i,\theta_i} - \nu_{i,\theta_i-1}) - \dot{\xi}_{i,\theta_i}$ have converged to 0 and $E_1 = 0$ for $0 < t < t_0$.

6. NUMERICAL RESULTS

In this section, the performance of the proposed observer is verified on the following system

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -2x_2 - x_1 - x_5 + u_1 \\
\dot{x}_3 &= x_4 + u_1 \\
\dot{x}_4 &= x_2 - x_3^3 - x_4 + u_2 \\
\dot{x}_5 &= -x_5^3 + (x_2^2 + 1)u_2 \\
y_1 &= x_1 \\
y_2 &= x_3
\end{aligned} \tag{24}
$$

from the formalism of (1)-(2) the g_1 and g_2 are:

$$
g_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ (x_2^2+1) \end{pmatrix}
$$

The Lie bracket $[g_1, g_2]$ is

$$
[g_1, g_2] = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2x_2 \end{pmatrix} \notin \text{span}\{g_1, g_2\}
$$

Thus, the distribution $span\{g_1, g_2\}$ is not involutive. From the algorithm given in Section 3, the diffeomorphism which transform the system in the form (11)-(13) is $\phi(x)$ with $\xi_{1,1} = x_1$, $\xi_{2,1} = x_3 - x_2$, $\xi_{1,2} = x_2$, $\xi_{2,2} = x_4 + 2x_2 + x_1 + x_5$ and $\eta = x_5$. This gives the following left invertible form

$$
\begin{aligned}
\dot{\xi}_{1,1} &= \xi_{1,2} \\
\dot{\xi}_{1,2} &= -2\xi_{1,2} - \xi_{1,1} - \eta + u_1 \\
\dot{\xi}_{2,1} &= \xi_{2,2} \\
\dot{\xi}_{2,2} &= -2\xi_{1,1} + 2\xi_{1,2} - (\xi_{2,1} - \xi_{1,2})^3 \\
&\quad - 5(\xi_{2,2} - 2\xi_{1,2} - \xi_{1,1}) \\
&\quad + 3\eta - \eta^3 + 2u_1 + (\xi_{1,2}^2 + 1)u_2 \\
\dot{\eta} &= -\eta^3 + (\xi_{1,2}^2 + 1)u_2 \\
y_1 &= \xi_{1,1} \\
y_2 &= \xi_{2,1} + \xi_{1,2}
\end{aligned} \tag{25}
$$

The dummy output is $\bar{y} = \xi_{2,1} = y_2 - \dot{y}_1$ and again from the algorithm the following can be obtained

$$
\Gamma_\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 2 & 1 \end{pmatrix} \text{ setting } P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

Now, using Section 5 and the super twisting algorithm Levant [1998], we can first observe $\xi_{1,1}$ $\xi_{1,2}$ as follows

$$
\begin{aligned}
\dot{\hat{\xi}}_{1,1} &= \nu_{1,1} \\
\nu_{1,1} &= \hat{\xi}_{1,2} - \lambda_{1,1}|\hat{\xi}_{1,1} - \xi_{1,1}|^{\frac{1}{2}} sign(\hat{\xi}_{1,1} - \xi_{1,1}) \\
\dot{\hat{\xi}}_{1,2} &= -\lambda_{1,2} sign(\hat{\xi}_{1,1} - \xi_{1,1})
\end{aligned}
$$

From y_2 and $\hat{\xi}_{1,2}$ the dummy output is defined $\bar{y} = y_2 - \hat{\xi}_{1,2} = \xi_{2,1}$ and in order to have sufficiently smooth dummy output $\bar{y}$ is filtered with a low pass filter of order at least equal to two (the order of the relative degree) and we obtain with this filtered output $\xi_{2,1}^f$.

$$
\begin{aligned}
\dot{\hat{\xi}}_{2,1} &= \nu_{2,1} \\
\nu_{2,1} &= \hat{\xi}_{2,2} - \lambda_{2,1}|\hat{\xi}_{2,1} - \xi_{2,1}^f|^{\frac{1}{2}} sign(\hat{\xi}_{2,1} - \xi_{2,1}^f) \\
\dot{\hat{\xi}}_{2,2} &= -\lambda_{2,2} sign(\hat{\xi}_{2,1} - \xi_{2,1}^f)
\end{aligned} \tag{26}
$$

As, there exists $t_0 > 0$ such that $\forall t > t_0$ $\hat{\xi}_{i,j}$ is equal to $\xi_{i,j}$ then $E_1 = 1$ for $t > t_0$ and $E_1 = 0$ for $t \leq t_0$. From these, the unknown input (22) $\hat{u}$ can be estimated as

$$
\hat{u} = E_1 \begin{pmatrix} 1 & 0 \\ \frac{-2}{\xi_{1,2}^2+2} & \frac{1}{\xi_{1,2}^2+2} \end{pmatrix}
\begin{pmatrix} \dot{\hat{\xi}}_{1,2} + 2\hat{\xi}_{1,2} + \hat{\xi}_{1,1} + \hat{\eta} \\ \dot{\hat{\xi}}_{2,2} - 3\hat{\xi}_{1,1} - 12\hat{\xi}_{1,2} + 5\hat{\xi}_{2,2} + (\hat{\xi}_{2,1} + \hat{\xi}_{1,2})^3 - 3\eta + \eta^3 \end{pmatrix}
$$

Now, including $\hat{u}$ in the $\hat{\eta}$ dynamic, the following estimator can be designed:

$$
\dot{\hat{\eta}} = E_1(-\hat{\eta}^3 + (\hat{\xi}_{1,2}^2 + 1)\kappa)
$$

with

$$
\begin{aligned}
\kappa &= \frac{-2}{\xi_{1,2}^2 + 2}(-\dot{\hat{\xi}}_{1,2} + 2\hat{\xi}_{1,2} + \hat{\xi}_{1,1} + \hat{\eta}) \\
&\quad + \frac{1}{\xi_{1,2}^2 + 2}\Big(\dot{\hat{\xi}}_{2,2} - 3\hat{\xi}_{1,1} - 12\hat{\xi}_{1,2} + 5\hat{\xi}_{2,2} \\
&\quad + (\hat{\xi}_{2,1} + \hat{\xi}_{1,2})^3 - 3\eta + \eta^3\Big)
\end{aligned}
$$

As for $t > t_0$ $\hat{\xi}_{i,j}$ is equal to $\xi_{i,j}$ then the estimator converge asymptotically to $\hat{\eta} = \eta$ and thus $\hat{u}$ converge to u. The proposed observer is tested on Simulink and plots are given in Figures 1-3. It can be seen from Figures 1-2 that actual and observed states are closed to each other, in fact the initial three states are on top of each other. The actual and observed commands are shown in Figure 3. Again, it can be seen that the observer is able to reconstruct the control commands accurately. This shows that the proposed higher order sliding mode observer is efficient and able to construct states and control commands accurately.

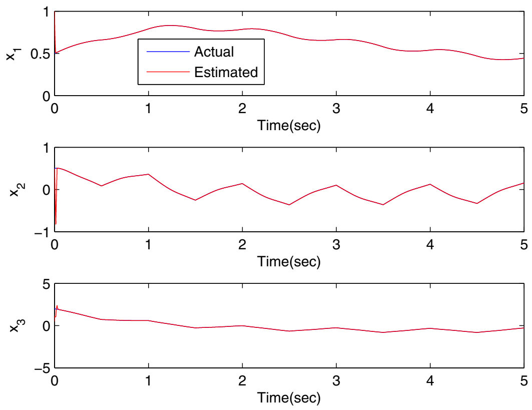

Fig. 1. Actual and observed states.

7. CONCLUSIONS

In this paper, we have proposed a higher order sliding mode observer to reconstruct states and control commands. The observer can deal with non-involutive systems and can construct non-smooth control inputs. This is

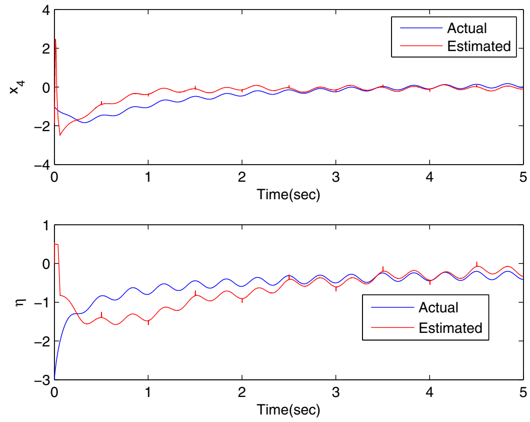

Fig. 2. Actual and observed states.

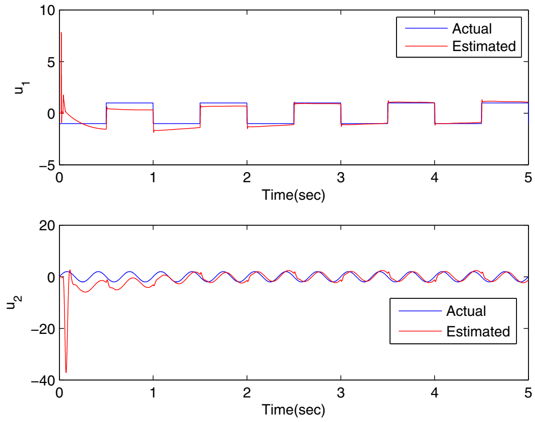

Fig. 3. Actual and reconstructed control commands.

achieved by using a new normal form for left inversion and using high order sliding mode theory. The efficiency of the proposed approach is verified by numerical results. In order to avoid chattering, we have used low pass filters that filters signals that coming directly from the equivalent vector. In future, we will address the problem of observability singularity and/or left invertibility singularity .

REFERENCES

J.-P. Barbot, D. Boutat, and T. Floquet. An observation algorithm for nonlinear systems with unknown inputs. *Automatica*, 45:1970–1974, 2009.

J.P. Barbot, T. Boukhobza, and M. Djemai. Sliding mode observer for triangular input form. *Processing of the 35th IEEE CDC-, Japan*, 1996.

D. Boutat, J-P. Barbot, and M. Darouach. A new algorithm to compute inverse dynamic of a class of nonlinear system. In *IEEE CDC 2013*, Florence, Italia, 2013.

P. Daoutidis and C. Kravaris. Inversion and zero dynamics in nonlinear multivarible control. *J. AIChE*, 527:538–603, 1991.

J. Davila, L. Fridman, and A. Levant. Second-order sliding-mode observer for mechanical systems. *IEEE TAC*, 50(11):1785–1789, 2005.

M. Fliess. A note on the invertibility of nonlinear input-output differential systems. *Syst. Control Lett.*, 8:147–151, 1986.

M. Fliess, J. Levine, P. Martin, and P. Rouchon. Flatness and defect of nonlinear systems: introductory theory and examples. *International journal of control*, 61(6):1327–1361, 1995.

T. Floquet and J.P Barbot. Super twisting algorithm based stepby- step sliding mode observers for nonlinear systems with unknown inputs. *International Journal of Systems Science*, 38(10), 2007.

L. Fridman, Y. Shtessel, C. Edwards, and X.G. Yan. Higher-order sliding-mode observer for state estimation and input reconstruction in nonlinear systems. *International Journal of Robust and Nonlinear Control*, 18(10): 399â412, 2008.

R. Hermann and A.J. Krener. Nonlinear controllability and observability. *IEEE Transactions on Automatic Control*, 22(5):728–740, 1977.

K.M. Hirschorn. Invertibility of multivariable nonlinear control systems. *IEEE Trans. Automat. Control*, 24: 855–865, 1979.

A. Isidori. Nonlinear control systems (3rd edition). *London: Springer-Verlag*, 1995.

A. Levant. Robust exact differentiation via sliding mode technique. *Automatica*, 34(3):379–384, 1998.

A. Levant. Homogeneity approach to high-order sliding mode design. *Automatica*, 41:823–830, 2005.

A. Levant. Finite differences in homogeneous discontinuous control. *IEEE TAC*, 52:1208–1217, 2007.

D.Y. Liu, O. Gibaru, and W. Perruquetti. Differentiation by integration with jacobi polynomials. *Journal of Computational and Applied Mathematics*, 235(9):3015–3032, 2011.

W. Perruquetti and J-P Barbot. *Sliding Mode Control in Engineering*. Marcel Dekker, 2002.

W. Respondek. Right and left invertibility of nonlinear control systems. *in Nonlinear Controllability and Optimal Control, ed., Sussmann H. J. (Marcel Dekker, New York*, 24:133–176, 1990.

E. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34: 435–443, 1989.