Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

On the use of algebraic geometry for the design of high-gain observers for continuous-time polynomial systems

Laura Menini * **Antonio Tornambè** *

* *Dip. Ingegneria Civile e Ingegneria Informatica Università di Roma Tor Vergata, Roma, Italy (e-mail: [menini,tornambe]@disp.uniroma2.it)*

Abstract: The goal of this paper is to apply some concepts and techniques from algebraic geometry to study the observability of nonlinear continuous-time polynomial systems. After deriving some new results for observability and embeddings, it is shown how to use such concepts to easily design high-gain observers. The proposed technique is illustrated by an application to the well known Rossler oscillator.

Keywords: Algebraic geometry, observability, high-gain observers, Rössler oscillator.

1. INTRODUCTION

One of the classical problems in control theory is that of designing state observers to obtain in real time estimates of unmeasurable state variables, using the available measured outputs. This has motivated the study of the structural property of observability and of various techniques for designing state observers, as in the works by Krener and Respondek (1985); Isidori (1995); Nijmeijer and van der Schaft (1990); Tsinias (1990); Kazantzis and Kravaris (1998); Menini and Tornambe (2002a,b, 2010a,b, 2011a).

Some of the works closer to the problems considered here are by Sussmann (1978); Sontag (1979); Takens (1981); Aeyels (1981a,b); Jouan and Gauthier (1996). In some of such works the problem of observability through sampled measurements is considered. In particular, an application using a CAS (Computer Algebra System) of the developments by Sontag (1979) was done by Nešić (1998).

When dealing with polynomial systems, it is very useful to use techniques from Algebraic Geometry, which provide the natural tools to state clear and easily testable results. This has been recognized by a variety of authors Diop (1991); Helmke et al. (2003); Tibken (2004); Menini and Tornambe (2014); Kawano and Ohtsuka (2013); Menini and Tornambe (2013a,b). The main novelty of this paper, with respect to such works, is that here observability and embedding problems are studied with the focus of observer design, whence results based on algebraic geometry are stated that allow the easy writing of explicit forms for an inverse of the observability map and of an embedding of the given system. Although we illustrate the application of our techniques with respect to high gain observers (see, e.g.,Tornambe (1989); Esfandiari and Khalil (1992); Gauthier and Kupka (2001); Hammouri et al. (2010)), it is stressed that, in principle, they can be applied to other ways of designing observers.

2. PRELIMINARIES AND NOTATION

The goal of this section is to briefly resume the basic notions of algebraic geometry (Cox et al. (1998, 2007)) that will be used in the sequel.

Let $x \in \mathbb{R}^n$, $x = [\, x_1 \ \dots \ x_n \,]^\top$, let $\mathbb{R}[x_1, ..., x_n]$ (or, shortly, $\mathbb{R}[x]$) be the commutative ring of all scalar polynomials in $x_1, ..., x_n$, and let $\mathbb{R}(x_1, ..., x_n)$ (or, shortly, $\mathbb{R}(x)$) be the field of all scalar rational functions of $x_1, ..., x_n$; $\mathbb{R}^n[x]$ denotes the set of vector functions with n entries in $\mathbb{R}[x]$.

Let $p_1, ..., p_m$ be polynomials in $\mathbb{R}[x]$; the set

$$\mathbf{V}_n(p_1, ..., p_m) := \{x \in \mathbb{R}^n : p_i(x) = 0 \text{ for all } i = 1, ..., m\}$$

is called the *affine variety* [1] (briefly, *variety*) of $\mathbb{R}^n$ defined by $p_1, ..., p_m$. As an example, $\mathbf{V}_1(x_1)$ is a point of $\mathbb{R}^1$, $\mathbf{V}_2(x_1)$ is a line of $\mathbb{R}^2$, and so on. If $\mathcal{V}_a = \mathbf{V}_n(p_1, ..., p_{m_a})$ and $\mathcal{V}_b = \mathbf{V}_n(q_1, ..., q_{m_b})$, for some polynomials p_i and q_j in $\mathbb{R}[x]$, then both $\mathcal{V}_a \cap \mathcal{V}_b$ and $\mathcal{V}_a \cup \mathcal{V}_b$ are varieties, and they are given by:

$$\mathcal{V}_a \cap \mathcal{V}_b = \mathbf{V}_n(p_1, ..., p_{m_a}, q_1, ..., q_{m_b}),$$
$$\mathcal{V}_a \cup \mathcal{V}_b = \mathbf{V}_n(p_1 q_1, ..., p_{m_a} q_1, ..., p_1 q_{m_b}, ..., p_{m_a} q_{m_b}).$$

A subset $\mathcal{I}$ of $\mathbb{R}[x]$ is a *polynomial ideal* (briefly, an *ideal*) if it satisfies:

(1) if $p, q \in \mathcal{I}$, then $p + q \in \mathcal{I}$;

(2) if $p \in \mathcal{I}$ and $q \in \mathbb{R}[x]$, then $pq \in \mathcal{I}$.

Let $p_1, ..., p_m$ be polynomials in $\mathbb{R}[x]$; it is possible to show that the set

$$\langle p_1, ..., p_m \rangle_n := \{q_1 p_1 + ... + q_m p_m : q_i \in \mathbb{R}[x], i = 1, ..., m\}$$

is an ideal of $\mathbb{R}[x]$ and, in particular, it is called the *ideal generated* by $p_1, ..., p_m$.

[1] In some texts $\mathbf{V}_n(p_1, ..., p_m)$ is called *algebraic set*, whereas the term affine variety is reserved for the case when $\mathbf{V}_n(p_1, ..., p_m)$ is irreducible.

An ideal $\mathcal{I}$ is said to be finitely *generated* if there exist polynomials $p_1, ..., p_m$ in $\mathbb{R}[x]$ such that $\mathcal{I} = \langle p_1, ..., p_m \rangle_n$; the set of the ideal *generators* $\{p_1, ..., p_m\}$ is called a *basis* of $\mathcal{I}$. By the Hilbert Basis Theorem (Cox et al., 2007), every ideal of $\mathbb{R}[x]$ is finitely generated. Clearly, if $\langle p_1, ..., p_{m_a} \rangle_n = \langle q_1, ..., q_{m_b} \rangle_n$, for some polynomials p_i and q_j in $\mathbb{R}[x]$, then $\mathbf{V}_n(p_1, ..., p_{m_a}) = \mathbf{V}_n(q_1, ..., q_{m_b})$.

Let $\mathcal{V} = \mathbf{V}_n(p_1, ..., p_m)$ be the variety defined by $p_1, ..., p_m$ in $\mathbb{R}[x]$. By definition, polynomials $p_1, ..., p_m$ vanish on $\mathcal{V}$, whence all polynomials belonging to $\langle p_1, ..., p_m \rangle_n$ vanish on $\mathcal{V}$. Note that there may be polynomials $p \notin \langle p_1, ..., p_m \rangle_n$ that vanish on $\mathcal{V}$; this renders necessary the following definitions.

Let $\mathcal{V}$ be a variety of $\mathbb{R}^n$; it is possible to show that the set

$$\mathbf{I}_n(\mathcal{V}) := \{p \in \mathbb{R}[x_1, ..., x_n] : p(x) = 0 \text{ for all } x \in \mathcal{V}\} \quad (1)$$

is an ideal and, in particular, it is called the *ideal of* $\mathcal{V}$; note that the empty set $\emptyset$ is a variety and that $\mathbf{I}_n(\emptyset) = \langle 1 \rangle_n = \mathbb{R}[x]$. Clearly, $\langle p_1, ..., p_m \rangle_n \subseteq \mathbf{I}_n(\mathbf{V}_n(p_1, ..., p_m))$, and the equality $\langle p_1, ..., p_m \rangle_n = \mathbf{I}_n(\mathbf{V}_n(p_1, ..., p_m))$ need not occur, e.g., if $n = 1$, all the polynomials $x_1^2 q(x_1)$ of $\langle x_1^2 \rangle_1$, with $q \in \mathbb{R}[x_1]$, vanish on the variety $\mathcal{V} = \mathbf{V}_1(x_1^2) = \{0\}$, but $\mathbf{I}_1(\mathbf{V}_1(x_1^2)) = \langle x_1 \rangle_1$ and $\langle x_1^2 \rangle_1 \neq \langle x_1 \rangle_1$ since $p(x_1) = x_1$ vanishes on $\{0\}$, but does not belong to $\langle x_1^2 \rangle_1$.

Let $\mathcal{V}_a$ and $\mathcal{V}_b$ be two varieties of $\mathbb{R}^n$, then

(1) $\mathcal{V}_a \subset \mathcal{V}_b$ if and only if $\mathbf{I}_n(\mathcal{V}_a) \supset \mathbf{I}_n(\mathcal{V}_b)$;

(2) $\mathcal{V}_a = \mathcal{V}_b$ if and only if $\mathbf{I}_n(\mathcal{V}_a) = \mathbf{I}_n(\mathcal{V}_b)$;

this, in particular, shows that the map $\mathbf{I}_n(\mathcal{V})$ is one-to-one.

For instance if $n = 1$, $\mathcal{V}_a = \{1\}$ and $\mathcal{V}_b = \{1, 2\}$, then $\mathbf{I}_1(\mathcal{V}_a) = \langle x_1 - 1 \rangle_1$ and $\mathbf{I}_1(\mathcal{V}_b) = \langle (x_1 - 1)(x_1 - 2) \rangle_1$, with $\mathbf{I}_1(\mathcal{V}_a) \supset \mathbf{I}_1(\mathcal{V}_b)$.

Let $\mathcal{I}$ be an ideal of $\mathbb{R}[x]$; it is possible to show that the set

$$\mathbf{V}_n(\mathcal{I}) := \{x \in \mathbb{R}^n : p(x) = 0 \text{ for all } p \in \mathcal{I}\} \quad (2)$$

is a variety and that $\mathbf{V}_n(\mathcal{I}) = \mathbf{V}_n(p_1, ..., p_m)$ for any basis $\{p_1, ..., p_m\}$ of $\mathcal{I}$.

A variety $\mathcal{V}$ can be studied through the corresponding ideal $\mathbf{I}_n(\mathcal{V})$ given by (1) and, conversely, an ideal $\mathcal{I}$ can be studied through the corresponding variety $\mathbf{V}_n(\mathcal{I})$ given by (2), but $\mathbf{I}_n(\mathbf{V}_n(\mathcal{I}))$ need not coincide with $\mathcal{I}$ (e.g., $\mathbf{I}_1(\mathbf{V}_1(\langle x_1^2 \rangle_1)) = \langle x_1 \rangle_1$), whence the map $\mathbf{V}_n(\mathcal{I})$ is not one-to-one (e.g., $\mathbf{V}_1(\langle 1 \rangle_1) = \mathbf{V}_1(\langle 1 + x_1^2 \rangle)$ and $\langle 1 \rangle_1 \neq \langle 1 + x_1^2 \rangle_1$).

Given any subset $\mathcal{S}$ of $\mathbb{R}^n$, it is easy to check that

$$\mathbf{I}_n(\mathcal{S}) := \{p \in \mathbb{R}[x_1, ..., x_n] : p(x) = 0 \text{ for all } x \in \mathcal{S}\}$$

is an ideal even if $\mathcal{S}$ is not a variety; hence, $\mathbf{V}_n(\mathbf{I}_n(\mathcal{S}))$ is a variety that contains $\mathcal{S}$. In particular, it is the smallest variety containing $\mathcal{S}$; $\mathbf{V}_n(\mathbf{I}_n(\mathcal{S}))$ is called the *Zariski closure* of $\mathcal{S}$. As an example that will be useful in the following, if $\mathcal{S} \subseteq \mathbb{R}^n$ is open, $\mathcal{S} \neq \emptyset$, then $\mathbf{I}_n(\mathcal{S}) = \{0\}$ and its Zariski closure coincides with the whole $\mathbb{R}^n$ ($\mathbf{V}_n(\mathbf{I}_n(\mathcal{S})) = \mathbb{R}^n$).

Some polynomials $q_1, ..., q_m \in \mathbb{R}[x]$ are *algebraically dependent* if there exists a non-zero polynomial $p \in \mathbb{R}[q]$, where $q = [\, q_1 \; \dots \; q_m \,]^\top$, such that $p(q_1(x), ..., q_m(x)) = 0$, $\forall x \in \mathbb{R}^n$ (*algebraically independent*, otherwise). Necessary and sufficient condition for $q_1, ..., q_m$ to be algebraically dependent is that $\operatorname{rank}_{\mathbb{R}(x)} \left(\frac{\partial q}{\partial x} \right) < m$.

Fix a total ordering $>$ of the monomials of $\mathbb{R}[x]$. For any $p \in \mathbb{R}[x]$, with $p \neq 0$, one can write

$$p(x) = a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + ... + a_\ell x^{\alpha_\ell},$$

where $a_i \in \mathbb{R}$, α_i is a multi-index, $i = 1, 2, ..., \ell$, and $x^{\alpha_1} > x^{\alpha_2} > ... > x^{\alpha_\ell}$; this allows one to define the *leading monomial* $\mathrm{LM}(p) = x^{\alpha_1}$ and the *leading coefficient* $\mathrm{LC}(p) = a_1$ of p.

Let $\{p_1, ..., p_m\}$ be a basis of an ideal $\mathcal{I}$ of $\mathbb{R}[x]$. A polynomial $r \in \mathbb{R}[x]$ is said to be *reduced* with respect to $\{p_1, ..., p_m\}$ if either $r = 0$ or no monomial that appears in r is divisible by $\mathrm{LM}(p_i), i = 1, ..., m$. A polynomial $r \in \mathbb{R}[x]$, which is reduced with respect to $\{p_1, ..., p_m\}$, is called a *remainder of the division of* $p \in \mathbb{R}[x]$ *by* $\{p_1, ..., p_m\}$ if $p - r \in \langle p_1, ..., p_m \rangle_n$.

Example 1. Fix the deglex (degree lexicographic) monomial ordering on $\mathbb{R}[x_1, x_2]$ with $x_2 > x_1$. Let $p(x) = x_1 x_2^2 - x_1$ and $\langle p_1, p_2 \rangle_2 = \langle x_1 x_2 - x_2, x_2^2 - x_1 \rangle_2$. It is easy to see that $r_1 = 0$ and $r_2 = x_1^2 - x_1$ are both reduced with respect to $\{p_1, p_2\}$ and that $p = x_2 p_1 + p_2 + r_1$ and $p = 0 p_1 + x_1 p_2 + r_2$, which show that the remainder of the division by an arbitrary basis need not be unique.

Let $\{g_1, ..., g_m\}$ be a set of non-zero polynomials in $\mathbb{R}[x]$. A basis $\{g_1, ..., g_m\}$ of an ideal $\mathcal{I}$ of $\mathbb{R}[x]$ is called a *Gröbner basis* of $\mathcal{I}$ if, for any $p \in \mathbb{R}[x]$, the remainder of the division of p by $\{g_1, ..., g_m\}$ is unique. Every non-zero ideal $\mathcal{I}$ of $\mathbb{R}[x]$ has a Gröbner basis, which need not be unique [2].

A monomial ordering on $\mathbb{R}[x_a, x_b]$, with $x_a \in \mathbb{R}^{n_a}$ and $x_b \in \mathbb{R}^{n_b}$, $n = n_a + n_b$, *eliminates* x_a if

$$x_a^\alpha > x_a^\beta \Rightarrow x_a^\alpha x_b^\gamma > x_a^\beta x_b^\delta,$$

for all multi-indices α, β for which $x_a^\alpha > x_a^\beta$ and for all multi-indices γ, δ. For instance, the lexicographic ordering with $x_1 > x_2 > ... > x_n$ eliminates $x_1, ..., x_\ell$, for all $1 \leq \ell < n$. Let $\mathcal{I}$ be an ideal of $\mathbb{R}[x_a, x_b]$. The elimination ideal of $\mathcal{I}$ that eliminates x_a is $\mathcal{I} \cap \mathbb{R}[x_b]$. Let $\{g_1, ..., g_m\}$ be a Gröbner basis for a monomial ordering that eliminates x_a. Then, the set obtained from $\{g_1, ..., g_m\}$ by retaining only the elements that do not depend on x_a (i.e., $\{g_1, ..., g_m\} \cap \mathbb{R}[x_b]$) is a Gröbner basis of the elimination ideal $\mathcal{I} \cap \mathbb{R}[x_b]$ for the monomial ordering on $\mathbb{R}[x_b]$ induced by $>$.

Example 2. Let $\mathcal{I} = \langle x_1 + x_2^2, x_1 - x_2 \rangle_2$. Fix the lexicographic ordering with $x_1 > x_2$, which eliminates x_1; a Gröbner basis of $\mathcal{I}$ for such an ordering is $\{x_2^2 + x_2, x_1 - x_2\}$, whence the elimination ideal of $\mathcal{I}$ that eliminates x_1 is $\mathcal{I} \cap \mathbb{R}[x_2] = \langle x_2^2 + x_2 \rangle_1$; in particular, $\{x_2^2 + x_2\}$ is a Gröbner basis of $\mathcal{I} \cap \mathbb{R}[x_2]$ for the induced ordering (in this case trivial).

3. OBSERVABILITY AND EMBEDDINGS OF POLYNOMIAL SYSTEMS

Consider the following polynomial system

$$\dot{x} = f(x), \quad (3a)$$

[2] By using the stronger notion of *reduced* Gröbner basis unicity is achieved, but this is not needed here.

$$y = h(x), \tag{3b}$$

where $h \in \mathbb{R}[x]$ and $f \in \mathbb{R}^n[x]$. Consider the *observability map of order* $h+1$, $h \in \mathbb{Z}$, $h \geq 0$, given by $y_{e,h} = O_h(x)$, where $y_{e,h} = [\, y_0 \ \dots \ y_h \,]^\top$, $y_i = \frac{\mathrm{d}^i y}{\mathrm{d}t^i}$, for $i = 0, 1, ..., h$, and

$$O_h(x) := \begin{bmatrix} L_f^0 h(x) \\ L_f h(x) \\ \vdots \\ L_f^h h(x) \end{bmatrix}, \tag{4}$$

with $L_f^{i+1} h(x) = \frac{\partial L_f^i h(x)}{\partial x} f(x)$ and $L_f^0 h(x) = h(x)$.

It is classical to define system (3) to be $(N+1)$-differentially observable if the observability map $y_{e,N} = O_N(x)$ is injective, i.e., if it is left invertible with a (not necessarily unique) left inverse $x = O_N^{-1}(y_{e,N})$ such that $O_N^{-1} \circ O_N(x) = x$, $\forall x \in \mathbb{R}^n$ (see Jouan and Gauthier (1996)). Such a notion can be extended as follows.

Definition 1. System (3) is $(N+1)$-*polynomially* (respectively, *rationally*) *observable* if it is $(N+1)$-differentially observable and $O_N^{-1}(y_{e,N})$ is a polynomial (respectively, rational) function.

Clearly, if system (3) is $(\overline{N}+1)$-differentially (respectively, polynomially or rationally) observable for a certain $\overline{N}$, then it is $(N+1)$-differentially (respectively, polynomially or rationally) observable for all $N \geq \overline{N}$.

Now we consider the problem of finding an embedding for system (3).

Problem 1. Given an integer $N \geq 0$, find, if any, a polynomial $p(y_{e,N}) \in \mathbb{R}[y_{e,N}]$ such that

$$p \circ O_N(x) = 0, \quad \forall x \in \mathbb{R}^n. \tag{5}$$

Note that, if the entries of $O_N(x)$ are algebraically dependent and have degree less than or equal to d, then there exists a polynomial $p(y_{e,N})$, having degree less than or equal to d^{N+1}, that solves Problem 1.

Any polynomial $p(y_{e,N}) \in \mathbb{R}[y_{e,N}]$ can be thought of as an ordinary differential equation,

$$p(y_{e,N}) = 0; \tag{6}$$

in particular, a polynomial $p(y_{e,N})$ solution of Problem 1 is called an *embedding* of (3). Any embedding has the following property: if $x(t)$ is a solution of (3a), then $y(t) = h(x(t))$ is a solution of (6), i.e., the corresponding $y_{e,N}(t)$ satisfies (6) identically.

About any regular point of $y_{e,N} = O_N(x)$, by the Implicit Function Theorem (see, e.g., Fleming (1987)), there exists locally a function $\varphi(y_{e,N-1})$ such that the following relation holds locally about such a regular point:

$$p(y_{e,N}) = 0 \Leftrightarrow y_N = \varphi(y_{e,N-1});$$

the differential equation corresponding to $y_N = \varphi(y_{e,N-1})$ is often called in normal form. Unfortunately, $\varphi(y_{e,N-1})$ need not exist about a singular point of $y_{e,N} = O_N(x)$ and, even in the case it exists, it need not be polynomial nor rational. Thus, the following particularization of Problem 1 will be considered in this paper.

Problem 2. Given an integer $N \geq 0$, find, if any, $\overline{p}(y_{e,N-1}) \in \mathbb{R}[y_{e,N-1}]$ (respectively, $\overline{p}(y_{e,N-1}) \in \mathbb{R}(y_{e,N-1})$) such that $p(y_{e,N}) = y_N - \overline{p}(y_{e,N-1})$ satisfies (5).

A solution of Problem 2 is called an *explicit embedding* of system (3).

Let $p(y_{e,N}) = y_N - \bar{p}(y_{e,N-1})$ be a solution to Problem 2; the differential equation corresponding to $y_N = \bar{p}(y_{e,N-1})$ can be rewritten in the first-order normal form (or state-space form) as follows:

$$\begin{aligned} \dot{y}_0 &= y_1, \\ &\vdots \\ \dot{y}_{N-1} &= y_N, \\ \dot{y}_N &= \bar{p}(y_{e,N-1}), \\ y &= y_0. \end{aligned}$$

Now, let the polynomial $\lambda^{N+1} + k_1 \lambda^N + ... + k_{N+1}$ have all roots with negative real part and let $\varepsilon > 0$ be a sufficiently small parameter. Under the assumptions and conditions of Theorems 1 and 2 of Tornambe (1992) (essentially, boundedness of $\bar{p}(y_{e,N-1}(t))$ as a function of t), a high-gain "practical" observer for such a system is given by

$$\dot{\hat{y}}_0 = \hat{y}_1 + \frac{k_1}{\varepsilon}(y_0 - \hat{y}_0), \tag{7a}$$

$$\vdots \tag{7b}$$

$$\dot{\hat{y}}_{N-1} = \hat{y}_N + \frac{k_N}{\varepsilon^N}(y_0 - \hat{y}_0), \tag{7c}$$

$$\dot{\hat{y}}_N = \frac{k_{N+1}}{\varepsilon^{N+1}}(y_0 - \hat{y}_0). \tag{7d}$$

The observer (7) guarantees that the estimation error $y_{e,N}(t) - \hat{y}_{e,N}(t)$ can be rendered arbitrarily small, by decreasing ε, and has the advantage that the obtained practical stability is global (i.e., the initial errors can be arbitrary).

Once the output and its derivatives have been estimated for a sufficiently high N, under the assumption that the system is $(N+1)$-differentially observable (hopefully, polynomially or rationally observable), an estimate of the state variables can be obtained by $\hat{x} = O_N^{-1}(\hat{y}_{e,N})$, especially when $y_{e,N}(t)$ is bounded as a function of t.

The following lemma follows easily from the definition of algebraic dependence (see e.g., Menini and Tornambe (2011b, 2009, 2010c,d); Menini and Tornambè (2012)).

Lemma 1. There exists a solution $p(y_{e,N}) \in \mathbb{R}[y_{e,N}]$ to Problem 1 with $p \neq 0$ if and only if the entries of the observability map $y_{e,N} = O_N(x)$ are algebraically dependent.

As well known, the entries of the observability map $y_{e,N} = O_N(x)$ are algebraically dependent if and only if $\mathrm{rank}_{\mathbb{R}(x)}\left(\frac{\partial O_N(x)}{\partial x}\right) \leq N$.

The following theorem gives, in terms of algebraic geometry, a characterization for the entries of the observability map to be algebraically independent, whence for Problem 1 to have no solution.

Theorem 1. Let $\mathcal{Y}_N = O_N(\mathbb{R}^n)$ be the image of $\mathbb{R}^n$ through the observability map; let $\mathbf{V}_{N+1}(\mathbf{I}_{N+1}(\mathcal{Y}_N))$ be the Zariski closure of $\mathcal{Y}_N$. The entries of $O_N(x)$ are algebraically independent (whence, there is no solution to Problem 1) if and only if $\mathbf{V}_{N+1}(\mathbf{I}_{N+1}(\mathcal{Y}_N)) = \mathbb{R}^{N+1}$.

Corollary 1. (1.1) If $N < n$, the entries of the observability map $y_{e,N} = O_N(x)$ are algebraically dependent if and only if system (3) is not $(N+1)$-differentially observable;

(1.2) If $N \geq n$, the entries of the observability map $y_{e,N} = O_N(x)$ are always algebraically dependent.

The following theorem gives two sufficient conditions under which there exists a solution to Problem 2.

Theorem 2. (2.1) If system (3) is N-polynomially (respectively, rationally) observable, then there exists $\overline{p}(y_{e,N-1}) \in \mathbb{R}[y_{e,N-1}]$ (respectively, $\overline{p}(y_{e,N-1}) \in \mathbb{R}(y_{e,N-1})$) such that $p(y_{e,N}) = y_N - \overline{p}(y_{e,N-1})$ satisfies (5).

(2.2) If $N \geq n+1$, then there exists $\overline{p}(y_{e,N-1}) \in \mathbb{R}(y_{e,N-1})$ such that $p(y_{e,N}) = y_N - \overline{p}(y_{e,N-1})$ satisfies (5).

Remark 1. Statement (2.1) of Theorem(2) guarantees that the N-polynomial (rational) observability, for some fixed N, implies the existence of a polynomial (rational) embedding, and Statement (2.2) guarantees that there always exists a rational embedding for $N \geq n+1$. As will be illustrated in Section 4, in practice, the rational embeddings can be more useful if the poles of the rational function $\overline{p}(y_{e,N-1})$ are guaranteed not to be attainable by the system trajectories. For some specific system, such an additional requirement may be possible only for values of N greater than $n+1$.

Example 3. If pair f, h is in the *observability form*,

$$f(x) = \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ \varphi(x_1, \dots, x_n) \end{bmatrix},$$
$$h(x) = x_1,$$

with $\varphi \in \mathbb{R}[x]$, then Problem 2 is solvable by

$$y_N = \varphi(y_0, \dots, y_{N-1}).$$

Example 4. Let $n = 1$, $f(x) = \frac{1}{2}x$ and $h(x) = x^2$. It is easy to see that $y_i = x^2$, for all $i = 0, 1, \dots$, whence the considered system is not N-differentially observable for any $N \geq 0$. Nevertheless, for such a system one has the linear explicit embedding $y_1 - y_0 = 0$.

Problems 1 and 2 have been extensively studied in the analytic case by Jouan and Gauthier (1996) and Gauthier and Kupka (2001) (earlier similar studies were given in Takens (1981)), showing that if $N \geq 2n+1$, then Problems 1 and 2 are generically solvable, thus implying the existence of an analytic function φ, locally about any regular point of $p(y_{e,N})$ (a point $y_{e,N} = O_N(x_r)$ is regular for $p(y_{e,N})$ if the Jacobian matrix of the observability mapping has full column rank about x_r), such that $y_N = \varphi(y_0, ..., y_{N-1})$. The above mentioned results do not give a constructive procedure for the computation of such a polynomial $p(y_{e,N})$, which can be determined by eliminating the x variables from the equations $y_0 = L_f^0 h(x), ..., y_N = L_f^N h(x)$ constituting the observability map $y_{e,N} = O_N(x)$.

Clearly, the elimination theory for polynomial ideals seems to be the correct instrument for such computations in the polynomial case. To pursue such an approach, let $\mathcal{I}_N$ be the set of all polynomials being solution of Problem 1, and define

$$\mathcal{J}_N := \left\langle y_0 - L_f^0 h(x), ..., y_N - L_f^N h(x) \right\rangle_{n+N+1}.$$

Theorem 3. (3.1) $\mathcal{I}_N$ is a polynomial ideal of $\mathbb{R}[y_{e,N}]$;

(3.2) $\mathcal{I}_N = \mathcal{J}_N \cap \mathbb{R}[y_{e,N}]$;

(3.3) if $\mathcal{Y}_N = O_N(\mathbb{R}^n)$ is the image of $\mathbb{R}^n$ through the observability map, then $\mathbf{V}_{N+1}(\mathcal{I}_N)$ is the Zariski closure of $\mathcal{Y}_N$, i.e., the smallest variety containing $\mathcal{Y}_N$; in addition, if system (3) is $(N+1)$-differentially observable, then $\mathbf{V}_{N+1}(\mathcal{I}_N) = \mathcal{Y}_N$.

According to the proof of Theorem 3, if one fixes the lexicographic monomial ordering with $x_1 > x_2 > \ldots > x_n > y_N > y_{N-1} > \ldots > y_0$, then a Gröbner basis of $\mathcal{I}_N$ can be obtained from a Gröbner basis G of $\mathcal{J}_N$ as $G \cap \mathbb{R}[y_{e,N}]$, i.e., simply by taking all the elements of G that do not depend on the entries of x.

By the second part of Statement (3.3), if system (3) is $(N+1)$-differentially observable, then $x(t)$ is a solution of (3a) if and only if $y(t) = h(x(t))$ is a solution of $p(y_{e,N}) = 0$, for all $p \in \mathcal{I}_N$.

4. APPLICATION TO THE ROSSLER OSCILLATOR

The Rössler oscillator is described by the equations:

$$\dot{x}_1 = x_1 x_2 + b - c x_1, \tag{8a}$$
$$\dot{x}_2 = -x_1 + x_3, \tag{8b}$$
$$\dot{x}_3 = -x_2 + a x_3, \tag{8c}$$
$$y = x_2, \tag{8d}$$

where a, b and c are three scalar positive parameters, and the choice of the measured output is just an example, the studies reported here below can be analogously developed for, e.g., $y = x_1$ and $y = x_2$. Despite the apparent simplicity of such dynamic equations, for many values of the parameters the Rössler oscillator exhibits a chaotic behaviour, and the presence of an attractor renders it a perfect benchmark for observer design, since it is easy to generate bounded chaotic trajectories not convergent to the origin. All the computations in this section, and in particular the computation of all the mentioned Gröbner bases, have been performed by using the freeware Macaulay 2, a Computer Algebra System (CAS) specialized in algebraic geometry computations (see Grayson and Stillman (2013)).

By computing $L_f h(x)$, $L_f^2 h(x)$ and $L_f^3 h(x)$, it can be seen that the observability map $O_3(x)$ is not injective. By computing $L_f^4 h(x)$ and a Gröbner basis G_4 of the ideal $\mathcal{J}_4 = \langle y_0 - h(x),\ y_1 - L_f h(x),\ y_2 - L_f^2 h(x),\ y_3 - L_f^3 h(x),\ y_4 - L_f^4 h(x) \rangle$ of $\mathbb{R}[x_1, x_2, x_3, y_4, y_3, y_2, y_1, y_0]$ w.r.t. the lexicographic monomial order (with $x_1 > x_2 > x_3 > y_4 > \ldots > y_0$), it can be seen that the observability map $y_{e,4} = O_4(x)$ is globally invertible. In fact, G_4 is constituted by 8 polynomials, of which the relevant ones are

$$g_{4,4} = x_3 y_0 + (-a - c)x_3 + \gamma_{4,4}(y_{e,4}),$$
$$g_{4,5} = x_3 y_1 + \gamma_{4,5}(y_{e,4}),$$

$$g_{4,6} = x_3 y_2 + \gamma_{4,6}(y_{e,4}),$$
$$g_{4,8} = x_1 - x_3 + y_1.$$

The complete expressions of the above polynomials are omitted for brevity. To see that $y_{e,4} = O_4(x)$ is injective, note that $h(x) - a - c$, $L_f h(x)$ and $L_f^2 h(x)$ do not vanish jointly at any $x \in \mathbb{R}^3$, being $a + b + c \neq 0$ by hypothesis. This can be checked immediately by computing the Gröbner basis of the ideal $\langle h(x) - a - c, L_f h(x), L_f^2 h(x)\rangle$, that is given by $\{a + b + c,\ x_2 + b,\ x_1 - x_3\}$, and noting that one of its elements never vanishes. Whence, by equating to zero $(y_0 - a - c)g_{4,4}$, $y_1 g_{4,5}$ and $y_2 g_{4,6}$, and summing up the three resulting equations, one obtains the following rational expression

$$x_3 = -\frac{(y_0 - a - c)\gamma_{4,4}(y_{e,4}) + y_1\gamma_{4,5}(y_{e,4}) + y_2\gamma_{4,6}(y_{e,4})}{(y_0 - a - c)^2 + y_1^2 + y_2^2}, \quad (9)$$

that is globally valid, along the trajectories of the system. Furthermore, by equating to zero $g_{4,8}$, one obtains:

$$x_1 = x_3 - y_1; \quad (10)$$

expressions (9), (10), with x_3 replaced by its expression given by (9), and the trivial $x_2 = y_0$, constitute an explicit expression of a left inverse $x = O_4^{-1}(y_{e,4})$, that is needed to design the fifth order "practical" observer (7).

To perform some simulations, the values of the system parameters have been chosen as $a = 0.1$, $b = 0.1$ and $c = 10$. The initial condition for the system has been chosen as

$$x(0) = [\,6\ \ 16\ \ -0.6\,]^\top,$$

which, in the time interval $[0, 40]$, gives the trajectory reported in Fig. 1. The initial condition for the observer has been chosen as $\hat{y}_i = 0$, $i = 0, \ldots, 4$. The polynomial $\lambda^{N+1} + k_1\lambda^N + \ldots + k_{N+1}$ has been chosen as

$$(\lambda + 1)(\lambda^2 + 2\lambda + 2)(\lambda^2 + 3\lambda + 4),$$

and the parameter ε has been set as $\varepsilon = 0.25 \cdot 10^{-3}$. The estimates of the state variables have been obtained by $\hat{x}(t) = O_4^{-1}(\hat{y}_{e,4}(t))$, being $\hat{y}_{e,4}(t)$ the observer state.

In Fig. 2 the time behaviour of the three state variables $x_1(t)$, $x_2(t)$ and $x_3(t)$ is reported together with the time behaviour of the respective estimates $\hat{x}_1(t)$, $\hat{x}_2(t)$ and $\hat{x}_3(t)$; they are practically undistinguishable. But, since the observer is just a "practical" one, a non vanishing estimation error is expected: such an error is shown in Fig. 3. The obtained error is acceptable, it can be reduced arbitrarily by decreasing ε, at the expense of a worst initial transient and more expensive computations.

5. CONCLUSIONS

After characterizing observability and high-gain observer design for polynomial systems in terms of algebraic geometry notions, the main theoretical result of this paper has been derived, that is Theorem 3, which allows to find all possible implicit embeddings for a polynomial system by using Gröbner bases and elimination theory. Then, through the application to the Rössler system, it has been shown that the same techniques, that can be used by means of powerful modern Computer Algebra Systems, effectively allow the design of high-gain observers for polynomial systems.

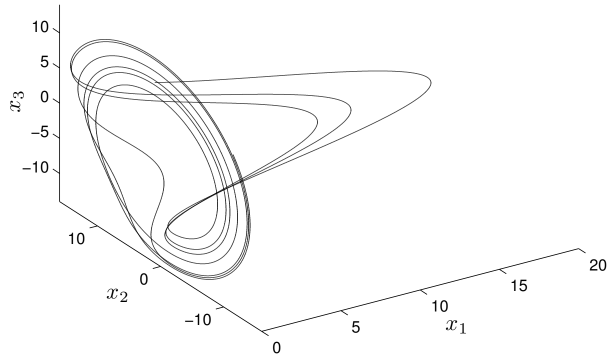

Fig. 1. The chosen trajectory of the Rössler system for $t \in [0, 40]$.

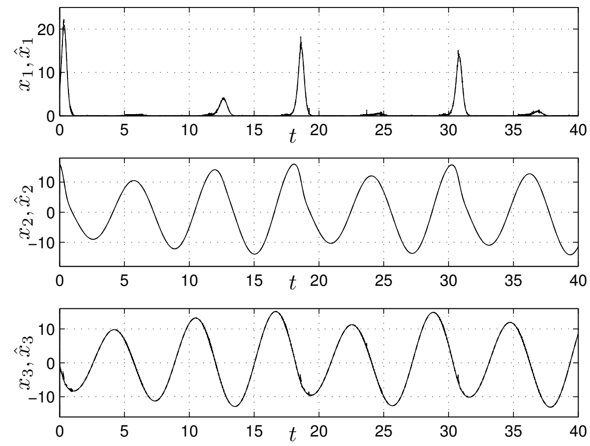

Fig. 2. The time behaviour of the three state variables and of their estimates, for the chosen trajectory.

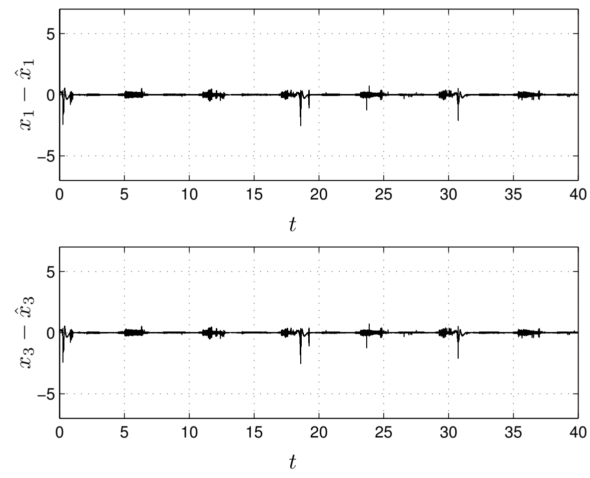

Fig. 3. The time behaviour of the estimation errors for the chosen test.

REFERENCES

Aeyels, D. (1981a). Generic observability of differentiable systems. *SIAM Journal on Control and Optimization*, 19(5), 595–603.

Aeyels, D. (1981b). On the number of samples necessary to achieve observability. *Systems & Control Letters*, 1(2), 92–94.

Cox, D., Little, J., and O'Shea, D. (1998). *Using algebraic geometry*, volume 185. Springer Verlag.

Cox, D., Little, J., and O'Shea, D. (2007). *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*, volume 10. Springer Verlag.

Diop, S. (1991). Elimination in control theory. *Mathematics of control, signals and systems*, 4(1), 17–32.

Esfandiari, F. and Khalil, H. (1992). Output feedback stabilization of fully linearizable systems. *International Journal of Control*, 56(5), 1007–1037.

Fleming, W.H. (1987). *Functions of several variables*. Springer-Verlag, New York Heidelberg Berlin, third edition.

Gauthier, J. and Kupka, I. (2001). *Deterministic observation theory and applications*. Cambridge Univ Pr.

Grayson, D.R. and Stillman, M.E. (2013). Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

Hammouri, H., Bornard, G., and Busawon, K. (2010). High gain observer for structured multi-output nonlinear systems. *Automatic Control, IEEE Transactions on*, 55(4), 987–992.

Helmke, U., Martin, C.F., and Zhou, Y. (2003). Genericity of algebraically observable polynomial systems. *preprint*.

Isidori, A. (1995). *Nonlinear control systems*. Communications and Control Engineering. Springer, Berlin Heidelberg New York, third edition.

Jouan, P. and Gauthier, J. (1996). Finite singularities of nonlinear systems. output stabilization, observability, and observers. *Journal of Dynamical and Control systems*, 2(2), 255–288.

Kawano, Y. and Ohtsuka, T. (2013). Observability at an initial state for polynomial systems. *Automatica*.

Kazantzis, N. and Kravaris, C. (1998). Nonlinear observer design using Lyapunov's auxiliary theorem. *Systems and Control Lett.*, 34, 241–247.

Krener, A. and Respondek, W. (1985). Nonlinear observers with linearizable error dynamics. *SIAM J. Control and Optimization*, 23, 197–216.

Menini, L. and Tornambe, A. (2002a). State estimation of (otherwise unobservable) linear mechanical systems through the use of non-smooth impacts: the case of two mating gears. *Automatica*, 38(10), 1823–1826.

Menini, L. and Tornambe, A. (2002b). Velocity observers for non-linear mechanical systems subject to non-smooth impacts. *Automatica*, 38(12), 2169–2175.

Menini, L. and Tornambe, A. (2009). Linearization through state immersion of nonlinear systems admitting Lie symmetries. *Automatica*, 45(8), 1873–1878.

Menini, L. and Tornambe, A. (2010a). Analysis and observer design for the Bullard and Rikitake dynamos. *IET Control Theory and Applications*, 4(8), 1353–1365.

Menini, L. and Tornambe, A. (2010b). Computation of the real logarithm for a discrete-time nonlinear system. *Systems & Control Letters*, 59(1), 33–41.

Menini, L. and Tornambe, A. (2010c). Semi-invariants and their use for stability analysis of planar systems. *International Journal of Control*, 83(1), 154–181.

Menini, L. and Tornambe, A. (2010d). Stability analysis of planar systems with nilpotent (non-zero) linear part. *Automatica*, 46(3), 537–542.

Menini, L. and Tornambe, A. (2011a). Design of state detectors for nonlinear systems using symmetries and semi-invariants. *Systems & Control Letters*, 60(2), 128–137.

Menini, L. and Tornambe, A. (2011b). *Symmetries and Semi-invariants in the Analysis of Nonlinear Systems*. Springer, London.

Menini, L. and Tornambè, A. (2012). Exact and approximate feedback linearization without the linear controllability assumption. *Automatica*, 48(9), 2221–2228.

Menini, L. and Tornambe, A. (2013a). Immersion and Darboux polynomials of Boolean networks with application to the Pseudomonas syringae HRP regulon. In *52th IEEE Conference on Decision and Control*, 4092–4097. Firenze, Italy.

Menini, L. and Tornambe, A. (2013b). Observability and dead-beat observers for Boolean networks modeled as polynomial discrete-time systems. In *52th IEEE Conference on Decision and Control*, 4428–4433. Firenze, Italy.

Menini, L. and Tornambe, A. (2014). On a Lyapunov equation for polynomial continuous-time systems. *International Journal of Control*, 87(2), 393–403.

Nešić, D. (1998). A note on observability tests for general polynomial and simple wiener–hammerstein systems. *Systems & control letters*, 35(4), 219–227.

Nijmeijer, H. and van der Schaft, A.J. (1990). *Nonlinear dynamical control systems*. Springer-Verlag, New York Heidelberg Berlin.

Sontag, E.D. (1979). On the observability of polynomial systems, i: Finite-time problems. *SIAM Journal on Control and Optimization*, 17(1), 139–151.

Sussmann, H.J. (1978). Generic single-input observability of continuous-time polynomial systems. In *Decision and Control including the 17th Symposium on Adaptive Processes, 1978 IEEE Conference on*, volume 17, 566–571. IEEE.

Takens, F. (1981). Detecting strange attractors in turbulence. In *Dynamical systems and turbulence, Warwick 1980*, 366–381. Springer.

Tibken, B. (2004). Observability of nonlinear systems-an algebraic approach. In *Decision and Control, 2004. CDC. 43rd IEEE Conference on*, volume 5, 4824–4825. IEEE.

Tornambe, A. (1989). Use of asymptotic observers having-high-gains in the state and parameter estimation. In *Decision and Control, 1989., Proceedings of the 28th IEEE Conference on*, 1791–1794. IEEE.

Tornambe, A. (1992). High-gain observers for non-linear systems. *International Journal of Systems Science*, 23(9), 1475–1489.

Tsinias, J. (1990). Further results on the observer design problem. *Syst. & Control Lett.*, 14, 411–418.