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On the Existence of a Mean-Square Stabilizing Solution to a Modified Algebraic Riccati Equation ⋆

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Abstract: In this paper, we investigate a mean-square stabilizing solution to a modified algebraic Riccati equation (MARE), which arises in our previous work on the linear quadratic optimal control for linear time-invariant discrete systems with random input gains. An explicit necessary and sufficient condition ensuring the existence of a mean-square stabilizing solution is given directly in terms of the system parameters with the help of theory of positive operators and the assumption of observability or detectability of certain stochastic systems is no longer needed. Such a necessary and sufficient condition is compatible with that for the existence of a stabilizing solution to the standard definite algebraic Riccati equation (ARE).

Keywords: modified algebraic Riccati equation, mean-square stabilizing solution, stochastic control, LQ optimal control

1. INTRODUCTION

Algebraic Riccati equations (AREs) play an important role in optimal control and filtering. They appear, for instance, in the solution for the infinite-horizon time-invariant linear-quadratic (LQ) and linear-quadratic-Gaussian (LQG) optimal control problems, as well as in the optimal linear filtering problems. Some important and essential concepts, such as stabilizability and detectability, were brought in and became the basis for the conventional control theory. The solutions, properties and applications of AREs, associated to control or filtering problems, are studied in a large number of research works, see e.g. [7, 8], and [16], and the references therein.

In our previous work [14, 15] on the LQ optimal control for linear time-invariant (LTI) discrete systems with random input gains, a modified algebraic Riccati equation (MARE), which plays a similar role to AREs, arises. In this paper, we aim to study the solutions to this MARE, particularly, the mean-square stabilizing solution, which is associated with the optimal controller for the stochastic LQ optimal control problem in [14] and [15]. The concepts of mean-square stabilizability and mean-square detectability for stochastic systems are defined. Actually, such kind of MAREs appear often in stochastic control theory, in particular, in the stochastic LQ and LQG optimal control problem [3, 5], as well as the stochastic optimal filtering problem [9]. In most existing research works, only sufficient conditions are given for the existence of a (mean-square) stabilizing solution to such kind of MAREs. In [5], a sufficient condition for the existence of a stabilizing solution to an MARE, which is associated with LQG control over erasure channels with perfect acknowledgment, is given in terms of the loss probabilities and the classical stabilizability and detectability. A similar condition is provided by [12]. In [13], by assuming stabilizability and exact detectability, which is identical to the mean-square detectability defined in this paper, an MARE is shown to have a stabilizing solution. The sufficient condition provided by [4] is given in terms of mean-square stabilizability and another definition of detectability, which is dual to the mean-square stabilizability. In our previous work [15], a sufficient condition is given in terms of mean-square stabilizability and mean-square detectability.

As can be seen from above, detectability for stochastic systems is always assumed. Is it necessary? Does there exist a necessary and sufficient condition to ensure the existence of a mean-square stabilizing solution? In the book [3], a numerical necessary and sufficient condition is given in terms of the feasibility of some linear matrix inequalities (LMIs). However, such a condition has no explicit interpretation with respect to the dynamical properties of stochastic systems. We are more interested in seeking an explicit necessary and sufficient condition given directly in terms of the system parameters. Such a condition is not current available, as far as we know. In this paper, an explicit necessary and sufficient condition is obtained with the help of theory of positive operators and the condition of (mean-square) detectability for a certain stochastic system is no longer needed. This condition is compatible with that for the existence of a stabilizing solution to the standard definite ARE when the random input gains become constant. Moreover, the approach can be applied to the standard definite ARE as well as generalized to finding out an explicit necessary and sufficient condition for many other MAREs in both the discrete-time and continuous-time cases.

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The remainder of this paper is organized as follows. The problem is stated in Section 2. In Section 3, some mathematical preliminaries are presented for used. The concepts of mean-square stabilizability and mean-square detectability for stochastic systems are given and some criteria are provided in Section 4. In Section 5, an explicit necessary and sufficient condition for the existence of a mean-square stabilizing solution to the considered MARE is obtained. A numerical example is worked out in Section 6, while a conclusion follows in Section 7. The proofs of lemmas are omitted due to space limitation.

2. PROBLEM FORMULATION

In this paper, we investigate the mean-square stabilizing solution, which is defined later, to the following MARE:

$$A'XA - X + C'C - (A'XB + C'D) \times [W \odot (B'XB + D'D)]^{-1}(B'XA + D'C) = 0, \quad (1)$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$ and $W \in \mathbb{R}^{m\times m}$, given in the form of (3). The symbol $\odot$ stands for the Hadamard product. The MARE (1) arises in the infinite-horizon LQ optimal control problem for discrete-time LTI systems with random input gains shown in Fig. 1.

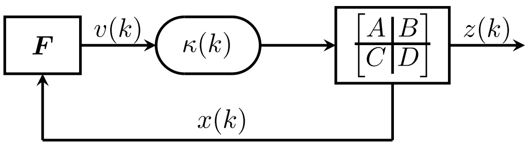

Fig. 1. LQ optimal control
for discrete-time LTI systems with random input gains

The stochastic system shown in Fig. 1 is described by

$$\begin{aligned} x(k+1) &= Ax(k) + B\kappa(k)v(k), \\ z(k) &= Cx(k) + D\kappa(k)v(k), \end{aligned} \quad (2)$$

where $x(k) \in \mathbb{R}^n$ is the system state, $v(k) \in \mathbb{R}^m$ is the control input generated by an LTI state feedback controller $\boldsymbol{F}$, and $z(k) \in \mathbb{R}^p$ is the system output. There are random input gains $\kappa(k)$ in the input channels. The parallel transmission strategy is adopted, i.e., each element of $v(k)$ is sent across an individual input channel. Then $\kappa(k)$ can be given by a diagonal random matrix $\mathrm{diag}\{\kappa_1(k), \ldots, \kappa_m(k)\}$, whose diagonal elements $\kappa_i(k)$ are mutually uncorrelated i.i.d random processes with mean $\mu_i = \boldsymbol{E}[\kappa_i(k)] \neq 0$ and variance $\sigma_i^2 = \boldsymbol{E}\left[(\kappa_i(k) - \mu_i)^2\right]$, respectively. The signal-to-noise ratio of the ith input channel is denoted by $\mathrm{SNR}_i \triangleq \frac{\mu_i^2}{\sigma_i^2}$. Denote

$$M \triangleq \mathrm{diag}\{\mu_1, \ldots, \mu_m\}, \quad \Sigma^2 \triangleq \mathrm{diag}\{\sigma_1^2, \ldots, \sigma_m^2\},$$
$$SNR \triangleq \mathrm{diag}\{SNR_1, \ldots, SNR_m\},$$

$$W \triangleq \begin{bmatrix} 1+\mathrm{SNR}_1^{-1} & 1 & \cdots & 1 \\ 1 & 1+\mathrm{SNR}_2^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 1+\mathrm{SNR}_m^{-1} \end{bmatrix}. \quad (3)$$

The LQ cost function is defined as

$$J(x(0), v(\cdot)) = \boldsymbol{E}\left[\sum_{k=0}^{\infty} z'(k)z(k)\right].$$

The aim of this LQ optimal control problem is to find an LTI state feedback controller to minimize the cost function as well as stabilize the closed-loop system in the mean-square sense.

It is shown in [14] and [15] that this stochastic LQ optimal control problem is solvable if and only if the MARE (1) has a mean-square stabilizing solution X with $X \geq 0$. Then the optimal controller is given by the associated static state feedback gain

$$F = -M^{-1}[W \odot (B'XB + D'D)]^{-1}(B'XA + D'C). \quad (4)$$

with the minimal cost

$$V(x(0)) \triangleq \inf_{\boldsymbol{F}} J(x(0), v(\cdot)) = x'(0)Xx(0).$$

Moreover, a sufficient condition ensuring the existence of a mean-square stabilizing solution is given in terms of the concepts of mean-square stabilizability and mean-square detectability, which are defined in Section 4. However, we are interested in seeking an explicit necessary and sufficient condition. In the following, we show the approach to find out such a necessary and sufficient condition.

3. MATHEMATICAL PRELIMINARIES

In this section, we present some preliminary mathematical information.

Let $\mathcal{S}_n \in \mathbb{R}^{n\times n}$ denote the vector space of $n \times n$ real symmetric matrices endowed with the inner product $\langle X, Y\rangle = \mathrm{tr}(XY') = \mathrm{tr}(XY)$. Denote by $\mathcal{P}_n \triangleq \{X \in \mathcal{S}_n : X \geq 0\}$ the subset of $n\times n$ real positive semi-definite matrices. A linear operator $\mathcal{T} : \mathcal{S}_n \mapsto \mathcal{S}_n$ is said to be positive if $\mathcal{T}(\mathcal{P}_n) \subset \mathcal{P}_n$. Denote the spectrum and the spectral radius of a linear operator $\mathcal{T}$ by $\sigma(\mathcal{T})$ and $\rho(\mathcal{T})$, respectively. One important property of a positive operator $\mathcal{T}$, which is known as the Krein-Rutman Theorem [1], is stated below.

Lemma 1. Let $\mathcal{T} : \mathcal{S}_n \mapsto \mathcal{S}_n$ be a positive operator. Then $\rho(\mathcal{T})$ is an eigenvalue of $\mathcal{T}$ together with an eigenvector $X \in \mathcal{P}_n$, $X \neq 0$, i.e., $\mathcal{T}(X) = \rho(\mathcal{T})X$.

The following two lemmas describe some useful properties of trace and Schur complement, respectively. They will be used often in the proof of the main Theorem.

Lemma 2. (Trace).

i) For any $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{m\times n}$, $\mathrm{tr}AB = \mathrm{tr}BA$.
ii) For any $A, B \in \mathcal{P}_n$, $\mathrm{tr}AB \geq 0$ with equality holding if and only if $AB = 0$. Moreover, when $A > 0$, the equality holds if and only if $B = 0$.

For any matrix $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ with G_{22} nonsingular, the Schur complement of G_{22} is defined as

$$G/_{22} \triangleq G_{11} - G_{12}G_{22}^{-1}G_{21}.$$

Lemma 3. Consider the above matrix G and its Schur complement $G/_{22}$, the following statements hold.

i) $G/_{22} = \begin{bmatrix} I & -G_{12}G_{22}^{-1} \end{bmatrix} G \begin{bmatrix} I \\ -G_{22}^{-1}G_{21} \end{bmatrix}$.
ii) $G \geq 0$ if and only if $G_{22} > 0$ and $G/_{22} \geq 0$.

4. MEAN-SQUARE STABILIZABILITY AND MEAN-SQUARE DETECTABILITY

Consider the following generalized stochastic system:

$$\begin{aligned} x(k+1) &= \left(A + \sum_{i=1}^{m} A_i p_i(k)\right) x(k), \\ z(k) &= \left(C + \sum_{i=1}^{m} C_i p_i(k)\right) x(k), \end{aligned} \tag{5}$$

where $p_1(k), \ldots, p_m(k)$ are mutually uncorrelated i.i.d random processes.

Definition 4. The stochastic system (5) is said to be mean-square stable, if for any initial state $x(0)$, $\boldsymbol{E}[x(k)x'(k)]$ is well-defined for any $k > 0$ and $\lim_{k\to\infty} \boldsymbol{E}[x(k)x'(k)] = 0$.

Definition 5. The stochastic system (5) is said to be mean-square detectable, if for any $x(0)$ such that $\boldsymbol{E}[z(k)z'(k)] \equiv 0$ for all $k > 0$, we have $\lim_{k\to\infty} \boldsymbol{E}[x(k)x'(k)] = 0$.

The above definition of mean-square detectability also appears in [2] and [13]. However, it will be shown that the detectability for stochastic systems is no longer needed for the existence of a mean-square stabilizing solution.

Referring back to the stochastic system (2), when $v(k)$ is generated by an LTI state feedback controller $\boldsymbol{F}$

$$\begin{aligned} x_K(k+1) &= A_K x_K(k) + B_K x(k), \\ v(k) &= C_K x_K(k) + D_K x(k), \end{aligned}$$

the closed-loop system is given by

$$\begin{aligned} \hat{x}(k+1) &= \left\{ \begin{bmatrix} A & 0 \\ B_K & A_K \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \kappa(k) \left[D_K \ C_K\right] \right\} \hat{x}(k), \\ z(k) &= \left\{ [C \ 0] + D\kappa(k) \left[D_K \ C_K\right] \right\} \hat{x}(k), \end{aligned} \tag{6}$$

where $\hat{x}(k) = [x'(k) \ x'_K(k)]'$. Then $[A|B]$ is said to be mean-square stabilizable with the random input gains $\kappa(k)$ if there exists an LTI feedback controller $\boldsymbol{F}$ such that the closed-loop system (6) is mean-square stable. In [10], it is shown that $[A|B]$ can be mean-square stabilizable with $\kappa(k)$ via dynamic state feedback if and only if $[A|B]$ can be mean-square stabilizable with $\kappa(k)$ via static state feedback. Hence, we limit our attention to the static state feedback in the rest of this section. Let $v(k)$ be generated by a static state feedback gain F. Then the closed-loop system (6) is given by

$$\begin{aligned} x(k+1) &= (A + B\kappa(k)F)\, x(k), \\ z(k) &= (C + D\kappa(k)F) x(k). \end{aligned} \tag{7}$$

Define the following positive operator:

$$\begin{aligned} \mathcal{L}_F : X \in \mathcal{S}_n \mapsto &(A + BMF)X(A + BMF)' \\ &+ B[\Sigma^2 \odot (FXF')]B' \in \mathcal{S}_n. \end{aligned}$$

Since $\mathcal{S}_n$ is a Hilbert space endowed with the inner product $\langle X, Y\rangle = \mathrm{tr}XY$, the adjoint operator $\mathcal{L}_F^*$ satisfying

$$\mathrm{tr}X\mathcal{L}_F(Y) = \mathrm{tr}\mathcal{L}_F^*(X)Y, \ \forall X, Y \in \mathcal{S}_n$$

is given by

$$\begin{aligned} \mathcal{L}_F^* : X \in \mathcal{S}_n \mapsto &(A + BMF)'X(A + BMF) \\ &+ F'[\Sigma^2 \odot (B'XB)]F \in \mathcal{S}_n. \end{aligned}$$

Evidently, $\boldsymbol{E}x(k+1)x'(k+1) = \mathcal{L}_F(\boldsymbol{E}x(k)x'(k))$. Moreover, it holds that $[A|B]$ is mean-square stabilizable with $\kappa(k)$ if and only if there exists F such that $\rho(\mathcal{L}_F) < 1$.

The real positive semi-definite eigenvectors of the operator $\mathcal{L}_F$ are essential in the subsequent developments. Then define the spectrum of $\mathcal{L}_F$ restricted to $\mathcal{P}_n$ as follows:

$$\sigma_{\mathcal{P}}(\mathcal{L}_F) = \{\lambda \in \sigma(\mathcal{L}_F) : \mathcal{L}_F(X) = \lambda X, X \in \mathcal{P}_n, X \neq 0\}.$$

For any $\lambda \in \sigma_{\mathcal{P}}(\mathcal{L}_F)$, $\lambda \geq 0$. By Lemma 1, $\rho(\mathcal{L}_F) \in \sigma_{\mathcal{P}}(\mathcal{L}_F)$, which implies that

$$\rho(\mathcal{L}_F) = \max_{\lambda \in \sigma_{\mathcal{P}}(\mathcal{L}_F)} \lambda.$$

For convenience, we call $\lambda \in \sigma_{\mathcal{P}}(\mathcal{L}_F)$ the mean-square nonnegative eigenvalues of the stochastic system (7). For a mean-square nonnegative eigenvalue λ, if $\lambda < 1$, it is said to be stable; otherwise, it is unstable.

Definition 6. A mean-square nonnegative eigenvalue λ of (7) is said to be mean-square observable if for the associated eigenvector $X \in \mathcal{P}_n$, we have that

$$\boldsymbol{E}(C + D\kappa(k)F)X(C + D\kappa(k)F)' \neq 0;$$

otherwise, it is said to be mean-square unobservable.

The following lemma provides a necessary and sufficient condition on the mean-square detectability of stochastic systems, which is given in terms of the mean-square observability of its mean-square nonnegative eigenvalues.

Lemma 7. The stochastic system (7) is mean-square detectable if and only if all the unstable mean-square nonnegative eigenvalues are mean-square observable.

A similar result is obtained in [2] for continuous-time stochastic systems. Note that

$$\begin{aligned} &\boldsymbol{E}(C + D\kappa(k)F)X(C + D\kappa(k)F)' \\ &= (C + DMF)X(C + DMF)' + D[\Sigma^2 \odot (FXF')]D'. \end{aligned}$$

Define the following operator

$$\begin{aligned} \mathcal{C}_F : X \in \mathcal{S}_n \mapsto &(C + DMF)X(C + DMF)' \\ &+ D[\Sigma^2 \odot (FXF')]D' \in \mathcal{S}_p. \end{aligned} \tag{8}$$

Then Lemma 7 can also be expressed as follows: the stochastic system (7) is mean-square detectable if and only if there do not exist $\lambda \geq 1$ and $X \in \mathcal{P}_n$ such that

$$\begin{aligned} \mathcal{L}_F(X) &= \lambda X, \\ \mathcal{C}_F(X) &= 0. \end{aligned}$$

Therefore Lemma 7 can be seen as a stochastic version of the classical Hautus-test in the mean-square sense.

5. SOLUTION TO THE MARE

In this section, we define the mean-square stabilizing solution and the maximal solution to the MARE (1). Several important lemmas are employed to find out an explicit necessary and sufficient condition ensuring the existence of a mean-square stabilizing solution.

Definition 8. A solution $X \in \mathcal{P}_n$ to the MARE (1) is said to be a mean-square stabilizing solution if $\rho(\mathcal{L}_F) < 1$, with the associated controller F given by (4). It is said to be a strong solution if $\rho(\mathcal{L}_F) \leq 1$.

To investigate the existence of a mean-square stabilizing solution to the MARE (1) and obtain an explicit necessary and sufficient condition ensuring the existence, we hope to get some insight from the theory of the standard definite ARE:

$$\begin{aligned} A'XA - X + C'C - (A'XB + C'D)& \\ \times (B'XB + D'D)^{-1}(B'XA + D'C) &= 0. \end{aligned} \tag{9}$$

Recall that an explicit necessary and sufficient condition ensuring the existence of a stabilizing solution to the ARE (9) when D has full column rank is that

a) $[A|B]$ is stabilizable;
b) $\begin{bmatrix} A-B(D'D)^{-1}D'C \\ \hline C-D(D'D)^{-1}D'C \end{bmatrix}$ has no unobservable eigenvalues on the unit circle.

The above condition can be found in [16]. Inspired by the above results, we assume that $W \odot (D'D) > 0$, which is weaker than the assumption that D has full column rank, and define the following stochastic system:

$$\begin{aligned} x(k+1) &= \{A - B\kappa(k)M^{-1}[W\odot(D'D)]^{-1}D'C\}x(k), \\ z(k) &= \{C - D\kappa(k)M^{-1}[W\odot(D'D)]^{-1}D'C\}x(k). \end{aligned} \quad (10)$$

The corresponding positive operator is $\mathcal{L}_{-M^{-1}[W\odot(D'D)]^{-1}D'C}$.

The following key lemma is obtained, which itself is an interesting result but also plays an essential role in deriving the necessary and sufficient condition.

Lemma 9. Given a solution $X \in \mathcal{P}_n$ to the MARE (1), any unstable mean-square nonnegative eigenvalue λ of the closed-loop system (7) with the associated controller F (4) is an unobservable mean-square nonnegative eigenvalue of the stochastic system (10).

The next lemma provides a condition under which the MARE (1) has at least a positive semi-definite solution.

Lemma 10. When $[A|B]$ is mean-square stabilizable with $\kappa(k)$, there exists $X \in \mathcal{P}_n$ such that the MARE (1) holds.

An application of Lemma 9 and Lemma 10 yields the next Theorem, which provides a sufficient condition for the existence of a mean-square stabilizing solution.

Theorem 11. When $[A|B]$ is mean-square stabilizable with $\kappa(k)$ and the stochastic system (10) is mean-square detectable, every positive semi-definite solution to the MARE (1) is mean-square stabilizing.

Evidently, the mean-square stabilizability of $[A|B]$ with $\kappa(k)$ is also a necessary condition. In what follows, it can be shown that the mean-square detectability of the stochastic system (10) is unnecessary; moreover, the mean-square stabilizing solution, if it exists, is unique.

Before proceeding, some special cases are given. When $\begin{bmatrix} C'C & C'D \\ D'C & D'D \end{bmatrix} > 0$, the stochastic system (10) is automatically mean-square detectable; therefore the mean-square stabilizing solution exists if and only if $[A|B]$ is mean-square stabilizable with $\kappa(k)$. This is the main result of our work in [14]. When $C'D = 0$, the sufficient condition becomes that $[A|B]$ is mean-square stabilizable with $\kappa(k)$ and the deterministic LTI system $\begin{bmatrix} A \\ \hline C \end{bmatrix}$ is detectable.

Now we are ready to state the necessary and sufficient condition.

Theorem 12. The mean-square stabilizing solution to the MARE (1) exists and is unique if and only if

i) $[A|B]$ is mean-square stabilizable with $\kappa(k)$;
ii) the stochastic system (10) has no unobservable mean-square nonnegative eigenvalue at 1.

By Theorem 12, the observability or detectability of a certain stochastic system is indeed unnecessary for the existence. Such a result is parallel to the theory of the standard definite ARE (9) mentioned before. More specifically, directly applying Theorem 12 when the random input gains $\kappa(k)$ are constant, we get that when D has full column rank, a necessary and sufficient condition for the existence of the stabilizing solution to the ARE (9) is given by

i) $[A|B]$ is stabilizable;
ii) $\begin{bmatrix} A-B(D'D)^{-1}D'C \\ \hline C-D(D'D)^{-1}D'C \end{bmatrix}$ has no unobservable mean-square nonnegative eigenvalue at 1.

The equivalence of condition b) and condition ii) is ensured by the following lemma.

Lemma 13. A deterministic system $\begin{bmatrix} A \\ \hline C \end{bmatrix}$ has no unobservable eigenvalues on the unit circle if and only if it has no unobservable mean-square nonnegative eigenvalue at 1.

When $C'D = 0$, which is a common assumption in many papers, we get the following condition which can be easily verified.

Corollary 14. There exists a mean-square stabilizing solution to the following MARE

$$A'XA - X + C'C - A'XB[W\odot(B'XB + D'D)]^{-1}B'XA = 0$$

if and only if

i) $[A|B]$ is mean-square stabilizable with $\kappa(k)$;
ii) $\begin{bmatrix} A \\ \hline C \end{bmatrix}$ has no unobservable eigenvalues on the unit circle.

Next, we will show the proof for Theorem 12. To this end, another important solution to the MARE (1) is introduced. For simplicity, denote

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \triangleq \begin{bmatrix} C'C & C'D \\ D'C & D'D \end{bmatrix} \quad \tilde{R} \triangleq W \odot R = R + SNR^{-1} \odot R,$$

$\mathcal{M}(X) \triangleq A'XA + Q - X$, $\mathcal{L}(X) \triangleq A'XB + S$,
$\mathcal{N}(X) \triangleq W \odot (R + B'XB)$.

The following set is defined:

$$\Gamma = \left\{ X \in \mathcal{S}^n | \begin{bmatrix} \mathcal{M}(X) & \mathcal{L}(X) \\ \mathcal{L}'(X) & \mathcal{N}(X) \end{bmatrix} \geq 0, \mathcal{N}(X) > 0 \right\}.$$

Note that the left hand side of the MARE (1) is the Schur complement of the block $\mathcal{N}(X)$ of $\begin{bmatrix} \mathcal{M}(X) & \mathcal{L}(X) \\ \mathcal{L}'(X) & \mathcal{N}(X) \end{bmatrix}$.

Definition 15. A solution $X \in \mathcal{S}_n$ to the MARE (1) is said to be a maximal solution if $X \geq \tilde{X}$ for any $\tilde{X} \in \Gamma$.

The maximal solution can be numerically computed by solving the following convex optimization problem:

$$\begin{aligned} &\max \quad \mathrm{tr}(X), \\ &\text{subject to} \quad \begin{bmatrix} \mathcal{M}(X) & \mathcal{L}(X) \\ \mathcal{L}'(X) & \mathcal{N}(X) \end{bmatrix} \geq 0, \mathcal{N}(X) > 0. \end{aligned} \quad (11)$$

The following lemma states the relationship between the maximal solution and the stochastic LQ optimal control problem.

Lemma 16. When $[A|B]$ is mean-square stabilizable with $\kappa(k)$, there exists the unique maximal solution $X_+ \in \mathcal{P}_n$ to the MARE (1) and $V(x(0)) = x'(0)X_+x(0)$. The maximal solution is a strong solution.

The relationship between the mean-square stabilizing solution and maximal solution is revealed next.

Lemma 17. When $[A|B]$ is mean-square stabilizable with $\kappa(k)$, the mean-square stabilizing solution, if it exists, coincides with the maximal solution.

Now we are prepared to prove Theorem 12.

Proof.

Sufficiency: By Lemma 16, the maximal solution X_+ exists and $\rho(\mathcal{L}_{F_+}) \leq 1$ with the associated controller F_+. Since $\lambda = 1$ is not an unobservable mean-square nonnegative eigenvalue of the stochastic system (10), by Lemma 9, $\rho(\mathcal{L}_{F_+}) < 1$. Therefore the maximal solution is exactly the mean-square stabilizing solution.

Necessity: Evidently, the mean-square stabilizabity of $[A|B]$ with $\kappa(k)$ is necessary. Suppose that the stochastic system (10) has an unobservable mean-square nonnegative eigenvalue at 1. We will show that in such a case, for any $X \in \mathcal{P}_n$ satisfying the MARE (1), it is not a mean-square stabilizing solution. Denote

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{12}' & T_{22} \end{bmatrix} \triangleq \begin{bmatrix} I & -S\tilde{R}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{M}(X) & \mathcal{L}(X) \\ \mathcal{L}'(X) & \mathcal{N}(X) \end{bmatrix} \begin{bmatrix} I & 0 \\ -\tilde{R}^{-1}S' & I \end{bmatrix},$$

with

$$T_{11} = \mathcal{L}^*_{-M^{-1}\tilde{R}^{-1}S'}(X) - X + Q - S\tilde{R}^{-1}S',$$
$$T_{12} = (A - B\tilde{R}^{-1}S')'XB - S\tilde{R}^{-1}[SNR^{-2} \odot (B'XB)],$$
$$T_{22} = \mathcal{N}(X).$$

According to Lemma 3, we have

$$\begin{aligned}
&\mathcal{M}(X) - \mathcal{L}(X)\mathcal{N}(X)^{-1}\mathcal{L}'(X) \\
&= \begin{bmatrix} I \\ -\mathcal{N}(X)^{-1}\mathcal{L}'(X) \end{bmatrix}' \begin{bmatrix} \mathcal{M}(X) & \mathcal{L}(X) \\ \mathcal{L}'(X) & \mathcal{N}(X) \end{bmatrix} \begin{bmatrix} I \\ -\mathcal{N}(X)^{-1}\mathcal{L}'(X) \end{bmatrix} \\
&= \begin{bmatrix} I \\ -\mathcal{N}(X)^{-1}\mathcal{L}'(X) \end{bmatrix}' \begin{bmatrix} I & S\tilde{R}^{-1} \\ 0 & I \end{bmatrix} \\
&\qquad \times T \begin{bmatrix} I & 0 \\ \tilde{R}^{-1}S' & I \end{bmatrix} \begin{bmatrix} I \\ -\mathcal{N}(X)^{-1}\mathcal{L}'(X) \end{bmatrix} \\
&= \begin{bmatrix} I \\ \tilde{R}^{-1}S' - \mathcal{N}(X)^{-1}\mathcal{L}'(X) \end{bmatrix}' T \begin{bmatrix} I \\ \tilde{R}^{-1}S' - \mathcal{N}(X)^{-1}\mathcal{L}'(X) \end{bmatrix} \\
&= \begin{bmatrix} I \\ -T_{22}^{-1}T_{12}' \end{bmatrix}' T \begin{bmatrix} I \\ -T_{22}^{-1}T_{12}' \end{bmatrix} \\
&= T/_{22}.
\end{aligned}$$

Since the system (10) has an unobservable mean-square nonnegative eigenvalue at 1, there exists $Y \geq 0$ such that

$$\mathcal{L}_{-M^{-1}\tilde{R}^{-1}S'}(Y) = Y,$$
$$\mathcal{C}_{-M^{-1}\tilde{R}^{-1}S'}(Y) = 0.$$

According to Lemma 2,

$$\begin{aligned}
&\mathrm{tr}(Q - S\tilde{R}^{-1}S')Y \\
&= \mathrm{tr}\boldsymbol{E}(C - D\kappa(k)M^{-1}\tilde{R}^{-1}S')'(C - D\kappa(k)M^{-1}\tilde{R}^{-1}S')Y \\
&= \mathrm{tr}\boldsymbol{E}(C - D\kappa(k)M^{-1}\tilde{R}^{-1}S')Y(C - D\kappa(k)M^{-1}\tilde{R}^{-1}S')' \\
&= \mathrm{tr}\mathcal{C}_{-M^{-1}\tilde{R}^{-1}S'}(Y) \\
&= 0,
\end{aligned}$$

then it follows that

$$\begin{aligned}
\mathrm{tr}T_{11}Y &= \mathrm{tr}[\mathcal{L}^*_{-M^{-1}\tilde{R}^{-1}S'}(X) - X]Y + \mathrm{tr}(Q - S\tilde{R}^{-1}S')Y \\
&= \mathrm{tr}X[\mathcal{L}_{-M^{-1}\tilde{R}^{-1}S'}(Y) - Y] \\
&= 0.
\end{aligned}$$

Hence we have

$$\begin{aligned}
\mathrm{tr}T_{12}T_{22}^{-1}T_{12}'Y &= -\mathrm{tr}(T_{11} - T_{12}T_{22}^{-1}T_{12}')Y = -\mathrm{tr}T/_{22}Y \\
&= -\mathrm{tr}[\mathcal{M}(X) - \mathcal{L}(X)\mathcal{N}(X)^{-1}\mathcal{L}'(X)]Y \\
&= 0,
\end{aligned}$$

which implies $T_{12}T_{22}^{-1}T_{12}'Y = 0$ by Lemma 2 due to $T_{22} > 0$ and $Y \geq 0$. It follows that $T_{12}'Y = 0$. On the other hand, denote by F the associated controller with X. Then

$$\begin{aligned}
&(\tilde{R}^{-1}S' + MF)Y \\
=&[\tilde{R}^{-1}S' - \mathcal{N}^{-1}(X)(B'XA + S')]Y \\
=&\mathcal{N}^{-1}(X)\{[W \odot (B'XB)]\tilde{R}^{-1}S' - B'XA\}Y \\
=&\mathcal{N}^{-1}(X)\{[SNR^{-2} \odot (B'XB)]\tilde{R}^{-1}S' \\
&\qquad - B'X(A - B\tilde{R}^{-1}S')\}Y \\
=& -\mathcal{N}^{-1}(X)T_{12}'Y \\
=&0,
\end{aligned}$$

which implies

$$MFY = -\tilde{R}^{-1}S'Y.$$

Therefore

$$\mathcal{L}_F(Y) = \mathcal{L}_{-M^{-1}\tilde{R}^{-1}S'}(Y) = Y,$$

i.e., $1 \in \sigma_{\mathcal{P}}(\mathcal{L}_F)$; then $\rho(\mathcal{L}_F) \geq 1$. Therefore the closed-loop system (7) is not mean-square stabilizing with the controller F. This shows the necessity. □

6. EXAMPLE

Consider an LTI system with

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Let the mean and variance of the random input gains $\kappa(k)$ be $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\Sigma^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}$, respectively. Then $SNR = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ and $W = \begin{bmatrix} 1.25 & 1 \\ 1 & 2 \end{bmatrix}$. By Theorem 3.1 in [10], $[A|B]$ is mean-square stabilizable with $\kappa(k)$ since $[A|B]$ is already in the Wonham decomposition form [12] and $1 + SNR_1 = 5 > (2 * 1)^2 = 4$ and $1 + SNR_2 = 2 > 1$.

According to Theorem 11 or Theorem 12, to determine the existence of the mean-square stabilizing solution, we need to examine the mean-square detectability of the stochastic system (10) or the mean-square observability of 1 if 1 is a mean-square nonnegative eigenvalue of the stochastic system (10). We have

$$\hat{F} \triangleq -M^{-1}[W \odot (D'D)]^{-1}D'C = \begin{bmatrix} 0 & -0.8 & 0 \\ 0 & 0 & -0.25 \end{bmatrix},$$

and the stochastic system (10) is equal to

$$\begin{aligned}
x(k+1) &= \begin{bmatrix} 2 & -0.8\kappa_1(k) & 0 \\ 0 & 1 - 0.8\kappa_1(k) & -0.25\kappa_2(k) \\ 0 & 0 & 1 - 0.25\kappa_2(k) \end{bmatrix} x(k), \\
z(k) &= \begin{bmatrix} 0 & 1 - 0.8\kappa_1(k) & 0 \\ 0 & 0 & 1 - 0.5\kappa_2(k) \end{bmatrix} x(k).
\end{aligned} \tag{12}$$

Then $\sigma(\mathcal{L}_{\hat{F}}) = \{4, 0.4, 1.5, 0.15, 0.2, 0.625\}$. A basis of the eigenspace corresponding to 4 is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, which is positive semi-definite. Hence 4 is mean-square nonnegative. A basis of the eigenspace corresponding to 0.4 with multiplicity 2 is $\begin{bmatrix} 0.4061 & 0.9138 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0.4061 & 0 & 0 \\ 0.9138 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. It is easy to tell that there does not exist a positive semi-definite eigenvector in this eigenspace. Therefore 0.4 is not mean-square nonnegative. Similarly, the remaining eigenvalues are not mean-square nonnegative by observing the corresponding eigenspace. Therefore the set of mean-square nonnegative eigenvalues of the stochastic system (12) is

$$\sigma_{\mathcal{P}}(\mathcal{L}_{\hat{F}}) = \{4\}.$$

The operator $\mathcal{C}_{\hat{F}}$ (8) is equal to

$$\mathcal{C}_{\hat{F}} : X \in \mathcal{S}_3 \mapsto \begin{bmatrix} 0 & 0 \\ 0.2 & 0 \\ 0 & 0.5 \end{bmatrix}' X \begin{bmatrix} 0 & 0 \\ 0.2 & 0 \\ 0 & 0.5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$
$$\times \left\{ \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix} \odot \left(\begin{bmatrix} 0 & 0 \\ -0.8 & 0 \\ 0 & -0.25 \end{bmatrix}' X \begin{bmatrix} 0 & 0 \\ -0.8 & 0 \\ 0 & -0.25 \end{bmatrix} \right) \right\} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \in \mathcal{S}_2.$$

Evidently, $\mathcal{C}_{\hat{F}} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = 0$, i.e., 4 is mean-square unobservable, or it is equivalent to say, the stochastic system (12) is not mean-square detectable. If we only get the sufficient condition shown in Theorem 11, we cannot conclude whether the mean-square stabilizing solution exists. Fortunately, we have an explicit necessary and sufficient condition shown in Theorem 12, which tells us that the mean-square stabilizing solution does exist since the system parameters satisfy the necessary and sufficient condition. By solving the convex optimization problem (11), we get the mean-square stabilizing solution

$$X = \begin{bmatrix} 75.9593 & -13.0253 & 7.6301 \\ -13.0253 & 5.2438 & -2.7106 \\ 7.6301 & -2.7106 & 3.6209 \end{bmatrix}$$

with the associated state feedback gain

$$F = \begin{bmatrix} -1.7776 & 0.0904 & -0.0787 \\ 0.3831 & -0.1528 & -0.2106 \end{bmatrix}.$$

Since $\rho(\mathcal{L}_F) < 1$, F is indeed mean-square stabilizing.

7. CONCLUSION

An explicit necessary and sufficient condition ensuring the existence of the mean-square stabilizing solution to the MARE (1) is investigated. It is exactly compatible with that for the standard definite ARE (9). The next step is to apply the approach given in this paper to obtain the mean-square stabilizing solution to some general MAREs.

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