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Distributed economic MPC: a framework for cooperative control problems ⋆

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Abstract: In this paper, we propose a distributed economic MPC algorithm for cooperative control of several self-interested interacting dynamical systems. Each system considers its own, local, performance criterion, and coordination between the systems is enforced via coupling constraints. The proposed control strategy consists of a distributed optimization algorithm, used to determine an overall optimal steady-state, whose current iterates at each time are then used by each system to compute a control input in an economic MPC framework. We analyze the properties of the proposed algorithm and prove convergence results for the resulting overall closed-loop system. Furthermore, we apply our results to the problem of synchonizing several agents with conflicting objective.

Keywords: Economic MPC, Distributed MPC, Cooperative Control

1. INTRODUCTION

In recent years, the control of networks of interacting dynamical systems has received significant attention and has gained importance in many application areas, such as the coordination of multiple robots or distributed power generation. For such networks of systems, model predictive control (MPC) is an appealing control technique due to its ability to explicitly handle state and input constraints and to incorporate some performance criterion. However, a centralized implementation of MPC is often not possible due to the large-scale nature of the problem and the limited amount of available information from other systems. To overcome these issues, many distributed MPC algorithms have been developed in the literature, an overview of which can, e.g., be found in Scattolini [2009] and Christofides et al. [2013]. For the above mentioned applications, a setting of particular interest is that where the system dynamics of the systems in the network are decoupled, but couplings between the systems are given via constraints and a common objective. For such a setting, stabilizing distributed MPC schemes have, e.g., been obtained by Dunbar and Murray [2006] with additional consistency constraints, by Richards and How [2007] and Grüne and Worthmann [2012] through the use of a certain sequential optimization algorithm, and by Spudić and Baotić [2013] via explicit MPC techniques; furthermore, distributed MPC algorithms for more general cooperative control problems than setpoint stabilization, such as consensus and synchronization, have, e.g., been treated by Keviczky and Johansson [2008] and Müller et al. [2012].

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All of the distributed MPC algorithms mentioned above have been formulated in the context of tracking MPC, meaning that the cost function used within the repeatedly solved optimization problem is assumed to be positive definite with respect to the specific setpoint or set to be stabilized. On the other hand, a more general MPC framework termed economic MPC [Angeli et al., 2012] was recently proposed, where this assumption is not needed, but an arbitrary cost function can be used, possibly resembling the economics related to the considered system. For such a framework, different properties such as average performance of the closed-loop system, (sub-)optimality of steady-state operation, convergence of the closed-loop system and fulfillment of average constraints are of interest and have recently been studied (see, e.g. [Angeli et al., 2012, Amrit et al., 2011, Grüne, 2013, Müller et al., 2014b]). Furthermore, a first result for distributed economic MPC was obtained by [Driessen et al., 2012, Lee and Angeli, 2011], where the overall optimal steady-state was assumed to be known and used as a terminal constraint.

In this paper, we propose a distributed economic MPC framework, which we believe to be well suited for cooperative control of several self-interested interacting systems. Namely, each system uses its own, local, objective function, which models its self-interest and need not be related to any specific setpoint as discussed above; coordination between the systems is then enforced by means of coupling constraints. In contrast to [Lee and Angeli, 2011, Driessen et al., 2012], our basic assumption is that the overall optimal steady-state (including the coupling constraints) is *not* known a priori, but has to be negotiated between the systems online by implementing some distributed optimization algorithm. We assume that communication between the systems requires time, i.e., the systems already have to perform control actions while still negotiating with their neighboring systems. This premise was also adopted in a similar context by Zelazo et al. [2013], where a shrinking-horizon preference agreement algorithm was developed for scalar single integrator systems with quadratic objectives. The remainder of this paper is structured as follows. In Section 2, we state the detalied problem setup and present the proposed distributed economic MPC algorithm. As indicated above, the proposed control structure will be hierarchical in the sense that it consists of a distributed optimization algorithm to determine the overall optimal steady-state, whose current iterates at each time step are then used to determine a suitable terminal constraint for the economic MPC problem solved by each system. In Section 3, we analyze the proposed algorithm and show that it has the desired properties. Section 4 illustrates the obtained results by considering the problem of synchonizing several agents with conflicting objective, before we give some concluding remarks in Section 5.

1.1 Notation

For a set $\mathcal{A} \subseteq \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, the distance of x from the set $\mathcal{A}$ is defined as $|x|_{\mathcal{A}} := \inf_{z\in\mathcal{A}} |x - z|$. Let $\mathbb{I}_{\geq 0}$ denote the set of nonnegative integers, and $\mathbb{I}_{[a,b]}$ the set of all integers in the interval $[a, b] \subseteq \mathbb{R}$. The unit ball in $\mathbb{R}^n$ is denoted by B_1, i.e., $B_1 := \{x \in \mathbb{R}^n : |x| \leq 1\}$. We say that a bounded sequence $v : \mathbb{I}_{\geq 0} \to \mathbb{R}^{n_v}$ is *essentially converging* to $\bar{v} \in \mathbb{R}^{n_v}$ if the following is true:

$$\forall\, \varepsilon > 0 : \quad \limsup_{T\to+\infty} \frac{\mathrm{card}(\{0 \leq t \leq T : |v(t) - \bar{v}| \geq \varepsilon\})}{T + 1} = 0.$$

Furthermore, as in [Angeli et al., 2012], the set of asymptotic averages of v is defined as

$$Av[v] := \{\bar{v} \in \mathbb{R}^{n_v} :\ \exists t_n \to +\infty : \lim_{n\to\infty} \frac{\sum_{k=0}^{t_n} v(k)}{t_n + 1} = \bar{v}\}.$$

Note that $Av[v]$ is nonempty (as bounded sequences in $\mathbb{R}^{n_v}$ have limit points), but it need not be a singleton in general.

2. DISTRIBUTED ECONOMIC MPC FOR SELF-INTERESTED AGENTS

We consider the problem of a network of n self-interested dynamical systems which have to fulfill some cooperative requirement imposed through coupling constraints. Each agent is modeled as a discrete-time linear system of the form

$$x_i(t + 1) = A_i x_i(t) + B_i u_i(t), \qquad x_i(0) = x_{i0}, \tag{1}$$

with $x_i(t) \in \mathbb{X}_i \subseteq \mathbb{R}^{n_i}$ and $u_i(t) \in \mathbb{U}_i \subseteq \mathbb{R}^{m_i}$ for all $t \in \mathbb{I}_{\geq 0}$, and the pair (A_i, B_i) is assumed to be stabilizable. Each of the systems is subject to local state and input constraints given by $(x_i(t), u_i(t)) \in \mathbb{Z}_i \subseteq \mathbb{X}_i \times \mathbb{U}_i$ for some convex and compact set $\mathbb{Z}_i$. The set of all steady-states for system i is defined as $\mathcal{S}_i := \{(x_i, u_i) \in \mathbb{X}_i \times \mathbb{U}_i : x_i = A_i x_i + B_i u_i\}$. The overall state and control vector for all n agents is given by $x(t) = [x_1(t)^T, \dots, x_n(t)^T]^T \in \mathbb{R}^p$ (with $p = \sum_{i=1}^n n_i$) and $u(t) = [u_1(t)^T, \dots, u_n(t)^T]^T \in \mathbb{R}^q$ (with $q = \sum_{i=1}^n m_i$), respectively, and the overall system dynamic is given by $x(t + 1) = Ax(t) + Bu(t)$, where $A := diag(A_1, \dots, A_n)$ and $B := diag(B_1, \dots, B_n)$. Furthermore, let $\mathbb{Z} := \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_n$ and $\mathcal{S} := \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$. Each system is equipped with a strictly convex and continuous objective function $\ell_i : \mathbb{Z}_i \to \mathbb{R}$, which models the self-interest of each agent. The cooperative requirement which the systems have to fulfill is given by coupling constraints of the form

$$x \in \mathcal{C} \tag{2}$$

for some convex set $\mathcal{C} \subseteq \mathbb{R}^p$. In this paper, we consider the case where these coupling constraints only have to be satisfied asymptotically, i.e., we require that $\lim_{t\to\infty} |x(t)|_{\mathcal{C}} = 0$. Note that as discussed in the Introduction, various application-related contexts fit into this framework, such as the synchonization of several agents with conflicting objective (see also Section 4).

Remark 1. For clarity of presentation, in this paper we consider coupling constraints (2) involving only the system states x_i; nevertheless, in a similar way, also coupling constraints involving both the system states x_i and the system inputs u_i can be treated. □

Each system computes its control input in an economic MPC fashion, i.e., at each time instant t, the following optimization problem is solved by each system i, where N denotes the prediction horizon:

$$\min_{\mathbf{u}_i} \sum_{k=0}^{N-1} \ell_i(x_i(k|t), u_i(k|t)) + V_i^f(x_i(N|t), x_i^s(t)) \tag{3}$$

subject to

$$x_i(0|t) = x_i(t) \tag{4a}$$

$$x_i(k + 1|t) = A_i x_i(k|t) + B_i u_i(k|t) \quad k \in \mathbb{I}_{[0,N-1]} \tag{4b}$$

$$(x_i(k|t), u_i(k|t)) \in \mathbb{Z}_i, \quad k \in \mathbb{I}_{[0,N-1]} \tag{4c}$$

$$x_i(N|t) \in \mathbb{X}_i^f(x_i^s(t), t) \tag{4d}$$

$$\sum_{k=0}^{N-1} h_i(x_i(k|t), u_i(k|t), t) \in \mathbb{Y}_i(t) \tag{4e}$$

Denote the optimal solution to problem (3)–(4) by $\mathbf{u}_i^0(t) := [u_i^0(0|t)^T, \dots, u_i^0(N - 1|t)^T]^T$ and the corresponding state sequence by $\mathbf{x}_i^0(t) := [x_i^0(0|t)^T, \dots, x_i^0(N|t)^T]^T$. A special feature of problem (3)–(4) is the terminal constraint (4d). Namely, both the terminal region $\mathbb{X}_i^f(x_i^s(t), t)$ as well as the steady-state $x_i^s(t)$ around which it is built are time-varying, which will be further specified in the following. Finally, constraint (4e) and its meaning will be described in more detail later; it will be used to ensure satisfaction of a certain average constraint, which in turn guarantees asymptotic fulfillment of the coupling constraints (2).

Now let (x^*, u^*) denote the overall optimal steady-state (including coupling constraints), defined as

$$(x^*, u^*) = \underset{(x,u)\in\mathbb{Z}\cap\mathcal{S}, x\in\mathcal{C}}{\arg\min} \sum_{i=1}^n \ell_i(x_i, u_i). \tag{5}$$

Note that due to strict convexity of the functions ℓ_i and convexity of the constraints, (x^*, u^*) is unique. Our basic prerequisite is that (x^*, u^*) is not known a priori, but has to be calculated online via a distributed optimization algorithm, which requires communication between the systems. We assume that this communication requires time, i.e., an update step in the distributed optimization algorithm is not instantaneous; in particular, we assume that each iteration of the distributed optimization algorithm solving (5) corresponds to one actual time step t in the evolution of the systems (1). This means that the systems "negotiate" about the overall optimal steady-state while already taking control actions (calculated via (3)–(4)). Let $\zeta(t) := (\xi(t), \eta(t))$ denote the iterate at time t of the distributed optimization algorithm solving (5), where $\xi(t) := [\xi_1(t)^T, \dots, \xi_n(t)^T]^T$ and $\eta(t) := [\eta_1(t)^T, \dots, \eta_n(t)^T]^T$ are the state and input components of $\zeta(t)$, respectively. We then impose the following assumption.

Assumption 1. The distributed optimization algorithm solving (5) is such that $\lim_{t\to\infty} \zeta(t) = (x^*, u^*)$. □

Remark 2. Thanks to convexity of problem (5), many distributed optimization algorithms exist satisfying Assumption 1. In settings where the coupling constraint set $\mathcal{C}$ consists of several coupling constraints each of which involves only a limited number of systems, dual subgradient methods [Ruszczyński, 2006] are, e.g., well suited; for problems where coupling constraints involve all systems, one can, for example, use the recently proposed cutting-plane consensus algorithm [Bürger et al., 2014]. □

Remark 3. When defining the overall optimal steady-state (x^*, u^*) in (5), one could also use different weighting factors $a_i > 0$ for each cost function ℓ_i, which would correspond to a certain priorization of the systems. □

Given the above, a first idea would be that each system uses its current iterate $\xi_i(t)$ of the distributed optimization algorithm as the steady-state $x_i^s(t)$ around which the terminal region in (4d) is built (or a projection of $\xi_i(t)$ on the feasible steady-state set in case that the current iterate $\xi_i(t)$ does not satisfy the local input and state constraints). However, this choice might not be feasible due to the following reasons. Namely, if $|\xi_i(t) - \xi_i(t-1)|$ is large (which can happen initially), recursive feasibility of problem (3)–(4) might be lost. Furthermore, $x_i^s(t)$ has to be chosen such that all states in the terminal region around $x_i^s(t)$ satisfy local input and state constraints (4c), which means that (for a given size of the terminal region) steady-states close to the boundary of $\mathbb{Z}_i$ cannot be used. Instead of using $x_i^s(t) := \xi_i(t)$ in (4d), in the following we propose a way to gradually change $x_i^s(t)$ such that recursive feasibility of problem (3)–(4) can be maintained and we have $\lim_{t\to\infty} x_i^s(t) = \lim_{t\to\infty} \xi_i(t) = x_i^*$. To this end, for each system $i \in \mathbb{I}_{[1,n]}$, let $P_i, Q_i > 0$, and define terminal regions of the form

$$\mathbb{X}_i^f(x_i^s, t) := \{x_i \in \mathbb{R}^{n_i} : E_i(x_i, x_i^s) \le \alpha_i(t)\} \tag{6}$$

with $E_i(x_i, x_i^s) := (x_i - x_i^s)^T P_i (x_i - x_i^s)$ and $\alpha_i(t) > 0$ for all $t \in \mathbb{I}_{\ge 0}$. Let $\overline{\mathbb{Z}}_i(t) := \mathbb{Z}_i \ominus (\mathbb{X}_i^f(0,t) \times K_i \mathbb{X}_i^f(0,t))$. We then impose the following assumption on the terminal regions $\mathbb{X}_i^f$ and the terminal cost V_i^f.

Assumption 2. The terminal regions $\mathbb{X}_i^f$, an auxiliary terminal control gain K_i and the terminal cost function $V_i^f(x_i, x_i^s)$ are computed such that the following is satisfied for each steady-state $(x_i^s, u_i^s) \in \mathcal{S}_i \cap \overline{\mathbb{Z}}_i(t)$ and all $x_i \in \mathbb{X}_i^f(x_i^s, t)$:

(i) $(x_i, K_i(x_i - x_i^s) + u_i^s) \in \mathbb{Z}_i$,

(ii) $E_i(A_i x_i + B_i(K_i(x_i - x_i^s) + u_i^s), x_i^s) - E_i(x, x_i^s) \le -(x_i - x_i^s)^T Q_i (x_i - x_i^s)$,

(iii) $V_i^f(A_i x_i + B_i(K_i(x_i - x_i^s) + u_i^s), x_i^s) - V_i^f(x_i, x_i^s) \le -\ell_i(x_i, K_i(x_i - x_i^s) + u_i^s) + \ell(x_i^s, u_i^s)$.

Remark 4. For a fixed (x_i^s, u_i^s), conditions (i)–(iii) of Assumption 2 are standard conditions imposed when using a terminal cost/region framework, both in the case of tracking and economic MPC [Rawlings and Mayne, 2009, Amrit et al., 2011]. In (ii), we actually require something slightly stronger than invariance of the terminal region, namely that it is contractive if the local controller is applied; this is crucial for our main results later on. Note that it is sufficient if this holds for some arbitrary positive definite Q_i. Note that condition (i) is satisfied due to the definition of the set $\overline{\mathbb{Z}}_i(t)$ and the fact that $\mathbb{X}_i^f(x_i^s, t) = \{x_i^s\} \oplus \mathbb{X}_i^f(0, t)$. Furthermore, in [Amrit et al., 2011, Section 4.1], a method was presented how P_i, K_i and V_i^f can be computed such that conditions (ii) and (iii) are satisfied for fixed (x_i^s, u_i^s). It is straightforward to show that this procedure can still be used in order to satisfy conditions (ii) and (iii) for all $(x_i^s, u_i^s) \in \mathcal{S}_i \cap \overline{\mathbb{Z}}_i(t)$. □

Now define $c_i := (1 - \frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P_i)})/\lambda_{\min}(P_i)$, fix $0 < \theta_i < 1$ and let for all $t \in \mathbb{I}_{\ge 0}$

$$\varepsilon_i(t) := \Big(-\sqrt{c_i} + \sqrt{c_i + \theta_i \frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P_i)^2}} \Big) \sqrt{\alpha_i(t)}. \tag{7}$$

Furthermore, denote by $(\hat{\xi}_i(t), \hat{\eta}_i(t))$ the projection of $(\xi_i(t), \eta_i(t))$ on the set $\mathcal{S}_i \cap \overline{\mathbb{Z}}_i(t)$. We now propose to use the following steady-state $x_i^s(t)$ within the terminal constraint (4d):

$$x_i^s(t) := (1 - \lambda_i(t)) x_i^s(t-1) + \lambda_i(t) \hat{\xi}_i(t), \tag{8}$$

$$\lambda_i(t) := \min \left\{ \frac{\varepsilon_i(t-1)}{|\hat{\xi}_i(t) - x_i^s(t-1)|}, 1 \right\} \tag{9}$$

for all $t \in \mathbb{I}_{\ge 1}$ and $x_i^s(0) = x_{i0}^s$, where x_{i0}^s is an arbitrary steady-state satisfying $(x_{i0}^s, u_{i0}^s) \in \mathcal{S}_i \cap \overline{\mathbb{Z}}_i(0)$. Due to convexity of $\mathcal{S}_i \cap \overline{\mathbb{Z}}_i(t)$ and the fact that $\overline{\mathbb{Z}}_i(t+1) \supseteq \overline{\mathbb{Z}}_i(t)$ for all $t \in \mathbb{I}_{\ge 0}$ (the latter will be established below), one can show by induction that $(x_i^s(t), u_i^s(t)) \in \mathcal{S}_i \cap \overline{\mathbb{Z}}_i(t)$ for all $t \in \mathbb{I}_{\ge 0}$, where $u_i^s(t)$ is given by $u_i^s(t) := (1 - \lambda_i(t)) u_i^s(t-1) + \lambda_i(t) \hat{\eta}_i(t)$. Furthermore, note that from (8)–(9) it follows that for all $t \in \mathbb{I}_{\ge 1}$ we have $|x_i^s(t) - x_i^s(t-1)| \le \varepsilon_i(t-1)$.

It remains to specify how $\alpha_i(t)$ in (6), i.e., the size of the terminal region, is updated. Namely, we propose to use the following update rule:

$$\alpha_i(t+1) = \begin{cases} \Big(1 - (1 - \theta_i) \frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P_i)}\Big) \alpha_i(t) & \text{if } \lambda_i(t) = 1 \\ \alpha_i(t) & \text{else} \end{cases} \tag{10}$$

for all $t \in \mathbb{I}_{\ge 1}$ and $\alpha_i(1) = \alpha_i(0) = \alpha_{i0} > 0$. Note that from (10), it follows that α_i is nonincreasing, as $0 < \theta_i < 1$ and $0 < \lambda_{\min}(Q_i)/\lambda_{\max}(P_i) \le 1$ (the latter inequality follows from Assumption 2(ii)). This means that the size of the terminal regions is nonincreasing, i.e., for each x_i^s we have $\mathbb{X}_i^f(x_i^s, t+1) \subseteq \mathbb{X}_i^f(x_i^s, t)$ and hence also $\overline{\mathbb{Z}}_i(t+1) \supseteq \overline{\mathbb{Z}}_i(t)$ for all $t \in \mathbb{I}_{\ge 0}$.

Finally, we need to specify the function h_i and the sets $\mathbb{Y}_i(t)$ appearing in (4e). Namely, we use

$$h_i(x_i, u_i, t) := |x_i - x_i^s(t)|, \tag{11}$$

and $\mathbb{Y}_i(t)$ is recursively defined as

$$\mathbb{Y}_i(0) := N\mathbb{Y}_i \oplus \mathbb{Y}_{i0}, \tag{12}$$

$$\mathbb{Y}_i(t+1) := \mathbb{Y}_i(t) \oplus \mathbb{Y}_i \oplus \overline{\mathbb{Y}}_i(t+1) \oplus \{-h_i(x_i(t), u_i(t), t)\}, \tag{13}$$

with $\mathbb{Y}_i := \mathbb{R}_{\leq 0}$, $\mathbb{Y}_{i0} \subseteq \mathbb{R}$ being some arbitrary convex and compact set such that (4e) is initially feasible, and

$$\overline{\mathbb{Y}}_i(t) := \overline{y}_i(t)B_1,$$
$$\overline{y}_i(t) := N|x_i^s(t) - x_i^s(t-1)| + |x_i^0(N|t-1) - x_i^s(t-1)| \tag{14}$$

for all $t \in \mathbb{I}_{\geq 1}$. Constraint (4e) ensures that the asymptotic average constraint $Av[h_i] \in \mathbb{Y}_i$ is satisfied and will be needed to ensure that the overall closed-loop system converges to x^*. For more details on economic MPC with average constraints, the interested reader is referred to [Angeli et al., 2012, Section V.B] and [Müller et al., 2014b].

To summarize, the proposed distributed economic MPC algorithm is as follwos.

Algorithm 1. (Distributed economic model predictive control for self-interested agents)
At each time $t \in \mathbb{I}_{\geq 0}$, all systems $i \in \mathbb{I}_{[1,n]}$

1) *communicate with neighboring systems,*
2) *perform an iterate of the distributed optimization algorithm solving* (5), *obtaining $\xi_i(t)$,*
3) *solve problem* (3)–(4), *where the terminal region $\mathbb{X}_i^f$ in* (4d) *is given by* (6)–(10) *and h_i and $\mathbb{Y}_i(t)$ in* (4e) *are given by* (11)–(14),
4) *apply $u_i(t) := u_i^0(0|t)$.* □

In Step 1, the set of neighboring systems with which system i has to communicate depends on the structure of the coupling constraint set $\mathcal{C}$ and the specific distributed optimization algorithm which is used in Step 2. The latter also determines what information has to be transmitted, such as, e.g., the latest estimate $\zeta_i(t-1)$ or certain dual variables. Within Algorithm 1, two optimization problems have to be solved by each system in each time step, one in Step 2 (when performing an iterate of the distributed optimization algorithm) and one in Step 3 when solving the economic MPC problem (3)–(4); both Steps 2 and 3 can be performed by all systems in parallel. Furthermore, note that the optimization problems solved by each system in Step 3 are completely decoupled from each other. Coordination between the systems is achieved via the distributed optimization algorithm (Steps 1 and 2), and information from other systems is injected in the local economic MPC problem (3)–(4) via the steady-state $x_i^s(t)$ appearing in the constraints (4d) and (4e).

3. ANALYSIS OF ALGORITHM 1

In the following, we analyze Algorithm 1 and show that its properties are as desired.

Theorem 1. Suppose that Assumptions 1 and 2 are satisfied, and the optimization problem (3)–(4) is initially feasible for all systems $i \in \mathbb{I}_{[1,n]}$. Then the following is satisfied when applying Algorithm 1.

(i) The optimization problem (3)–(4) is recursively feasible for all systems $i \in \mathbb{I}_{[1,n]}$.
(ii) $\lim_{t\to\infty} x_i^s(t) = \lim_{t\to\infty} \xi_i(t) = x_i^*$ for all $i \in \mathbb{I}_{[1,n]}$.
(iii) The overall closed-loop system essentially converges to x^*.

Due to the fact that $x^* \in \mathcal{C}$, we have the following corollary of Theorem 1.

Corollary 1. Suppose the conditions of Theorem 1 are satisfied. Then the closed-loop system essentially converges to the set $\mathcal{C}$.

Remark 5. In Theorem 1 and Corollary 1, only essential convergence of the overall closed-loop system to x^* and $\mathcal{C}$, respectively, could be established, which is a slightly weaker notion than asymptotic convergence. Under some additional conditions on the convergence rate of $\overline{y}_i$ in (14) and hence on the convergence rate of the specific distributed optimization algorithm used in Step 2 of Algorithm 1, also asymptotic convergence of the overall closed-loop system to x^* and $\mathcal{C}$, respectively, can be established (see [Müller et al., 2014b] and [Angeli et al., 2011] for more details on essential versus asymptotic convergence in economic MPC with average constraints). □

Remark 6. While the overall closed-loop system (essentially) converges to x^* as shown in Theorem 1, the transient performance of each system can be much better than $\ell_i(x_i^*, u_i^*)$. Namely, one can typically observe (compare Section 4) that the systems initially "spend time" in a region where the cost ℓ_i is lower than $\ell_i(x_i^*, u_i^*)$, before they converge to (x_i^*, u_i^*) in order to satisfy the cooperative requirement (2). This behaviour depends on various parameteres such as the initial condition, the prediction horizon and the size of the set $\mathbb{Y}_{i0}$ (see [Müller et al., 2014b] for a more detailed discussion on this issue). □

In the following, we establish two auxiliary results which are needed in order to prove Theorem 1.

Proposition 1. Consider an economic MPC algorithm for a (single) system i, where the repeatedly solved optimization problem is given by (3)–(4), where h_i in (4e) is bounded on $\mathbb{Z}_i \times \mathbb{I}_{\geq 0}$. Suppose the terminal regions $\mathbb{X}_i^f(x_i^s(t), t)$ in (4d) are defined such that for all $t \in \mathbb{I}_{\geq 0}$ and all $x_i \in \mathbb{X}_i^f(x_i^s(t), t)$, Assumption 2(i) is satisfied and $A_i x_i + B_i(K_i(x_i - x_i^s(t)) + u_i^s(t)) \in \mathbb{X}_i^f(x_i^s(t+1), t+1)$. Furthermore, let $\rho(t)$ be any sequence such that h_i in (4e) satisfies $h(x_i, u_i, t+1) - h(x_i, u_i, t) \in \rho(t)B_1$ for all $t \in \mathbb{I}_{\geq 0}$ and all $(x_i, u_i) \in \mathbb{Z}_i$, and suppose that the set $\mathbb{Y}_i(t)$ in (4e) is defined via (12)–(13) for some convex set $\mathbb{Y}_i$, some compact set $\mathbb{Y}_{i0}$, and $\overline{\mathbb{Y}}_i(t+1)$ such that $\{h_i(x_i^0(N|t), K_i(x_i^0(N|t) - x_i^s(t)) + u_i^s(t), t)\} \oplus N\rho(t)B_1 \subseteq \mathbb{Y}_i \oplus \overline{\mathbb{Y}}_i(t+1)$ for all $t \in \mathbb{I}_{\geq 0}$. We then have the following.

(i) If problem (3)–(4) is initially feasible, then it is recursively feasible.
(ii) If there exists a sequence $\sigma : \mathbb{I}_{\geq 0} \to \mathbb{R}$ with $\lim_{t\to\infty} \sigma(t) = 0$ such that $\overline{\mathbb{Y}}_i(t) \subseteq \sigma(t)B_1$ for all $t \in \mathbb{I}_{\geq 0}$, then the average constraint $Av[h_i] \subseteq \mathbb{Y}_i$ is satisfied for the resulting closed-loop system.
(iii) Suppose that $h_i(x_i, u_i, t) := \hat{h}_i(x_i, u_i) + \varphi(x_i, u_i, t)$ with $|\varphi(x_i, u_i, t)| \leq \hat{\varphi}(t)$ for all $(x_i, u_i) \in \mathbb{Z}_i$ and all $t \in \mathbb{I}_{\geq 0}$, and some $\hat{\varphi} : \mathbb{I}_{\geq 0} \to \mathbb{R}$. If there exists a sequence $\hat{\sigma} : \mathbb{I}_{\geq 0} \to \mathbb{R}$ with $\lim_{t\to\infty} \hat{\sigma}(t) = 0$ such that $\overline{\mathbb{Y}}_i(t+1) \oplus \hat{\varphi}(t)B_1 \subseteq \hat{\sigma}(t)B_1$ for all $t \in \mathbb{I}_{\geq 0}$, then the average constraint $Av[\hat{h}_i] \subseteq \mathbb{Y}_i$ is satisfied for the resulting closed-loop system.

Proposition 1 is an extension of [Müller et al., 2014b, Theorem 1] to the case of time-varying output functions h_i, and its proof is omitted in this conference paper due to space limitations. The next auxiliary result will be needed later in order to show that the terminal regions $\mathbb{X}_i^f$ defined via (6)–(10) change "slow enough" such that recursive feasibility of problem (3)–(4) can be established.

Lemma 1. Consider the terminal regions $\mathbb{X}_i^f(x_i^s(t), t)$ given by (6)–(10) and suppose that Assumption 2(ii) is satisified. Then for each $t \in \mathbb{I}_{\geq 0}$ and each $x_i \in \mathbb{X}_i^f(x_i^s(t), t)$, it holds that $x_i^+ := A_i x_i + B_i(K_i(x_i - x_i^s(t)) + u_i^s(t)) \in \mathbb{X}_i^f(x_i^s(t+1), t+1)$.

Lemma 1 is a slight extension of Lemma 1 in [Müller et al., 2014a], and its proof is omitted in this conference paper due to space limitations.

Proof of Theorem 1: To prove statement (i) of Theorem 1, we show that all conditions for Proposition 1(i) are satisfied and hence the conclusion follows. Namely, h_i defined by (11) is bounded on $\mathbb{Z}_i \times \mathbb{I}_{\geq 0}$ as $\mathbb{Z}_i$ is compact. By Lemma 1, we conclude that for the terminal regions given by (6)–(10), we have that $A_i x_i + B_i(K_i(x_i - x_i^s(t)) + u_i^s(t)) \in \mathbb{X}_i^f(x_i^s(t+1), t+1)$ for each $t \in \mathbb{I}_{\geq 0}$ and each $x_i \in \mathbb{X}_i^f(x_i^s(t), t)$. Furthermore, for each $t \in \mathbb{I}_{\geq 0}$ and each $(x_i, u_i) \in \mathbb{Z}_i$, one obtains

$$\begin{aligned} h_i(x_i, u_i, t+1) &= |x_i - x_i^s(t+1)| \\ &= |x_i - x_i^s(t) + x_i^s(t) - x_i^s(t+1)| \\ &\leq |x_i - x_i^s(t)| + |x_i^s(t) - x_i^s(t+1)| \\ &=: h_i(x_i, u_i, t) + \rho(t). \end{aligned} \tag{15}$$

Note that (15) implies that $h_i(x_i, u_i, t+1) - h_i(x_i, u_i, t) \in \rho(t)B_1$ for each $t \in \mathbb{I}_{\geq 0}$ and each $(x_i, u_i) \in \mathbb{Z}_i$. Moreover, from the definition of $\overline{\mathbb{Y}}_i(t)$ in (14), we obtain $\{h_i(x_i^0(N|t), K_i(x_i^0(N|t) - x_i^s(t)) + u_i^s(t), t)\} \oplus N\rho(t)B_1 \subseteq \overline{\mathbb{Y}}_i(t+1) \subseteq \mathbb{Y}_i \oplus \overline{\mathbb{Y}}_i(t+1)$ for all $t \in \mathbb{I}_{\geq 0}$. To summarize, all of the conditions for Proposition 1(i) are satisfied for all systems $i \in \mathbb{I}_{[1,n]}$ and hence we conclude that the optimization problem (3)–(4) is recursively feasible for all systems $i \in \mathbb{I}_{[1,n]}$.

Next, consider statement (ii) of Theorem 1. Let $i \in \mathbb{I}_{[1,n]}$. According to (10), the sequence $\alpha_i(t)$ is nonincreasing and bounded from below (by zero), hence it converges. Denote its limit by $\alpha_i^{\min} \geq 0$. Then from (7) it follows that also $\varepsilon_i(t)$ converges to $\varepsilon_i^{\min}$, where $\varepsilon_i^{\min}$ is given by (7) with $\alpha_i(t)$ replaced by $\alpha_i^{\min}$. We now show by contradiction that $\alpha_i^{\min} = 0$. Namely, assume it was not, i.e., $\alpha_i^{\min} > 0$ and hence also $\varepsilon_i^{\min} > 0$. By (10), $\alpha_i^{\min} > 0$ is only possible if there exists a finite time $t(\alpha_i^{\min}) \in \mathbb{I}_{\geq 0}$ such that $\lambda_i(t) \neq 1$ for all $t \in \mathbb{I}_{\geq t(\alpha_i^{\min})}$, which means that $\alpha_i(t) = \alpha_i^{\min}$ for all such t. By (9), this implies that $|\hat{\xi}_i(t) - x_i^s(t-1)| > \varepsilon_i(t-1) = \varepsilon_i^{\min}$ for all $t \in \mathbb{I}_{\geq t(\alpha_i^{\min})+1}$ and, as $\varepsilon_i(t)$ is nonincreasing, $\lambda_i(t) \geq \lambda_i^{\min} := \varepsilon_i^{\min}/(\max_{r,s \in \mathbb{Z}_i} |r - s|)$ for all $t \in \mathbb{I}_{\geq 1}$. Furthermore, $\mathbb{Z}_i(t) = \mathbb{Z}_i(t(\alpha_i^{\min}))$ for all $t \in \mathbb{I}_{\geq t(\alpha_i^{\min})}$. By Assumption 1, $\xi_i(t)$ converges to x_i^*, and hence, thanks to convexity of $\mathcal{S}_i \cap \mathbb{Z}_i(t(\alpha_i^{\min}))$, also $\hat{\xi}_i(t)$ converges to some $\hat{\xi}_i^\alpha$, where $\hat{\xi}_i^\alpha$ is the projection of x_i^* on the set $\mathcal{S}_i \cap \mathbb{Z}_i(t(\alpha_i^{\min}))$. This means that $\lim_{t\to\infty} \psi_i(t) = 0$, where $\psi_i(t) := \hat{\xi}_i(t) - \hat{\xi}_i^\alpha$. Now consider the sequence $\tilde{x}_i(t) := x_i^s(t) - \hat{\xi}_i^\alpha$. By subtracting $\hat{\xi}_i^\alpha$ on both sides of equation (8), we obtain

$$\tilde{x}_i(t) = (1 - \lambda_i(t))\tilde{x}_i(t-1) + \lambda_i(t)\psi_i(t),$$

for all $t \in \mathbb{I}_{\geq 1}$ and hence also

$$|\tilde{x}_i(t)| \leq (1 - \lambda_i^{\min})|\tilde{x}_i(t-1)| + |\psi_i(t)|, \tag{16}$$

as $\lambda_i^{\min} \leq \lambda_i(t) \leq 1$ for all $t \in \mathbb{I}_{\geq 1}$ as established above. As $\lim_{t\to\infty} \psi_i(t) = 0$, for each $\delta_\psi > 0$ there exists a $t_\psi \in \mathbb{I}_{\geq 0}$ such that $|\psi_i(t)| \leq \delta_\psi$ for all $t \in \mathbb{I}_{\geq t_\psi}$. Fix $\delta_\psi > 0$ such that $\delta_\psi + 2\delta_\psi/\lambda_i^{\min} \leq \varepsilon_i^{\min}$. From (16), it then follows that there exists a finite time $t_{\tilde{x}} \in \mathbb{I}_{\geq t_\psi}$ such that $|\tilde{x}_i(t)| \leq 2\delta_\psi/\lambda_i^{\min}$ for all $t \in \mathbb{I}_{\geq t_{\tilde{x}}}$. Namely, for all $t \in \mathbb{I}_{\geq t_\psi}$, if $|\tilde{x}_i(t)| > 2\delta_\psi/\lambda_i^{\min}$ it follows from (16) that $|\tilde{x}_i(t+1)| - |\tilde{x}_i(t)| \leq -\delta_\psi$, and hence $|\tilde{x}_i(t_{\tilde{x}})| \leq 2\delta_\psi/\lambda_i^{\min}$ for some finite time $t_{\tilde{x}} \in \mathbb{I}_{\geq t_\psi}$. But then, from (16) it follows that also $|\tilde{x}_i(t)| \leq 2\delta_\psi/\lambda_i^{\min}$ for all $t \in \mathbb{I}_{\geq t_{\tilde{x}}}$. Summarizing the above, there exists a time $t' \in \mathbb{I}_{\geq t(\alpha_i^{\min})}$ such that

$$\begin{aligned} |\hat{\xi}_i(t') - x_i^s(t'-1)| &= |\psi_i(t') - \tilde{x}_i(t'-1)| \\ &\leq |\psi_i(t')| + |\tilde{x}_i(t'-1)| \leq \delta_\psi + 2\delta_\psi/\lambda_i^{\min} \leq \varepsilon_i^{\min}, \end{aligned}$$

which is a contradiction to the fact established above that $|\hat{\xi}_i(t) - x_i^s(t-1)| > \varepsilon_i^{\min}$ for all $t \in \mathbb{I}_{\geq t(\alpha_i^{\min})}$. Hence we conclude that $\alpha_i^{\min} = 0$. But then $\lim_{t\to\infty} \mathcal{S}_i \cap \overline{\mathbb{Z}}_i(t) = \mathcal{S}_i \cap \mathbb{Z}_i$ and hence $\lim_{t\to\infty} \hat{\xi}_i(t) = \lim_{t\to\infty} \xi_i(t) = x_i^*$ due to Assumption 1 and the definition of $\hat{\xi}_i$. Furthermore, by (10), there exists an infinite subsequence $\{t_r\}$ such that $\lambda(t_r) = 1$ and hence $x_i^s(t_r) = \hat{\xi}_i(t_r)$. As $\hat{\xi}_i(t)$ converges to x_i^*, for each $\delta > 0$ there exists a $t(\delta) \in \mathbb{I}_{\geq 0}$ such that $|\hat{\xi}_i(t) - x_i^*| \leq \delta$ for all $t \in \mathbb{I}_{\geq t(\delta)}$. Let $r(\delta) := \min_{t_r \geq t(\delta)} r$; then it follows from (8) that also $|x_i^s(t) - x_i^*| \leq \delta$ for all $t \in \mathbb{I}_{\geq t_{r(\delta)}}$. This can be shown by induction as $|x_i^s(t_{r(\delta)}) - x_i^*| \leq \delta$ and from (8) it follows that if both $|x_i^s(t-1) - x_i^*| \leq \delta$ and $|\hat{\xi}_i(t) - x_i^*| \leq \delta$ for some t, then also the convex combination $x_i^s(t)$ satisfies $|x_i^s(t) - x_i^*| \leq \delta$. But as $\delta > 0$ was arbitrary, it follows that $\lim_{t\to\infty} x_i^s(t) = x_i^*$, as claimed.

Finally, we prove statement (iii) of Theorem 1. To this end, note that

$$\begin{aligned} h(x_i, u_i, t) &= |x_i - x_i^s(t)| = |x_i - x_i^* + x_i^* - x_i^s(t)| \\ &\leq |x_i - x_i^*| + |x_i^* - x_i^s(t)| =: |x_i - x_i^*| + \hat{\varphi}(t). \end{aligned}$$

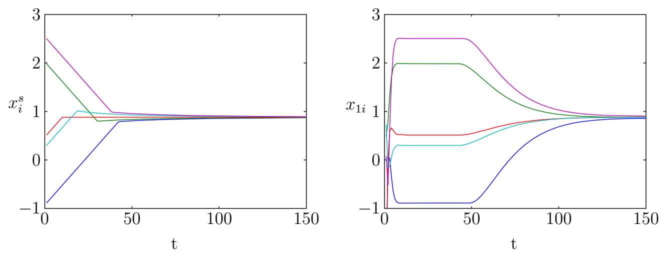
Defining $\hat{\sigma}(t) := \overline{y}_i(t+1) + \hat{\varphi}(t)$ with $\overline{y}_i(t)$ as given in (14), it follows that $\overline{\mathbb{Y}}_i(t+1) \oplus \hat{\varphi}(t)B_1 = \hat{\sigma}(t)B_1$ for all $t \in \mathbb{I}_{\geq 0}$. Furthermore, as $\lim_{t\to\infty} \alpha_i(t) = \lim_{t\to\infty} \varepsilon_i(t) = 0$, one can show that $\lim_{t\to\infty} \hat{\sigma}(t) = 0$. Hence we can apply Proposition 1(iii) with $\hat{h}_i(x_i, u_i) := |x_i - x_i^*|$ to conclude that for the closed-loop system i resulting from application of Algorithm 1 we have $Av[\hat{h}_i] \subseteq \mathbb{Y}_i$, which translates into

$$\lim_{T\to\infty} \frac{\sum_{t=0}^T |x_i(t) - x_i^*|}{T+1} = 0.$$

We can now apply Lemma 1 (together with Remark 12) from Müller et al. [2014b] (see also Lemma 2.2 in Angeli et al. [2011]) to conclude that the closed-loop system i essentially converges to x_i^*. As this holds for all systems $i \in \mathbb{I}_{[1,n]}$, it follows that the overall closed-loop system resulting from application of Algorithm 1 essentially converges to x^* as claimed. □

4. EXAMPLE: CONSENSUS UNDER CONFLICTING OBJECTIVE

We consider five discrete-time double integrator systems of the form (1) with $A_i = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B_i = [0\ 1]^T$,



(a) Terminal steady-states x_i^s. (b) Closed-loop sequences of x_{1i}.

Fig. 1. Closed-loop sequences resulting from Algorithm 1 for five double integrators connected over a line graph.

and $\mathbb{Z}_i := \mathbb{X}_i \times \mathbb{U}_i = [-5\ 5]^3$. The cost function for each system is given by $\ell_i(x_i, u_i) = (x_{1i} - a_i)^2 + (x_{2i} - b_i)^2 + (u_i - d_i)^2$, where $x_i = [x_{1i}\ x_{2i}]^T$ and a_i, b_i, d_i are randomly chosen within the interval $[-3\ 3]$. The coupling constraints (2) are given as $(E(\mathcal{G})^T \otimes I_2)x = 0$, where $E(\mathcal{G})$ is the incidence matrix of the graph $\mathcal{G}$ describing the interconnection topology of the systems, which we chose to be a line graph. This means that the systems asysmptotically have to reach consensus, i.e., $x_1 = \cdots = x_5$ asymptotically. Without the coupling constraints (2), the optimal steady-state for each system is given by $x_{1i}^{opt} = a_i$ and $x_{2i}^{opt} = u_i^{opt} = 0$. In case that the values a_i are not the same for all five systems, the individual objectives of the systems (given by ℓ_i) are conflicting with the requirement of reaching consensus, as $x^{opt} = [(x_1^{opt})^T\ \ldots\ (x_5^{opt})^T]^T \notin \mathcal{C}$. Figure 1 shows simulation results obtained by applying Algorithm 1, where in Step 2 a distributed dual subgradient algorithm is used. The prediction horizon used in problem (3)–(4) is $N = 15$, and $a_1 = -0.89$, $a_2 = 1.99$, $a_3 = 0.51$, $a_5 = 0.30$, and $a_5 = 2.50$. The overall optimal steady-state (5) is such that $x_{1i}^* = 0.88$, $x_{2i}^* = 0$ and $u_i^* = 0$ for all $i \in \mathbb{I}_{[1,5]}$. Both ξ_i and x_i^s are initialized with $\xi_i(0) = x_i^s(0) = x_i^{opt}$. As guaranteed by Theorem 1, one obtains that ξ_i, x_i^s and x_i converge to x_i^*. Furthermore, as discussed in Remark 6, during the transient phase each system "spends time" in a region where its cost ℓ_i is lower than $\ell(x_i^*, u_i^*)$, and in particular approaches $x_i^{opt} = [a_i\ 0]^T$ (see Figure 1(b)), before it is forced to converge to x_i^* in order to asymptotically reach consensus.

5. CONCLUSIONS

In this paper, we presented a distributed economic MPC algorithm for cooperative control problems involving self-interested agents, where coordination between the systems is enforced via coupling constraints. The proposed algorithm is such that these coupling constraints are satisfied asymptotically, while each system acts according to its own objective during the transient phase. The results were illustrated with the problem of reaching consensus among double integrator systems with conflicting objective.

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