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Large Time Simulation Reduction for Solving the Mechanical Contact Problem: A Fuzzy Control Approach.

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Abstract: In this paper, a fuzzy control-based method is provided to solve the unilateral contact between two elastic structures under static loads and multiple contact pairs, modeled by finite element methods. To the best author's knowledge, this problem has been not exploited in previous literature under this approach. The objective is to achieve an important reduction of time simulation with respect to previous existing approaches for this problem. As seen in the paper, the complexity of controller is independent of the number of degree of freedom and its design does not depend on the value of external static loads.

Keywords: Mechanical contact problem; linear matrix inequality; nonlinear control system; fuzzy model

1. INTRODUCTION

In the context of mechanical engineering, the unilateral contact between two elastic structures under external static loads is recognized as one of the most common contact problems. The main difficulty is to determine rapidly the value of the existing contact loads in the contact surface such that both structures are well balanced**.**

In this field, many available solutions based on Finite Element Method (FEM) can be found in the literature: classical approaches, such as the popular penalty approximation (Arnold, 1982) and «mixed » or « trial-and-error » methods, and Lagrange multiplier methods (Carpenter et al., 1991) (Hild and Renard, 2010) and their modifications (Bussetta et al., 2012). However, these methods require high cost simulation for complex problems: for instance, the number of Lagrange multipliers is essentially determined by the number of contacting entity pairs and can considerably increase the size of the system to solve. Although in the past decade, substantial progresses have been made in the analysis of contact problems using finite element procedures (Wriggers, 1995), it is well known that this kind of solutions for this problem are still time consuming: although the exponential growth of the computer power allows to perform more sophisticated problems, the size (the number of unknowns) of industrial models and the complexity and the cost of simulations (multi-scale, multi-physics, uncertainty, …) increase at the same time.

Recently, in (Gonzalez et al., 2013), a first approach based on robust control theory was proposed to solve this problem, illustrating promising results in terms of reduction of time simulation. However, this work deals with the case of one single contact pair in the contact surface. The proposed control model is formed by a discrete-time uncertain linear system in which all the nonlinearities are embedded into uncertainties in the system matrices and a static state feedback control. The control law is optimized in terms of decay rate performance, with the aim to reduce the number of iterations.

The objective of this work is to extend the ideas of (Gonzalez et al., 2013) to the case of multiple contact pairs. Moreover, an output transformation is proposed with the aim to reduce the complexity of the control design, by means of the separation principle. On the other hand, it is well known the advantages of Takagi-Sugeno (T-S) models (Takagi and Sugeno, 1985): (i) T-S fuzzy models provide a systematic procedure to exactly represent in a compact set nonlinear models by means of the sector nonlinearity approach (Tanaka and Wang, 2001) and (ii) the last advances in linear system theories can be integrated into the framework of nonlinear control. Therefore, the problem is addressed in two steps: first, a discrete-time fuzzy model with uncertainties of the contact problem will be obtained via sector nonlinearity, and second, a fuzzy parallel distributed compensation control PDC scheme (see (Cao and Frank, 2000) and references therein). Finally, after time simulation, the steady-state will provide the contact loads of the well balanced setting. Recall that this control takes advantage of the knowledge of the nonlinearities, achieving a good compromise between complexity and decay-rate performance in the control design.

Notation and materials: In this paper, the shortcut X_z is used to denote $X_z = \sum_{i=1}^{r} \theta_i(.) X_i$, where X_i denotes any matrices and $\theta_i(.)$ denotes a positive scalar function such $0 \le \theta_i(.) \le 1$ that and $\sum_{i=1}^{r} \theta_i(.) = 1$. Analogously, $X_{zz} = \sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i(.) \theta_j(.) X_i X_j$.

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The next lemma is a sufficient condition to ensure the negativeness (or positiveness) of multiple convex sum:

Lemma 1: (Tanaka and Wang, 2001) The inequality $\Gamma_{zz} < 0$ is fulfilled providing the following conditions hold, for $i, j \in \mathbb{S}_r^2$, $\mathbb{S}_r^2 = \{1,..,r\} \times \{1,..,r\}$

$$\Gamma_{ii} < 0\,, \qquad \frac{2}{r-1}\Gamma_{ii} + \Gamma_{ij} + \Gamma_{ji} < 0, \quad i \neq j \tag{1}$$

Remark 1: The relaxation in Lemma 1 is chosen in this paper because it is a good compromise between accuracy and complexity. However, other more powerful relaxations might be chosen leading to less conservativeness (Thierry Marie Guerra et al., 2009).

2. PROBLEM STATEMENT

Let us consider two mechanical structures S1 and S2 in contact (see Fig. 1). After applying the finite element method, we obtain a set of nodes that defines the discrete finite element mesh for both structures. Every node is described by its x-y coordinates in 2D plane $(x_i, y_i), i = 1,2,..,N$. , where N is the total number of nodes of the finite element mesh corresponding to both structures. Then, we define the initial position vector $\widehat{P}_i$ containing all the x-y coordinates of every node as:

$$\widehat{P}_i = \begin{pmatrix} x_{i1} & y_{i1} & x_{i2} & y_{i2} & \dots & x_{iN} & y_{iN} \end{pmatrix}^T \tag{2}$$

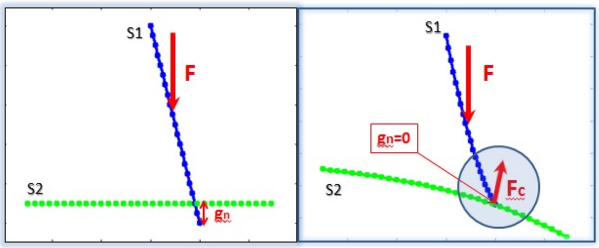

Fig. 1. Left-side, FEM of two elastic beams (initial position), right-side, FEM of two elastic beams (well balanced).

Let us introduce the following additional parameters: (i) The well-balanced position vector $\widehat{P}$, (ii) A constant and known matrix R, whose value depends on the stiffness and mass matrices and modal bases of structures in contact (Cazier et al., 2012), (iii) The vector $\widehat{F}$ containing the x-y components of the external loads, and (iv) The vector containing the x-y components of the contact loads $\widehat{F}_c$ applied over every node.

The discrete equilibrium equation is:

$$\widehat{P} - \widehat{P}_i = R(\widehat{F} + \widehat{F}_c)\,, \tag{3}$$

$$\widehat{P} = \begin{pmatrix} x_1 & y_1 & x_2 & y_2 & \dots & x_N & y_N \end{pmatrix}^T$$

The contact pair (see Fig. 2) is defined by the following contact nodes:

$$P_M = \begin{pmatrix} x_P & y_P \end{pmatrix}^T,\ Q_M = \begin{pmatrix} x_q & y_q \end{pmatrix}^T,\ P_S = \begin{pmatrix} x_s & y_s \end{pmatrix}^T$$

Let us define their respective index nodes $\iota_p, \iota_q, \iota_s$ such that: $x_\zeta = \widehat{P}(2\iota_\zeta), y_\zeta = \widehat{P}(2\iota_\zeta + 1)$, $\quad \zeta = p, q, s$

The nodes P_M and Q_M belong to the master structure (S2) and both define the master segment $\overrightarrow{P_M Q_M}$. The node P_S belongs to the slave structure S1. Now, the following parameters are defined (see Fig. 2)

$$d = \left\| P_M Q_M \right\| = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2} \tag{4}$$

$$\vec{n} = n_x \vec{x} + n_y \vec{y}, \qquad n_x = (y_p - y_q)/d \quad , \ n_y = (x_q - x_p)/d$$

$$\alpha = \frac{1}{d}\left((x_p - x_s) n_y - (y_p - y_s) n_x \right)$$

The vector $\vec{n}$ is defined as the normal vector. Note that $\|\vec{n}\| = 1$.

The normal gap g_n is defined as the scalar product between $\vec{n}$ and $\overrightarrow{P_M Q_M}$:

$$g_n = \left\langle \overrightarrow{P_m P_s} \middle| \vec{n} \right\rangle = (x_s - x_p) n_x + (y_s - y_p) n_y$$

Now, accordingly to the notation in Fig. 2, we define the contact load F_c as a vector containing the x-y components of the components f_{CP}, f_{CS}, f_{CQ} applied over the contact nodes P_M, P_S, Q_M respectively:

$$\begin{aligned} &F_c = f_n F_N, \qquad F_N = \begin{pmatrix} f_{CP}^T & f_{CS}^T & f_{CQ}^T \end{pmatrix}^T \\ &f_{CP} = \begin{pmatrix} -(1-\alpha) \times n_x & -(1-\alpha) \times n_y \end{pmatrix}^T \\ &f_{CS} = \begin{pmatrix} n_x & n_y \end{pmatrix}^T \\ &f_{CQ} = \begin{pmatrix} -\alpha \times n_x & -\alpha \times n_y \end{pmatrix}^T \end{aligned} \tag{5}$$

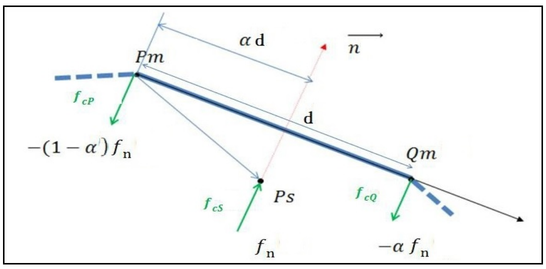

Fig. 2. Geometric description of the contact pair**.**

The scalar parameter f_n is defined as the normal load. Note that the problem of finding the contact load F_c is reduced to the problem of finding the normal load f_n .

On the other hand, the localisation matrix L is defined as a matrix of dimension $6 \times 2N$ formed by 0/1 entries. This matrix is built accordingly to the index nodes $\iota_p, \iota_q, \iota_s$. Then, by means of L is possible to find the x-y components of the contact nodes P_M, P_S and P_Q, respectively by searching L such that $0 \le \alpha \le 1$, where:

$$P_c = L\,\widehat{P}, \qquad P_c = \begin{pmatrix} x_p & y_p & x_s & y_s & x_q & y_q \end{pmatrix}^T \tag{6}$$

The condition $0 \le \alpha \le 1$ means that the slave node P_S must be contained into the master segment $\overrightarrow{P_M Q_M}$. Note also that the contact surface might be formed by more than one contact pair. So, we define N_C as the number of contact pairs. Then, all the defined parameters linked to the p_{th} contact pair can be identified by the superscript (p). For instance: $\left(\iota_p, \iota_q, \iota_s\right)^{(p)}$, $g_n^{(p)}$ and $f_n^{(p)}$ are the index nodes, the normal gap and the normal load of the p_{th} contact pair, respectively, where $p = 1, 2, ..., N_c$

Definition 1: The structures S1 and S2 are defined to be well balanced if and only if the following equations are satisfied for $p = 1, 2, ..., N_c$:

$$L^{(p)}\widehat{P} - L^{(p)}\widehat{P}_i = L^{(p)}R\left(\widehat{F} + \widehat{F}_c\right), \tag{7}$$

$$\widehat{F}_c = \sum_{j=1}^{Nc} f_n^{(j)} \left(L^{(j)}\right)^T F_N^{(j)}, \qquad g_n^{(p)} = 0, \qquad 0 \le \alpha^{(p)} \le 1$$

Therefore, this problem is reduced to find $L^{(p)}$ and $f_n^{(p)}$ such that equations in (7) are satisfied.

3. MAIN RESULTS

Let us introduce the following notation ($p = 1, 2, ..., N_c$):

$$\phi = \left(\left(\phi_{xy}\right)^T \quad \left(\phi_f\right)^T\right)^T \tag{8}$$

$$\phi_{xy} = \left(\left(\phi_{xy}^{(1)}\right)^T, ..., \left(\phi_{xy}^{(p)}\right)^T, ..., \left(\phi_{xy}^{(Nc)}\right)^T\right)^T$$

$$\phi_f = \left(f_n^{(1)}, ..., f_n^{(p)}, ..., f_n^{(Nc)}\right)^T$$

$$\phi_{xy}^{(p)} = L^{(p)}\widehat{P} = \begin{pmatrix} x_p^{(p)} & y_p^{(p)} & x_s^{(p)} & y_s^{(p)} & x_q^{(p)} & y_q^{(p)} \end{pmatrix}^T$$

Then, (7) can be rewritten in a compact form as:

$$C(\phi_{xy})\phi + W = 0 \tag{9}$$

$$C(\phi_{xy}) = \begin{pmatrix} I_{g1xg1} & F\left(\phi_{xy}\right) \\ G\left(\phi_{xy}\right) & 0_{g2xg2} \end{pmatrix}, \qquad g_1 = 6N_c, \qquad g_2 = N_c$$

$$F\left(\phi_{xy}\right) = -\widetilde{R}\left(G_1\left(\phi_{xy}\right) + G_2\left(\phi_{xy}\right)\Lambda\right)$$

$$G\left(\phi_{xy}\right) = diag(G^{(1)}(\phi_{xy}), ..., G^{(p)}(\phi_{xy}), ..., G^{(Nc)}(\phi_{xy}))$$

$$\widetilde{R} = \left(\widetilde{L}\right)^T R\widetilde{L}, \qquad \widetilde{L} = \left(\left(L^{(1)}\right)^T \quad ... \quad \left(L^{(N_c)}\right)^T\right)^T$$

$$G_d\left(\phi_{xy}\right) = diag\left(G_d^{(1)}\left(\phi_{xy}\right), ..., G_d^{(N_c)}\left(\phi_{xy}\right)\right), \quad d = 1, 2$$

$$G_1^{(p)}\left(\phi_{xy}\right) = \begin{pmatrix} -n_x^{(p)} & -n_y^{(p)} & n_x^{(p)} & n_y^{(p)} & 0 & 0 \end{pmatrix}$$

$$G_2^{(p)}\left(\phi_{xy}\right) = \begin{pmatrix} n_x^{(p)} & n_y^{(p)} & 0 & 0 & -n_x^{(p)} & -n_y^{(p)} \end{pmatrix}$$

$$G^{(p)}(\phi_{xy}) = \begin{pmatrix} -n_x^{(p)} & -n_y^{(p)} & n_x^{(p)} & n_y^{(p)} & 0 & 0 \end{pmatrix}$$

$$\Lambda = diag(\alpha^{(1)}, ..., \alpha^{(N_c)})$$

$$W = \begin{pmatrix} W_{xy}^T & 0 \end{pmatrix}^T, \qquad W_{xy} = -\left(\widetilde{L}\right)^T\left(\widehat{P}_i + R\widehat{F}\right)$$

3.1 A closed-loop control model

The objective of this section is to provide a control method based on a reduced closed-loop discrete-time model to obtain the normal loads $f_n^{(p)}$ satisfying (9).

Let us define the following discrete-time system:

$$\phi_{k+1} = \phi_k + u_k, \qquad e_k = C(\phi_{xy,k})\phi_k + W \tag{10}$$

Note that the steady-state ϕ_k of the stabilized system (10) contains the contact loads $f_n^{(p)}$ that satisfy (9) since $\lim_{k\to\infty} e_k \to 0$. Therefore, a stabilizing control law $u_k = f_{uc}(e_k)$ must be obtained such that $f_{uc}(0) = 0$ and $f_{uc}(e) \ne 0, \forall e \ne 0$.

3.2 Model reduction for stabilization

Note that $C(\phi_{xy,k}) \in R^{g3\times g3}$, where $g_3 = 7N_c$. In order to simplify the control design, the following reduction method is proposed: first, the output error e_k is pre-multiplied by the following regular matrix:

$$T(\phi_{xy,k}) = \begin{pmatrix} I_{g1xg1} & 0_{g1xg2} \\ -G\left(\phi_{xy,k}\right) & I_{g2xg2} \end{pmatrix}$$

Thus, the new matrix $C_t(\phi_{xy,k}) = T(\phi_{xy,k})C(\phi_{xy,k})$ is

$$C_t(\phi_{xy,k}) = \begin{pmatrix} I_{g1xg1} & F\left(\phi_{xy,k}\right) \\ 0 & C_r\left(\phi_{xy,k}\right) \end{pmatrix}$$

$$C_r\left(\phi_{xy,k}\right) = -G\left(\phi_{xy,k}\right) \times F\left(\phi_{xy,k}\right)$$

So, the new output error can be redefined as:

$$\varepsilon_k = T(\phi_{xy,k})e_k = C_t(\phi_{xy,k})\phi_k + W_t(\phi_{xy,k}) \tag{11}$$

$$W_t(\phi_{xy,k}) = T(\phi_{xy,k})W$$

Taking into account that $\exists T^{-1}(\phi_{xy,k})$, $\forall \phi_{xy,k}$, it is easy to see that $\lim_{k\to\infty} \varepsilon_k \to 0$ if and only if $\lim_{k\to\infty} e_k \to 0$. Taking a control law such as $u_k = Y(\phi_{xy,k})\varepsilon_k$, the closed-loop control of system (10) with the new output error (11) and the above defined u_k, can be expressed as:

$$\phi_{k+1} = \left(I + Y(\phi_{xy,k})C_t(\phi_{xy,k})\right)\phi_k + w_k, \quad w_k = Y(\phi_{xy,k})W_t(\phi_{xy,k})$$

Note that $\|w_k\| \le \delta$ for some positive $\delta < \infty$, since $\left|n_x^{(p)}\right| \le 1$ and $\left|n_y^{(p)}\right| \le 1$. Choosing $Y(\phi_{xy,k}) = diag(Y_{xy}(\phi_{xy,k}), Y_f(\phi_{xy,k}))$, by virtue of the block-triangular form of $C_t(\phi_{xy,k})$, the separation principle (Yoneyama et al., 2001) can be applied. On the other hand, take into account that: (i) the input w_k is bounded and (ii) when $k \to \infty$, w_k reaches a constant steady-value because it depends on $\phi_{xy,k}$. Then, the input w_k does not affect the stability of the system (10). Therefore, the stability of (10) is ensured if the following subsystems are stable:

$$\phi_{xy,k+1} = \left(I + Y_{xy}(\phi_{xy,k})\right)\phi_{xy,k} \tag{12}$$

$$\phi_{f,k+1} = \phi_{f,k} + u_{f,k}, \qquad u_{f,k} = Y_f(\phi_{xy,k})\zeta_{f,k} \tag{13}$$

Where $\zeta_{f,k} = C_r(\phi_{xy,k})\phi_{f,k}$. Therefore, $Y_{xy}(\phi_{xy,k})$ and $Y_f(\phi_{xy,k})$ can be designed separately: the control gain $Y_{xy}(\phi_k)$ can be fixed arbitrarily such that the poles of system (12) are located inside the unit circle (thereafter, we choose $Y_{xy}(\phi_{xy,k}) = -I$ to ensure the maximum convergence speed), and then, only the control $u_{f,k}$ must be designed such as (13) is stabilized.

3.3 A fuzzy PDC-based control

Let us write $C_r\left(\phi_{xy}\right)$ as:

$$C_r\left(\phi_{xy,k}\right) = G\left(\phi_{xy,k}\right)\widetilde{R}\left(G_1\left(\phi_{xy,k}\right) + G_2\left(\phi_{xy,k}\right)\Lambda_k\right)$$

$$G_d\left(\phi_{xy,k}\right) = diag\left(G_d^{(1)}\left(\phi_{xy,k}\right), \ldots, G_d^{(N_c)}\left(\phi_{xy,k}\right)\right), \quad d = 1,2$$

$$G_1^{(p)}\left(\phi_{xy,k}\right) = \begin{pmatrix} -n_{xk}^{(p)} & -n_{yk}^{(p)} & n_{xk}^{(p)} & n_{yk}^{(p)} & 0 & 0 \end{pmatrix}$$

$$G_2^{(p)}\left(\phi_{xy,k}\right) = \begin{pmatrix} n_{xk}^{(p)} & n_{yk}^{(p)} & 0 & 0 & -n_{xk}^{(p)} & -n_{yk}^{(p)} \end{pmatrix}$$

$$\widetilde{R} = \widetilde{L}^T R \widetilde{L}, \qquad \widetilde{L} = \begin{pmatrix} \left(L^{(1)}\right)^T & \ldots & \left(L^{(N_c)}\right)^T \end{pmatrix}^T$$

$$\Lambda_k = diag(\alpha_k^{(1)}, \ldots, \alpha_k^{(N_c)}), \; n_{xk}^{(p)} = \cos(\tau_{nk}^{(p)}), \; n_{yk}^{(p)} = \sin(\tau_{nk}^{(p)})$$

Now, consider the following assumptions:

Assumption 1: Let us define the angle of the normal vector as $\tau_{nk}^{(p)} = arctg(n_{yk}^{(p)} / n_{xk}^{(p)})$. The normal angle deviation along all possible trajectories is assumed to be bounded: $\left|\tau_{nk}^{(p)} - \tau_{n0}^{(p)}\right| \le \lambda$, for a given $\lambda > 0$.

Assumption 2: The parameter $\alpha_k^{(p)} \in [-\chi, 1+\chi] \; \forall k \ge 0$, for a given $\chi \ge 0$. (Introduction of parameter χ is discussed Remark 3).

The following uncertain T-S model with $r = 2^{N_c}$ rules is built for system (13). The sector nonlinearity method (Tanaka and Wang, 2001) is applied here considering the state subspace Ω_{xy} ($\phi_{xy,k} \in \Omega_{xy}, \forall k \ge 0$) defined from Assumptions 1 and 2:

$$C_r(\phi_{xy,k}) = \sum_{i=1}^{r} \theta_i(\phi_{xy,k})\left(\overline{C}_i + \overline{M}_i \Delta_k \overline{N}_i\right) \tag{14}$$

$$\overline{C}_i = \frac{1}{2}\left(\overline{C}_{Mi} + \overline{C}_{mi}\right), \qquad \overline{M}_i = I, \qquad \overline{N}_i = \gamma_i I$$

$$\gamma_i = \frac{1}{2}\max\left\|\overline{C}_{Mi} - \overline{C}_{mi}\right\|_\infty$$

$$\overline{C}_{Mi} = \max_{\phi_{xy} \in \Omega_{xy}} (C_r(\phi_{xy}))\Big|_{\Lambda = \Lambda^{(i)}}, \qquad \overline{C}_{mi} = \min_{\phi_{xy} \in \Omega_{xy}} (C_r(\phi_{xy}))\Big|_{\Lambda = \Lambda^{(i)}}$$

$$\Lambda^{(1)} = diag(-\chi, -\chi .., -\chi), \; \Lambda^{(2)} = diag(1+\chi, -\chi ..., -\chi)$$
$$\Lambda^{(3)} = diag(-\chi, 1+\chi, .., -\chi), \; \ldots, \; \Lambda^{(r)} = diag(1+\chi, .., 1+\chi)$$

Where the membership functions $\theta_i(\phi_{xy,k})$ are built by taking the 2^p possible combinations of Λ_k by taking the minimum and maximum value for every $\alpha_k^{(p)}$, $p = 1, \ldots, N_c$

$$\theta_1(.) = \Pi_{p=1}^{Nc}\mu_{1k}^{(p)}, \qquad \theta_2(.) = \mu_{2k}^{(1)}\Pi_{p=2}^{Nc}\mu_{1k}^{(p)}$$

$$\theta_3(.) = \mu_{1k}^{(1)}\mu_{2k}^{(2)}\Pi_{p=3}^{Nc}\mu_{1k}^{(p)}, \; \ldots, \; \theta_r(.) = \Pi_{p=1}^{Nc}\mu_{2k}^{(p)}$$

$$\mu_{1k}^{(p)} = \frac{1 - \alpha_k^{(p)} + \chi}{1 + 2\chi}, \qquad \mu_{2k}^{(p)} = 1 - \mu_{1k}^{(p)}$$

The proposed control law to stabilize (13) is the following PDC control scheme:

$$u_{f,k} = Y_f(\phi_{xy,k})\zeta_{f,k}, \qquad Y_f(\phi_k) = \sum_{i=1}^{r}\theta_i(\phi_{xy,k})\widehat{Y}_i \tag{15}$$

3.4 Closed-loop stabilization

This section provides a LMI-based sufficient condition to design $Y_f(\phi_{xy,k})$:

Theorem 1: System (10) with control law $u_k = Y\left(\phi_{xy,k}\right)\varepsilon_k$, ε_k defined in (11) and $Y\left(\phi_{xy,k}\right) = diag\left(-I, Y_f\left(\phi_{xy,k}\right)\right)$ is stable and guarantees a decay-rate performance β if there exists a symmetric matrix $Q>0$, and matrices X_i such that LMI's in (1) are satisfied, with Γ_{ij} defined as:

$$\Gamma_{ij}=\begin{pmatrix} -\beta^2 Q & (*) & (*) & (*) \\ Q+X_j\overline{C_i} & -Q & (*) & (*) \\ 0 & M_i^T X_j^T & -I & (*) \\ N_i & 0 & 0 & -I \end{pmatrix}$$

Furthermore, the stabilizing controller gain (15) can be computed as $\widehat{Y}_i = Q^{-1}X_i$.

Proof: Let us consider the quadratic Lyapunov function $V_k=\phi_{xy,k}^T Q \phi_{xy,k}$ and the return difference defined as $\Delta^{(\beta)}V_k = V_{k+1}-\beta^2 V_k$. The condition $\Delta^{(\beta)}V_k<0$ ensures the stability of system (13) with decay rate performance β, that is, $\|\phi_{xy,k}\| \le \beta^2 \|\phi_{xy,0}\|$. Taking into account (13), denoting $X_i = Q\widehat{Y}_i$, and applying the cross-product lemma, the return difference $\Delta^{(\beta)}V_k$ can be expressed as:

$$\Delta^{(\beta)}V=\begin{pmatrix} -\beta^2 Q & (*) \\ Q+X_z\overline{C_z} & -Q \end{pmatrix}+\begin{pmatrix} 0 \\ X_z M_z \end{pmatrix}\Delta_k\begin{pmatrix} N_z & 0 \end{pmatrix}+(*)<$$
$$\begin{pmatrix} -\beta^2 Q & (*) \\ Q+X_z\overline{C_z} & -Q \end{pmatrix}+\nu\begin{pmatrix} 0 \\ X_z M_z \end{pmatrix}\begin{pmatrix} 0 \\ X_z M_z \end{pmatrix}^T+\nu^{-1}\begin{pmatrix} N_z^T \\ 0 \end{pmatrix}\begin{pmatrix} N_z^T \\ 0 \end{pmatrix}^T$$

Where ν is some positive scalar. Note that, without loss of generality, ν can be set to 1. Then, applying Schur complement, we obtain:

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\theta_i(.)\theta_j(.)\begin{pmatrix} -\beta^2 Q & (*) & (*) & (*) \\ Q+X_j\overline{C_i} & -Q & (*) & (*) \\ 0 & M_i^T X_j^T & -I & (*) \\ N_i & 0 & 0 & -I \end{pmatrix}<0 \qquad (16)$$

The negativeness of the double convex sum (16) is ensured by virtue of Lemma 1. Finally, applying the separation principle discussed in Section 3.2, the stability of (10) is ensured. The proof is completed.

Remark 2: The localisation matrix $\widetilde{L}$ has been assumed to be time-constant in the discrete-time system (10). Note that $\widetilde{L}$ can be set in advance by searching, $\forall p=1,..,N_c$, the initial master segments $\overrightarrow{P_m Q_m}^{(p)}$ (computed by means of the initial coordinates, available in $\widehat{P}_i$) with matches with the initial slave nodes $P_S^{(p)}$, such that $0\le \alpha_0^{(p)} \le 1$. Therefore, it is reasonable to set the initial state condition $\phi_{xy,0}=\widetilde{L}\widehat{P}_i$. Note also that $\phi_{xy,0}\in\Omega_{xy}$ for any arbitrarily values for $\chi,\lambda>0$, so the proposed $\phi_{xy,0}$ always verify Assumptions 1 and 2.

Remark 3: If the localisation matrix of the well-balanced structures is different from the initial one obtained as explained in Remark 2, after simulation we will obtain $\alpha^{(p)}\notin[0,1]$ for some $p=1,...,N_c$. This fact explains why is interesting to choose $\chi>0$. Thus, the convergence of the proposed algorithm is ensured even if the localisation matrix $\widetilde{L}$ of the well-balanced setting is located at the neighbourhood of the initial $\widetilde{L}$. In this case, the simulation should be repeated by setting a new initial localisation matrix $\widetilde{L}$ depending on $\alpha^{(p)}$. The discussion about how to address this topic has been intentionally left out of the scope of this paper for lack of space.

4. EXAMPLES

Let us consider the following structures, modelled by 2D finite elements in plane stress of Fig. 3. The master and slave structures (S1 and S2), consist of an elastic and a rigid beam, respectively.

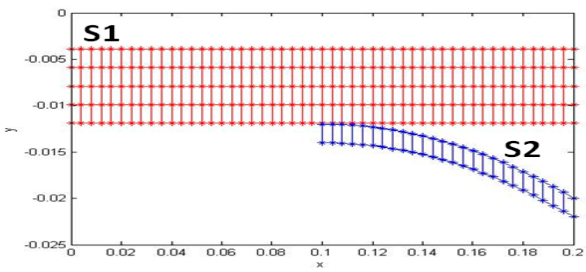

Fig. 3. Structures S1 and S2 before applying the external load.

The system is discretized by Finite Element method obtaining N=307 degrees of freedom for both structures (255 for S1 and 52 for S2). The external load F is applied over the node 100 of S1 and $F(100)=-2800N$ and $N_c=2$.

After search the index nodes such that $0\le\alpha_0^{(1)}\le 1$, we build the localisation matrix $\widetilde{L}$ with the index nodes: $(\iota_p,\iota_q,\iota_s)^{(1)}=(256,257,231)$; $(\iota_p,\iota_q,\iota_s)^{(1)}=(257,258,232)$ obtaining $\alpha_0^{(1)}=0.956$ and $\alpha_0^{(2)}=0.959$. Then, we compute $\tau_{n0}^{(1)}=1.5676rad$, $\tau_{n0}^{(2)}=1.5613rad$ by means of $\phi_{xy,0}=\widetilde{L}\widehat{P}_i$. Now, we define Ω_{xy} by virtue of Assumptions 1 and 2, with $\lambda=4.49\bullet10^{-5}rad$ and $\chi=0.1$. Then, applying the sector nonlinearity method, a T-S uncertain model (14) with four rules ($r=4$) is built. By Theorem 1, a stabilizing control law on the form (15) is found with a minimum decay rate of $\beta=0.019$ with:

$$Y_1=10^9\begin{pmatrix} -2.01 & 1.90 \\ 1.90 & -1.80 \end{pmatrix}, Y_2=10^9\begin{pmatrix} -2.39 & 2.25 \\ 2.25 & -2.13 \end{pmatrix}$$
$$Y_3=10^9\begin{pmatrix} -2.16 & 2.04 \\ 2.04 & -1.93 \end{pmatrix}, Y_4=10^9\begin{pmatrix} -2.55 & 2.41 \\ 2.41 & -2.27 \end{pmatrix}$$

Finally, the normal loads that achieve the well balanced setting (Fig. 4) ($f_n^{(1)} = 4436.6\text{N}$ and $f_n^{(2)} = 1839.3N$) are obtained by simulation of system (10) with control law $u_k = Y(\phi_{xy,k})\varepsilon_k$ (see Fig. 5): The time simulation is stopped after 13 iterations, when the norm of the error e_k defined in (10) is checked to be smaller than a prescribed tolerance $\delta_e = 10^{-12}$, that is $\|e_k\| \le \delta_e$. On the other hand, the maximum deviation of the angle of normal vectors and the limits of $\alpha_n^{(p)}$ are checked to satisfy Assumptions 1 and 2: $\left|\tau_{nk}^{(p)} - \tau_{n0}^{(p)}\right| \le 1.78 \bullet 10^{-5} rad < \lambda$, and $0.957 \le \alpha_n^{(p)} \le 0.991$, respectively.

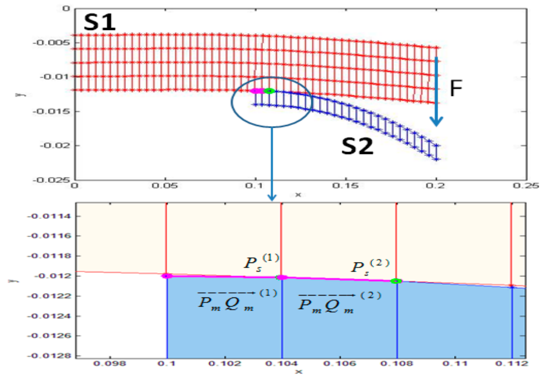

Fig 4. Structures S1 and S2 after computing the normal load (well-balanced) under external static load F.

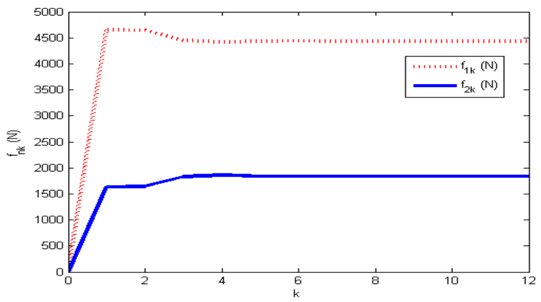

Fig 5. Time response of the state corresponding to the normal loads.

The CPU time required for the simulation was 46ms meanwhile the same problem solved with a classical Lagrange multipliers formulation required 293ms, illustrating a time reduction around 84%. For both methods, the used software was Matlab R2007b with the Structural Dynamic Toolbox (SDT) V6.1 on an Intel Core 2 Duo P9600 CPU at 2.54 GHz.

4. CONCLUSIONS

In this paper, a fuzzy control approach has been proposed to solve the unilateral contact problem considering multiple contact pairs, obtaining a good compromise between cost in control design and performance has been achieved. Finally, a numerical result has been proposed to illustrate the effectiveness of this method and promising results in terms of time simulation reduction.

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