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Optimal Synthesis to Inverse Problems of Dynamics

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Abstract: Perturbed inverse problems are under consideration for dynamical systems linear relative controls. It is assumed that sampling history and sampling error estimate are known. Auxiliary optimal control problems are introduced to minimize a regularized integral discrepancy functional. The trajectories of the system are constructed with the help of Optimal Synthesis in the domain of admissible motions. It is proven that realizations of Optimal Synthesis generating trajectories of the system and minimizing the discrepancy functional in the domain are solutions of the perturbed inverse problem of dynamics.

1. INTRODUCTION

Perturbed inverse problems for controlled dynamical systems are under consideration. A sampling history of the real trajectory is known. This trajectory is generated by the control, which isn't known. Moreover, the deviation of the samples from the real trajectory satisfies the known estimate of sampling error. The inverse problem with perturbed (inaccurate) sampling of trajectory is to rebuild such controls that would be close to the control generated the real trajectory and that would have the least norm in L_2.

A solution of the problem was considered, for example, in the works Osipov et al. [1999, 2011]. The approach, suggested in the papers, is based on a feedback constructed in forward time using a dynamical guide. The approach has roots in works by N.N. Krasovskii's school on the optimal feedbacks theory. This method can be considered as a variation of the regularization method by A.N. Tikhonov (Tikhonov, Arsenin [1977]).

In the presented work another approach is suggested. It is as well based on the optimal feedbacks theory to problems with regularized integral discrepancy functionals. The approach uses backward procedures of dynamic programming, the characteristics method and optimal synthesis, too (see Subbotina, Tokmantsev [2011]).

2. INVERSE PROBLEMS

We consider the following controlled system

$$\frac{dx(t)}{dt} = f(t,x) + G(t,x)u, \qquad t \in [0,T], \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, controls $u \in \mathbb{R}^n$ are restricted, namely

$$u \in U = \{u_i \in [a_i^-, a_i^+],\ a_i^- < a_i^+,\ i = 1,2,\ldots,n\}. \tag{2}$$

We denote the set of admissible controls by the symbol $\mathcal{U}$:

$$\mathcal{U} = \{u(t) \in U, t \in [0,T] \colon u(\cdot) \text{ is measurable}\}.$$

2.1 Assumptions

We use the symbols $f(t,x) = (f_1(t,x),\ldots,f_n(t,x))$ for an n-dimensional vector, and

$$G(t,x) = (g_{i,j}(t,x)), \quad i,j \in \overline{1,n}$$

for an $n \times n$ - matrix.

We denote by the symbol Π_T the strip $\Pi_T = [0,T] \times \mathbb{R}^n$.

We assume that the conditions are satisfied.

A. Functions $f_i(t,x)$, $g_{i,j}(t,x)$, $i,j \in \overline{1,n}$ are continuously differentiable in $(0,T) \times \mathbb{R}^n$ and have the sublinear growth property

$$\|f_i(t,x)\| \le K_1(1 + \|x\|)$$
$$\|g_{i,j}(t,x)\| \le K_1(1 + \|x\|),$$

where $K_1 > 0$, $(t,x) \in \mathrm{cl}\,\Pi_T$, while symbol $\|x\|$ means the Euclidean norm of the finite dimensional vector x.

We consider the compact set $\Psi^r \subset \Pi_T$, $r > 0$, of the form

$$\Psi^r = \{(t,x(t)) :\ t \in [t_0,T],\ t_0 \in [0,T],$$
$$x(t) = x(t;t_0,x_0,u(\cdot)),\ u(\cdot) \in \mathcal{U}\}.$$

As the domain Ψ^r is compact there are the following restrictions in the domain Ψ^r (7):

$$\max_{i,j,k \in \overline{1,n}} \max_{(t,x) \in \Phi(\delta)} \{|f_i(t,x)|, \quad |g_{i,j}(t,x)|\} \le K_2, \tag{3}$$

$$\max_{i,j,k \in \overline{1,n}} \max_{(t,x) \in \Phi(\delta)} \left\{ \left|\frac{\partial f_i(t,x)}{\partial t}\right|, \left|\frac{\partial f_i(t,x)}{\partial x_k}\right|, \right. \tag{4}$$
$$\left. \left|\frac{\partial g_{ij}(t,x)}{\partial t}\right|, \left|\frac{\partial g_{ij}(t,x)}{\partial x_k}\right| \right\} \le K_3,$$

where $K_2 > 0$, $K_3 > 0$.

2.2 Unperturbed inverse problems

Let $x_*(\cdot) : [0,T] \to \mathbb{R}^n$ be the real trajectory of the system (1). The inverse problem of dynamics consists in finding the control $u(\cdot) \in \mathcal{U}$, generating such trajectory $x(t)$ of the system (1) $x(t) = x(t; 0, x_*(0), u(\cdot))$ that

$$x(t) = x_*(t) \text{ for all } t \in [0, T].$$

One can easy prove the assertions using Analysis (Kolmogorov, Fomin [1999]) and the strong convexity of the square of the norm $\|u(\cdot)\|_{L_2}$ in L_2.

Lemma 1. The set $\mathcal{U}$ is closed in L_2.

Lemma 2. The set $\mathbb{U}$ is nonempty, convex and bounded in L_2. Moreover, there exists the only element $u_*(\cdot) \in \mathbb{U}$ with the minimal norm in L_2.

This element is called the normal solution of the inverse problem.

2.3 Perturbed inverse problems

Usually we don't know the real trajectory $x_*(\cdot)$ of the system (1) in a perturbed inverse problem. Instead of it we know continuous function $y(\cdot) : [0, T] \to \mathbb{R}^n$, which is the sampling history of the state variable $x_*(t)$. It is known also that the real trajectory of the system (1) belongs to the domain Ω_δ of admissible errors of the samples

$$(t, x_*(t)) \in \Omega_\delta = \{(t, x) : t \in [0, T],\ \|x - y(t)\| \le \delta\}, \quad (5)$$

where $\delta > 0$ is the parameter of sampling error.

The goal of the perturbed inverse problem is to reconstruct the real trajectory of the system $x_*(\cdot)$ as accurate as possible, using the sampling history $y(\cdot)$.

In other words, we need to build the control $u^\delta(\cdot) : [0, T] \to U$, the compact domain $\Phi(\delta) \subset \Psi^r$ and such a trajectory $x^\delta(\cdot) : [0, T] \to \mathbb{R}^n$ of the system (1), that the relations hold

$$\Omega_\delta \subset \Phi(\delta) \subset \Psi^r \quad (6)$$

$$\Phi_0(\delta) = \{x_0 \colon \|x_0 - y(0)\| \le \delta\}, \quad (7)$$

$$(t, x^\delta(t)) \in \Phi(\delta),\ \forall t \in [0, T], \quad (8)$$

that in case $\delta \to 0$

$$(t, x^\delta(t)) \in \Phi(\delta),\ \forall t \in [0, T], \quad (9)$$

$$d_H(\Phi(\delta), \operatorname{gr} x_*(\cdot)) \to 0,\ \delta \to 0, \quad (10)$$

$$\|x^\delta(\cdot) - x_*(\cdot)\|_C = \max_{t \in [0,T]} \|x^\delta(t) - x_*(t)\| \to 0, \quad (11)$$

$$\|u^\delta(\cdot) - u_*(\cdot)\|^2_{L_2} = \int_0^T \|u^\delta(t) - u_*(t)\|^2 dt \to 0, \quad (12)$$

where $d_H(\cdot)$ is the Hausdorff distance, symbol $\operatorname{gr} x_*(\cdot)$ means the graph of the function $x_*(\cdot)$, $\|\cdot\|_C$ is the norm in the space of continuous functions, $\|\cdot\|_{L_2}$ is the norm in the space L_2.

3. A CONSTRUCTION OF SOLUTIONS OF THE PERTURBED INVERSE PROBLEM

3.1 Regularized optimal control problems

We introduce the auxiliary optimal control problems of the system (1), (2) in domain (7) to minimize the discrepancy functionals

$$I_{t_0,x_0}(u(\cdot)) = \int_{t_0}^T \left[\frac{\|x(t) - y(t)\|^2}{2} + \frac{\alpha}{2}\|u(t)\|^2 \right] dt, \quad (13)$$

where $\alpha > 0$ is a regularizing parameter, $(t_0, x_0) \in \Phi(\delta)$, $x(t) = x(t; t_0, x_0, u(\cdot))$ is the trajectory of the system (1), starting at the point $x(t_0) = x_0$ and generating by an admissible control $u(\cdot) : [t_0, T] \to U$.

The Hamiltonian $H^\alpha(t, x, s)$ for this problem (1), (2), (13) has the form

$$H^\alpha(t, x, s) = \min_{u \in U} \Big[\langle s, f(t, x)\rangle + \langle s, G(t, x)u\rangle + \frac{\alpha}{2}\|u\|^2 + \frac{\|x - y(t)\|^2}{2} \Big], \quad (14)$$

where the symbol $\langle \cdot, \cdot \rangle$ means the inner product of two finite dimension vectors. It's not difficult to see that the following relations are true:

$$H^\alpha(t, x, s) = \langle s, f(t, x)\rangle + \frac{\|x - y(t)\|^2}{2} + \langle s, G(t, x)u^\alpha\rangle + \frac{\alpha}{2}\|u^\alpha\|^2. \quad (15)$$

Here $u^\alpha = (u^\alpha_1, \ldots, u^\alpha_n)$:

$$u^\alpha_i = \begin{cases} a^-_i, & \text{if} \quad r^\alpha_i(t, x, s) \le a^-_i, \\ r^\alpha_i(t, x, s), & \text{if} \quad r^\alpha_i(t, x, s) \in [a^-_i, a^+_i], \\ a^+_i, & \text{if} \quad r^\alpha_i(t, x, s) \ge a^+_i, \end{cases} \quad (16)$$

where

$$r^\alpha_i(t, x, s) = -\frac{1}{\alpha} \sum_{j=1}^n s_j G_{j,i}(t, x), \quad i \in \overline{1, n}.$$

Introduction of the regularizing parameter α in (13) implies that the argument $u^\alpha(t, x, s)$ for the minimization operation in the expression for the hamiltonian $H^\alpha(t, x, s)$ is the singleton for any vector $s \in \mathbb{R}^n$. Also it provides continuity for the extremal control $u^\alpha(t, x, s)$ on $\operatorname{cl} \Pi_T \times \mathbb{R}^n$.

3.2 The characteristic system

We consider the characteristic system for the problem (1), (2), (13)

$$\frac{dx_i}{dt} = \frac{\partial H^\alpha(t, x, s)}{\partial s_i}, \quad i \in \overline{1, n}, \quad t \in [0, T], \quad (17)$$

$$\frac{ds_i}{dt} = -\frac{\partial H^\alpha(t, x, s)}{\partial x_i}, \quad i \in \overline{1, n}, \quad t \in [0, T], \quad (18)$$

with boundary conditions

$$x_i(T) = \xi_i,\ s_i(T) = 0,\ i \in \overline{1, n},\ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n. \quad (19)$$

Here

$$\frac{\partial H^\alpha(t, x, s)}{\partial s_i} = f_i(t, x) + \sum_{j=1}^n G_{i,j}(t, x) u^\alpha_j(t, x, s), \quad (20)$$

$$\frac{\partial H^\alpha(t, x, s)}{\partial x_i} = [x_i - y_i(t)] + \langle s, D_{x_i} f(t, x)\rangle + \langle s, D_{x_i} G(t, x) u^\alpha(t, x, s)\rangle + R_i(t, x, s), \quad (21)$$

where

$$R_i(t, x, s) = \big\langle \big(G^\top(t, x)s + \alpha u^\alpha(t, x, s)\big), D_{x_i} u^\alpha(t, x, s) \big\rangle, \quad (22)$$

$$D_{x_i} f(t, x) = \left(\frac{\partial f_1(t, x)}{\partial x_i}, \ldots, \frac{\partial f_n(t, x)}{\partial x_i}, \right)$$

$$D_{x_i} G(t, x) = \left\{ \left(\frac{\partial G_{k,j}(t, x)}{\partial x_i} \right) \right\}, \quad k, j \in \overline{1, n},$$

$$D_{x_i}u^\alpha(t,x,s) = \left(\frac{\partial u_1^\alpha(t,x,s)}{\partial x_i},\ldots,\frac{\partial u_n^\alpha(t,x,s)}{\partial x_i},\right)$$
notation $^\top$ means the operation of transposition.

The following relations imply from (16) for any $j \in \overline{1,n}$:

• If the relations
$$u_j^\alpha(t,x,s) = r_j^\alpha(t,x,s) \in [a_j^-, a_j^+]$$
are true on the time interval $[t_1,t_2] \subset [0,T]$, then the j-th element of the summa in the expression of scalar multiplication for $R_i(t,x,s)$ $i = 1,\ldots,n$ (22) is equal to zero, since
$$G_j^\top(t,x)s + \alpha u_j^\alpha(t,x,s) = 0, \tag{23}$$
where the symbol $G_j^\top(t,x)$ means the j-th row of the matrix $G^\top(t,x)$.

• In the case of either the relations
$$r_j^\alpha(t,x(t),s(t)) \le a_j^-, \qquad u_j^\alpha(t,x(t),s(t)) \equiv a_j^-,$$
or the relations
$$r_j^\alpha(t,x,s) \ge a_j^+, \qquad u_j^\alpha(t,x(t),s(t)) \equiv a_j^+$$
hold on the interval $(t_1,t_2) \subset [0,T]$, then
$$D_{x_i}u_j^\alpha(t,x(t),s(t)) = 0. \tag{24}$$

Thus, from (22), (23), (24), it follows that
$$R_i(t,x,s) = 0, \qquad \forall\, i \in \overline{1,n}. \tag{25}$$

One can summarize the foregoing in the following statement:

Lemma 3. The characteristic system (17), (18) for the problem (1), (2), (13) has the form
$$\frac{dx}{dt} = f(t,x) + G(t,x)u^\alpha(t,x,s), \tag{26}$$
$$\frac{ds}{dt} = -\big[(x-y(t)) + s^\top D_x f(t,x) + s^\top D_x G(t,x)u^\alpha(t,x,s)\big], \tag{27}$$
where
$$s^\top D_x G(t,x)u^\alpha(t,x,s) = = \left(s^\top D_{x_1}G(t,x)u^\alpha(t,x,s),\ldots,s^\top D_{x_n}G(t,x)u^\alpha(t,x,s)\right).$$

3.3 Properties of solutions of the characteristic system

We consider state components $x(t,\xi)$ of the solutions of the characteristic system (17) (hereinafter — state characteristics). We denote the set of the state characteristics, graphs of which $\{(t,x(t,\xi))\}$, $t \in [0,T]$, belong to the domain $\Phi(\delta)$ (7), by the symbol $X(\alpha,\delta)$.

The following statements are true:

Lemma 4. Let the conditions **A** be true in problem (1), (2), (13), then the constant $K_4 = K_4(\delta,\alpha) > 0$ exists such that the following relations for all characteristics (26), (27) satisfying inclusions $(t,x(t,\xi)) \in \Phi(\delta)$, $t \in [0,T]$ are true:
$$|x_i(t,\xi)| \le K_4, \ \left|\frac{dx_i(t,\xi)}{dt}\right| \le K_4, \ t \in [0,T], \ i \in \overline{1,n}; \tag{28}$$
$$|s_i(t,\xi)| \le K_4, \ \left|\frac{ds_i(t,\xi)}{dt}\right| \le K_4, \ t \in [0,T] \ i \in \overline{1,n}. \tag{29}$$

It is easy to prove the following statements.

Lemma 5. Let the conditions **A** be true in problem (1), (2), (13), then the extremal control $u^\alpha(t,x,s)$ (16) has the following properties:

• The functions $(t,x,s) \to u_i^\alpha(t,x,s)$, $i \in \overline{1,n}$ are continuous in $\Phi(\delta) \times S_\delta$, where
$$S_\delta = \{s \in \mathbb{R}^n \colon |s_i| \le K_2, \ i \in \overline{1,n}\}. \tag{30}$$

• for any $i \in \overline{1,n}$, $(t',x'),(t'',x'') \in \Phi(\delta)$, $s',s'' \in S_\delta$ the following estimates are true
$$|u_i^\alpha(t',x',s') - u_i^\alpha(t'',x'',s'')| \le \le \frac{1}{\alpha}\varphi(|t'-t''|,\|x'-x''\|,\|s'-s''\|), \tag{31}$$
where
$$\varphi(|t'-t''|,\|x'-x''\|,\|s'-s''\|) \to 0 \tag{32}$$
in case $|t'-t''| \to 0$, $\|x'-x''\| \to 0$, $\|s'-s''\| \to 0$.

3.4 A solution of the optimal feedback problem

We construct the optimal feedback (the optimal synthesis) $(t,x) \to u^0(t,x):\ \Phi(\delta) \to U$ for the problem (1), (2), (13). We introduce the set $X^0(\alpha,\delta)$ of optimal trajectories $x^0(\cdot)$ for the problem (1), (2), (13), generated by this synthesis and satisfied the condition
$$(t,x^0(t)) \in \Phi(\delta).$$
It follows from the condition **A** that the set $X^0(\alpha,\delta)$ is nonempty and compact in $C^n[0,T]$.

It follows from the definition of $\Phi(\delta)$ and the paper Subbotina [2006], this set $X^0(\alpha,\delta)$ is the subset of $X(\alpha,\delta)$ of all such state characteristics $x(\cdot,\xi)$ (17) that
$$(t,x(t,\xi)) \in \Phi(\delta).$$
We should finally pick such characteristics $x^\delta(\cdot) = x(\cdot,\xi) \in X^0(\alpha,\delta) \subset X(\alpha,\delta)$ and the realizations of extremal feedbacks $u_\delta^\alpha[t] = u^\alpha(t,x^\delta(t),s^\delta(t))$, generating them, which satisfy the relations:
$$I_{0,x^\delta(0)}(u_\delta^\alpha(\cdot)) = \min_{x(\cdot,\xi)\in X(\alpha,\delta)} I_{0,x(0)}(u^\alpha[\cdot]) = V(\alpha,\delta), \tag{33}$$
$$u^\alpha[\cdot] = u^\alpha(t,x(t,\xi),s(t,\xi)).$$
We prove that these characteristics $x^\delta(\cdot,\xi)$ and the realizations of extremal feedbacks $u_\delta^\alpha[t] = u^\delta(t,x^\delta(t),s^\delta(t))$, generating them, give the solution of the perturbed inverse problem of dynamics.

4. JUSTIFICATION FOR THE SOLUTION OF THE PERTURBED INVERSE PROBLEMS

4.1 The numerical method

We consider the procedure of numerical solution of characteristic system (17), (18). We denote the step of numerical integration of the characteristic system by h,
$$\Gamma = \{t_i = ih\}, \ i \in \overline{0,N(h)}, \ (N(h)+1)h = T.$$
Let $x_h(\cdot)$, $s_h(\cdot)$ be the numerical approximations of the solutions $x(t) = x(t,\xi)$, $s(t) = s(t,\xi)$ of the characteristic system (17), (18) with boundary conditions
$$x_h(T) = \xi, \ \xi \in \Phi_T(\delta), \ s_h(T) = 0$$
and with the restriction
$$(t,x(t)) \in \Phi(\delta), \ t \in [0,T].$$
The numerical solutions of the characteristic system (17), (18) are called the Euler polygonal paths.

Results of the works Subbotina, Tokmantsev [2011], Subbotina [2006] implies the following statement:

Lemma 6. Let $x_h(t)$, $s_h(t)$ be such numerical approximations of the accurate solutions of the characteristic system (17), (18), which approximate the accurate solutions $x(t)$, $s(t)$ such that $(t, x(t)) \in \Phi(\delta)$. Then such constants M_1, M_2, where $M_1 = M_1(\Phi(2\delta))$, $M_2 = M_2(\Phi(2\delta))$, exist that for any $t \in [0, T]$ the following estimates are true:

$$(t, x_h(t)) \in \Phi(2\delta),$$
$$\|x_h(t) - x(t)\| \le M_1 h,$$
$$\|s_h(t) - s(t))\| \le M_2 h,$$
$$M_i(\Phi(2\delta)) \le M_i(\Phi(2\delta_0)) \quad \forall \delta \in (0, \delta_0), \quad i = 1, 2.$$

4.2 The main result

We will consider the characteristics $x^\delta(\cdot)$ and the realizations of extremal feedbacks $u_\delta^\alpha[t] = u^\delta(t, x^\delta(t), s^\delta(t))$, generating them, which satisfy the condition (33). And let $x_h^\delta(\cdot)$, $u_h^\delta(\cdot)$ be the numerical approximations of the inverse problem of the dynamics (1) – (5), approximating the accurate solution $x^\delta(\cdot)$, $u_\delta^\alpha(\cdot)$.

We prove the following result.

Theorem 7. Let the conditions **A** be true in the perturbed inverse problems of dynamics (1) – (5), while the parameters of the problem $h = h(\delta) > 0$, $\alpha = \alpha(\delta) > 0$, $\delta > 0$ satisfy the conditions

$$\lim_{\delta\to 0} \frac{2}{\alpha}\left(\phi(\delta, h) + \rho(\delta, h) + \frac{T\delta^2}{2}\right) = 0, \tag{34}$$
$$\lim_{\delta\to 0} h(\delta) = 0, \ \lim_{\delta\to 0} \alpha(\delta) = 0,$$

then the following relations are true

$$\lim_{\delta\to 0} \|x_{h(\delta)}^\delta - x_*\|_C = 0, \lim_{\delta\to 0} \|u_{h(\delta)}^\delta - u_*\|_{L_2} = 0 \tag{35}$$

for the functions $x_h^\delta(\cdot)$, $u_h^\delta(\cdot)$.

Proof:

As the trajectory $x^\delta(\cdot)$ belongs to the domain $\Phi(\delta)$ of admissible motions and lemma 6 is true, we can estimate the deviation of the numerical approximation of the solution $x_{h(\delta)}^\delta(\cdot)$ from the real trajectory $x_*(\cdot)$.

$$\|x_h^\delta(t) - x_*(t)\| \le \|x_h^\delta(t) - x^\delta(t)\| + \|x^\delta(t) - x_*(t)\| \le$$
$$\le M_1 h + d_H(\Phi(\delta), \operatorname{gr} x_*(\cdot)).$$

If $h \to 0$, $\delta \to 0$, then

$$\|x_{h(\delta)}^\delta(\cdot) - x_*(\cdot)\|_C = \max_{t\in[0,T]} \|x_{h(\delta)}^\delta(t) - x_*(t)\| \le$$
$$\le M_1 h + d_H(\Phi(\delta), \operatorname{gr} x_*(\cdot)) \to 0. \tag{36}$$

We estimate the numerical approximation $\widetilde{V}3(\alpha, \delta, h)$ of the value $V(\alpha, \delta)$ (33)

$$\widetilde{V}(\alpha, \delta, h) = \int_0^T \frac{\|x_h^\delta(t) - y(t)\|^2}{2} + \frac{\alpha\|u_h^\delta(t)\|^2}{2} dt \le$$
$$\le \frac{1}{2}\int_0^T \|x_h^\delta(t) - x^\delta(t)\|^2 +$$
$$+2\|x_h^\delta(t) - x^\delta(t)\|\|x^\delta(t) - y(t)\|dt +$$
$$+\int_0^T \frac{\|x^\delta(t) - y(t)\|^2}{2} + \frac{\alpha\|u_h^\delta(t)\|^2}{2} dt.$$

Using inclusion $(t, x(t)) \in \Phi(\delta)$ and lemma 6, we get

$$\frac{1}{2}\int_0^T \|x_h^\delta(t) - x^\delta(t)\|^2 + 2\|x_h^\delta(t) - x(t)\|\|x^\delta(t) - y(t)\|dt$$
$$\le \frac{T(M_1^\delta h)^2}{2} + T(\delta + d_H(\Phi(\delta), \operatorname{gr} x_*(\cdot)))M_1 h.$$

We introduce the symbol

$$\phi(\delta, h) = TM_1 h\left(\frac{M_1 h}{2} + \delta + d_H(\Phi(\delta), \operatorname{gr} x_*(\cdot))\right)$$

and provide the relations

$$\widetilde{V}(\alpha, \delta, h) = \int_0^T \frac{\|x_h^\delta(t) - y(t)\|^2}{2} + \frac{\alpha\|u_h^\delta(t)\|^2}{2} dt \le$$
$$\le \phi(\delta, h) + \int_0^T \frac{\|x^\delta(t) - y(t)\|^2}{2} + \frac{\alpha\|u_h^\delta(t)\|^2}{2} dt.$$

Using estimates for ΔI_1, ΔI_2, we obtain

$$\frac{\alpha}{2}\int_0^T \|u_h^\delta(t)\|^2 \le \frac{\alpha}{2}\int_0^T \|u_\delta^\alpha[t]\|^2 dt +$$
$$+\frac{\alpha}{2}\int_0^T \|u_h^\delta(t) - u_\delta^\alpha[t]\|^2 dt,$$

where
$u_h^\delta(t) = u^\alpha(t_i, x_h^\delta(t_i), s_h^\delta(t_i))$, $u_\delta^\alpha[t] = u^\alpha(t, x_h^\delta(t), s_h^\delta(t))$,

$$\frac{\alpha}{2}\int\limits_0^T \|u_h^\delta(t)-u_\delta^\alpha[t]\|^2dt$$

$$\leq \frac{\alpha}{2}\sum_{i=0}^{N(h)}\int\limits_{ih}^{(i+1)h}\Big\|u^\alpha(t_i,x_h^\delta(t_i),s_h^\delta(t_i))$$

$$-u^\alpha(t,x_h^\delta(t),s_h^\delta(t))\Big\|^2dt$$

$$+\frac{\alpha}{2}\int\limits_0^T\left\|u^\alpha(t,x_h^\delta(t),s_h^\delta(t))-u^\alpha(t,x^\delta(t),s^\delta(t))\right\|^2dt$$

$$\leq \frac{\alpha}{2}\sum_{i=0}^{N(h)}\int\limits_{ih}^{(i+1)h}\Big\|u^\alpha(t_i,x_h^\delta(t_i),s_h^\delta(t_i))$$

$$-u^\alpha(t,x_h^\delta(t),s_h^\delta(t))\Big\|^2dt$$

$$+\frac{1}{2}\varphi(0,\|x_h^\delta(\cdot)-x^\delta(\cdot)\|_C,\|s_h^\delta(\cdot)-s^\delta(\cdot)\|_C)$$

$$\leq \frac{1}{2}T\varphi(h,K_4h,K_4h)+\frac{1}{2}\varphi(0,M_1h,M_2h)=\rho(h,\delta).$$

Finally, we have

$$\int\limits_0^T\frac{\|x_h^\delta(t)-y(t)\|^2}{2}+\frac{\alpha\|u_h^\delta(t)\|^2}{2}dt\leq$$

$$\leq \phi(\delta,h)+\rho(\delta,h)+\int\limits_0^T\frac{\|x^\delta(t)-y(t)\|^2}{2}+\frac{\alpha\|u_\delta^\alpha(t)\|^2}{2}dt.$$

The integral in the last expression coincides with the functional $I_{0,x^\delta(0)}(u_\delta^\alpha(\cdot))$ (33). Since the trajectory $x^\delta(\cdot)$ and the control $u_\delta^\alpha[\cdot]$ are the solution of the optimal control problem (1), (2), (13) and satisfy the condition (33), so the functional $I_{0,x^\delta(0)}(u_\delta^\alpha(\cdot))$ isn't greater then $I_{0,x_*(0)}(u_*(\cdot))$. So, the following inequalities are true

$$\int\limits_0^T\frac{\|x^\delta(t)-y(t)\|^2}{2}+\frac{\alpha\|u_\delta^\alpha[t]\|^2}{2}dt$$

$$\leq\int\limits_0^T\frac{\|x_*(t)-y(t)\|^2}{2}+$$

$$+\frac{\alpha}{2}\|u_*(t)\|^2dt\leq\frac{T\delta^2}{2}+\frac{\alpha}{2}\|u_*\|_{L_2}^2.$$

We gather the estimations and obtain

$$\widetilde{V}(\alpha,\delta,h)=\int\limits_0^T\frac{\|x_h^\delta(t)-y(t)\|^2}{2}+\frac{\alpha\|u_h^\delta(t)\|^2}{2}dt\leq$$

$$\leq\phi(\delta,h)+\rho(\delta,h)+\frac{T\delta^2}{2}+\frac{\alpha}{2}\|u_*\|_{L_2}^2.$$

We minorize the integral $\widetilde{V}(\alpha,\delta,h)$

$$\widetilde{V}(\alpha,\delta,h)=\int\limits_0^T\frac{\|x_h^\delta(t)-y(t)\|^2}{2}+\frac{\alpha\|u_h^\delta(t)\|^2}{2}dt\geq$$

$$\geq\int\limits_0^T\frac{\alpha\|u_h^\delta(t)\|^2}{2}dt=\frac{\alpha}{2}\|u_h^\delta\|_{L_2}^2.$$

As a result, we get

$$\|u_h^\delta\|_{L_2}^2\leq\frac{2}{\alpha}\left(\phi(\delta,h)+\rho(\delta,h)+\frac{T\delta^2}{2}\right)+\|u_*\|_{L_2}^2. \quad (37)$$

The collection of functions $u_h^\delta(\cdot)\in\mathcal{U}$, $h>0$, $\delta>0$ is bounded in L_2, so there exists such sequence $\delta_k\to 0, h_k=h(\delta_k)\to 0, k\to\infty$ that $u_k(\cdot)=u_{h_k}^{\delta_k}(\cdot)$ weakly converges to an element $v_*(\cdot)$ in L_2. We will show that $v_*(\cdot)\in\mathbb{U}$.

Let's prove that $v_*(\cdot)\in\mathbb{U}=\{u\in\mathcal{U}\colon x(t,u)=x_*(t)\}$.

$$x_k(t_{i+1})=x_k(t_i)-h_k[G(t_i,x_k(t_i))u_k(t_i)+f(t_i,x_k(t_i))],$$

$$x_k(t_{i+1})=x_k(T)-h_k\sum_{j=i}^{N}[G(t_j,x_k(t_j))u_k(t_j)+$$

$$+f(t_j,x_k(t_j))].$$

We pass to the limit as $k\to\infty$. Since $\|x_k(\cdot)-x_*\|_C\to 0$ and $u_k(\cdot)$ weakly converges to $v_*(\cdot)$ in L_2, , it follows from the lemma 1 and the equalities that $v_*(\cdot)\in\mathcal{U}$ and

$$x_*(t)=x_*(T)-\int\limits_t^T[G(\tau,x_*(\tau))v_*(\tau)+f(\tau,x_*(\tau))]d\tau, \quad (38)$$

$$t\in[0,T].$$

The equality (38) means that $v_*(\cdot)\in\mathbb{U}$.

According to weakly lower semicontinuity of the square of the norm in L_2, we obtain

$$\|v_*(\cdot)\|_{L_2}^2\leq\liminf_{k\to\infty}\|u_k(\cdot)\|_{L_2}^2. \quad (39)$$

From inequalities (37), (39) and from lemma (2), we get the following estimates

$$\|u_*(\cdot)\|_{L_2}^2\leq\|v_*(\cdot)\|_{L_2}^2\leq\liminf_{k\to\infty}\|u_k(\cdot)\|_{L_2}^2\leq$$

$$\leq\limsup_{k\to\infty}\|u_k(\cdot)\|_{L_2}^2\leq\|u_*(\cdot)\|_{L_2}^2.$$

Therefore,

$$\lim_{k\to\infty}\|u_k(\cdot)\|_{L_2}^2=\|u_*(\cdot)\|_{L_2}^2=\|v_*(\cdot)\|_{L_2}^2.$$

As the normal solution $u_*(\cdot)$ is unique in $\mathbb{U}$ (see lemma 2), then

$$u_*(\cdot)=v_*(\cdot).$$

So, the collection of functions $u_h^\delta(\cdot)$, $h>0$ has the only weak limit point $u_*(\cdot)$, moreover

$$\lim_{h\to 0}\|u_h^\delta(\cdot)\|_{L_2}=\|u_*(\cdot)\|_{L_2}.$$

Then, using the equality and the weak convergency of $u_k(\cdot)$ to $u_*(\cdot)$ in L_2, we finally get

$$\|u_h^\delta(\cdot)-u_*(\cdot)\|_{L_2}^2\leq$$

$$\leq\|u_h^\delta(\cdot)\|_{L_2}^2-2\langle u_h^\delta(\cdot),u_*(\cdot)\rangle+\|u_*(\cdot)\|_{L_2}^2\to 0,$$

as $h\to 0$.

□

5. EXAMPLE 1

Consider the system

$$\dot{x}_1 = x_2 + u_1, \quad \dot{x}_2 = u_2,$$

$u_1 \in [0, 1]$, $u_2 \in [-9, -10]$, $t \in [0, 10]$. Let the sampling history be the following one (free fall)

$$y_1(t) = -\frac{9.8t^2}{2} + 480, \quad y_2(t) = -9.8t, \quad t \in [0, 10].$$

We put $\delta = 0.01$, $h = 0.001$, $\alpha = 0.01$.

The picture Fig.1 shows the real trajectory $x_1^*(t) = y_1(t)$ (red color) and the reconstructed trajectory $x_{1h}^{\delta}(t)$ (blue color).

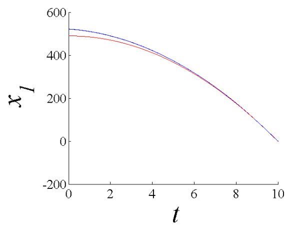

Fig 1.

The picture Fig.2 shows the reconstructed control $u_{2h}^{\delta}(t)$.

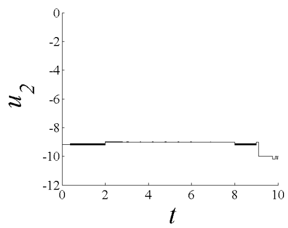

Fig 2.

We note that the reconstructed controls $u = (u_1(t), u_2(t))$ should be constant and $u_1(t) = 0$, $u_2(t) = -9.8$ because the given real trajectory $(y_1(t), y_2(t))$ is the trajectory of free fall.

6. EXAMPLE 2

Consider the system

$$\dot{x}_1 = x_2 + u_1, \quad \dot{x}_2 = -sin(x_1) + u_2,$$

$u_1 \in [0, 1]$, $u_2 \in [-0.5, 20]$, $t \in [0, 10]$. Let the sampling history be the following one (free fall)

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = -sin(y_1) + 9.8.$$

We put $\delta = 0.01$, $h = 0.0001$, $\alpha = 0.001$. The picture Fig.3 shows the real trajectory $x_2^*(t) = y_2(t)$.

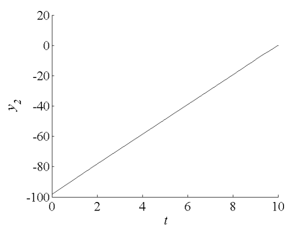

Fig 3.

The picture Fig.4 shows the difference between the reconstructed trajectory $x_{2h}^{\delta}(t)$ and $x_2^*(t) = y_2(t)$.

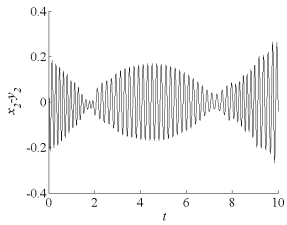

Fig 4.

The picture Fig.5 shows the reconstructed control $u_{2h}^{\delta}(t)$.

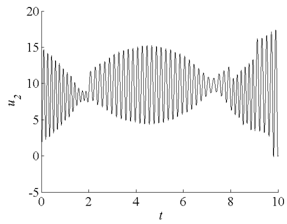

Fig 5.

7. CONCLUSION

In the paper the new method for solving inverse problems is suggested and justified. It can be useful in problems of modeling and identification of parameters of models in mechanics, engineering, economics, biology and so on, when statistic data is given.

The method will be developed to problems with more complete dynamics, to problems in higher dimension state space and to problems with uncomplete statistic data.

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