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On Extended LMI Conditions for $\mathcal{H}_2/\mathcal{H}_\infty$ Control of Discrete-Time Linear Systems ⋆

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Abstract: This paper provides new insights into the currently available extended linear matrix inequality (LMI) conditions for control of discrete-time linear systems, motivating the use of two intrinsically different extended LMI characterizations for $\mathcal{H}_2$ performance. While these conditions are equivalent for $\mathcal{H}_2$ analysis and many $\mathcal{H}_2$ control problems related to precisely known linear time-invariant (LTI) systems, they generally yield different results when employed for multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ control and $\mathcal{H}_2$ analysis and control of uncertain linear systems. The advantage of considering both the $\mathcal{H}_2$ LMIs for robust $\mathcal{H}_2$ state feedback and multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ control design is demonstrated by means of exhaustive numerical comparisons.

Keywords: linear systems, discrete-time systems, linear matrix inequalities, $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance, robust control, multi-objective control

1. INTRODUCTION

Based on the pioneering work of Lyapunov (1892), LMI formulations emerged for various important analysis and control problems related to linear systems, see Boyd et al. (1994); Scherer and Weiland (2004) for an overview. The main advantage of LMI formulations is that they are convex, and are therefore solved effectively using interior-point methods (Boyd and Vandenberghe (2004)). While some analysis and control problems are directly formulated in terms of LMIs, others remain hard to solve (Gahinet and Apkarian (1994); Scherer et al. (1997); Masubuchi et al. (1998); Apkarian and Adams (1998)). For example, even for precisely known LTI systems the existence of an LMI formulation for multi-objective dynamic output-feedback controller synthesis is unknown. In addition, robust analysis and control design problems for uncertain systems are NP-hard in general, requiring assumptions on the space where the uncertain parameters lie and the introduction of conservatism to arrive at tractable convex formulations.

So-called *extended* LMI conditions proved very useful to reduce conservatism in, amongst others, multi-objective control problems for LTI systems, robust analysis problems and robust state feedback design. The extended $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance characterizations proposed in de Oliveira et al. (2002) improved the conditions for multi-objective control that were proposed earlier in Scherer et al. (1997), and were also extended to reduce conservatism in robust analysis and state feedback design problems, see also de Oliveira et al. (1999). More recently, a more general approach for deriving extended LMIs based on the projection lemma (see Pipeleers et al. (2009)) opened the door to even further reduce conservatism in LMI conditions.

The contribution of this paper is to provide new insights into the currently available convex approaches for control of discrete-time linear systems. Specifically for control problems involving $\mathcal{H}_2$ performance, it is important to consider two extended LMI conditions to avoid introducing unnecessary conservatism (Paganini and Feron (2000)). Namely, as is also discussed in Pipeleers et al. (2009), extended LMI conditions for $\mathcal{H}_2$ performance analysis can be derived starting either from the controllability or the observability gramian based standard LMI formulation. In this paper, it is shown that the two resulting conditions generally provide different performance upper bounds when employed for, amongst others, multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ and robust $\mathcal{H}_2$ control design. Theoretical clarifications and several numerical comparisons demonstrate the significance of both the extended $\mathcal{H}_2$ LMIs.

The paper is organized as follows. First, standard and extended LMI characterizations for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance are reviewed in Section 2, and their intrinsic similarities and differences are discussed. Then, the different extended LMIs are exploited for robust $\mathcal{H}_2$ state feedback and multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ control design in Section 3, respectively Section 4. Finally, the conclusions follow in Section 5.

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Notation The set of real (symmetric) matrices of dimension $m \times n$ (n) is denoted by $\mathbb{R}^{m\times n}$ $(\mathbb{S}^n)$. The transpose of a matrix X is written as X^T, while $\mathrm{Tr}\{X\}$ denotes its trace. For a symmetric matrix X, positive definiteness is indicated by $X \succ 0$. A star $(\star)$ indicates symmetric terms in matrix inequalities.

2. $\mathcal{H}_2$ AND $\mathcal{H}_\infty$ PERFORMANCE CHARACTERIZATIONS

Standard and extended LMI characterizations for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance analysis are reviewed and compared in this section. It is shown that, while the considered $\mathcal{H}_\infty$ performance conditions are directly related through a congruence transformation, the $\mathcal{H}_2$ conditions feature an intrinsic difference.

Consider the finite-dimensional discrete-time multiple-input multiple-output LTI state-space realization

$$H : \begin{cases} x(k+1) = Ax(k) + Bw(k), \\ z(k) = Cx(k) + Dw(k), \end{cases} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ denotes the state, $w \in \mathbb{R}^{n_w}$ is the exogenous input and $z \in \mathbb{R}^{n_z}$ is the regulated output. k denotes discrete-time. It is well-known that system (1) is asymptotically stable if, and only if, there exists a symmetric matrix P such that the matrix inequalities

$$P \succ 0, \quad A^TPA - P \prec 0$$

hold. In addition to this stability condition, the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance of system (1), denoted by $\|H\|_2$, respectively $\|H\|_\infty$, are characterized below.

2.1 $\mathcal{H}_2$ performance

There are essentially two standard LMI characterizations for the $\mathcal{H}_2$ performance of system (1), relying on either the controllability or the observability gramian based formulation of the $\mathcal{H}_2$ norm. These two characterizations, which are directly related through the concept of duality, are presented in Lemma 1 and Lemma 2 (Scherer and Weiland (2004); Boyd et al. (1994)).

Lemma 1. $\|H\|_2 < \mu$ if, and only if, there exist matrices $P \in \mathbb{S}^{n_x}$ and $W \in \mathbb{S}^{n_z}$ such that $\mathrm{Tr}\{W\} < \mu^2$ and

$$\begin{bmatrix} P & \star & \star \\ A^TP & P & \star \\ B^TP & 0 & I \end{bmatrix} \succ 0, \quad \begin{bmatrix} W & \star & \star \\ C^T & P & \star \\ D^T & 0 & I \end{bmatrix} \succ 0. \tag{2}$$

Lemma 2. $\|H\|_2 < \mu$ if, and only if, there exist matrices $P \in \mathbb{S}^{n_x}$ and $W \in \mathbb{S}^{n_w}$ such that $\mathrm{Tr}\{W\} < \mu^2$ and

$$\begin{bmatrix} P & \star & \star \\ AP & P & \star \\ CP & 0 & I \end{bmatrix} \succ 0, \quad \begin{bmatrix} W & \star & \star \\ B & P & \star \\ D & 0 & I \end{bmatrix} \succ 0. \tag{3}$$

Applying the approach of Pipeleers et al. (2009) to the standard LMI conditions (2) and (3) yields the extended $\mathcal{H}_2$ analysis LMIs that are presented in Lemma 3 and Lemma 4, featuring an additional slack variable.

Lemma 3. $\|H\|_2 < \mu$ if, and only if, there exist matrices $P \in \mathbb{S}^{n_x}$, $G \in \mathbb{R}^{n_x\times n_x}$ and $W \in \mathbb{S}^{n_z}$ such that $\mathrm{Tr}\{W\} < \mu^2$ and

$$\begin{bmatrix} G+G^T-P & \star & \star \\ A^TG & P & \star \\ B^TG & 0 & I \end{bmatrix} \succ 0, \quad \begin{bmatrix} W & \star & \star \\ C^T & P & \star \\ D^T & 0 & I \end{bmatrix} \succ 0. \tag{4}$$

Lemma 4. $\|H\|_2 < \mu$ if, and only if, there exist matrices $P \in \mathbb{S}^{n_x}$, $G \in \mathbb{R}^{n_x\times n_x}$ and $W \in \mathbb{S}^{n_w}$ such that $\mathrm{Tr}\{W\} < \mu^2$ and

$$\begin{bmatrix} G+G^T-P & \star & \star \\ AG & P & \star \\ CG & 0 & I \end{bmatrix} \succ 0, \quad \begin{bmatrix} W & \star & \star \\ B & P & \star \\ D & 0 & I \end{bmatrix} \succ 0. \tag{5}$$

The conditions from Lemma 3 are equivalent to the extended LMIs presented in de Oliveira et al. (2002), which is shown by application of a congruence transformation and a nonlinear change of variables, see also the proof of Theorem 9 in Section 3. While the use of both Lemma 3 and Lemma 4 is beneficial for robust- and multi-objective control design, currently only Lemma 3 is used to derive conservative $\mathcal{H}_2$ upper bounds.

2.2 $\mathcal{H}_\infty$ performance

Similar as for $\mathcal{H}_2$ performance, the concept of duality can be used to derive two standard LMIs for $\mathcal{H}_\infty$ performance analysis, which are presented in Lemma 5 and Lemma 6. The $\mathcal{H}_\infty$ norm is interpreted as the worst-case induced ℓ_2-norm, defined as

$$\|H\|_\infty := \sup_{w(k)\neq 0} \frac{\|z(k)\|_2}{\|w(k)\|_2}.$$

Lemma 5. $\|H\|_\infty < \gamma$ if, and only if, there exists a matrix $P \in \mathbb{S}^{n_x}$ such that

$$\begin{bmatrix} P & \star & \star & \star \\ A^TP & P & \star & \star \\ B^TP & 0 & \gamma I & \star \\ 0 & C & D & \gamma I \end{bmatrix} \succ 0. \tag{6}$$

Lemma 6. $\|H\|_\infty < \gamma$ if, and only if, there exists a matrix $P \in \mathbb{S}^{n_x}$ such that

$$\begin{bmatrix} P & \star & \star & \star \\ AP & P & \star & \star \\ CP & 0 & \gamma I & \star \\ 0 & B^T & D^T & \gamma I \end{bmatrix} \succ 0. \tag{7}$$

Contrary to the two standard $\mathcal{H}_2$ LMI formulations that are presented in Lemma 1 and Lemma 2, the LMIs (6) and (7) are related by a congruence transformation involving only LMI variables, matrix P in Lemma 5 being the inverse of matrix P in Lemma 6. Following the approach in Pipeleers et al. (2009) results in the two extended LMIs for $\mathcal{H}_\infty$ analysis in Lemma 7 and Lemma 8.

Lemma 7. $\|H\|_\infty < \gamma$ if, and only if, there exist matrices $P \in \mathbb{S}^{n_x}$ and $G \in \mathbb{R}^{n_x\times n_x}$ such that

$$\begin{bmatrix} G+G^T-P & \star & \star & \star \\ A^TG & P & \star & \star \\ B^TG & 0 & \gamma I & \star \\ 0 & C & D & \gamma I \end{bmatrix} \succ 0. \tag{8}$$

Lemma 8. $\|H\|_\infty < \gamma$ if, and only if, there exist matrices $P \in \mathbb{S}^{n_x}$ and $G \in \mathbb{R}^{n_x\times n_x}$ such that

$$\begin{bmatrix} G+G^T-P & \star & \star & \star \\ AG & P & \star & \star \\ CG & 0 & \gamma I & \star \\ 0 & B^T & D^T & \gamma I \end{bmatrix} \succ 0. \tag{9}$$

Again, the LMIs (8) and (9) are related by a congruence transformation that solely depends on LMI variables. Namely, multiply (8) on the right by

$$\mathcal{Z} := \begin{bmatrix} 0 & G^{-1} & 0 & 0 \\ G^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix}$$

and on the left by $\mathcal{Z}^T$, and define the Lyapunov matrix and slack variable in Lemma 8 as $G^{-T}+G^{-1}-G^{-T}PG^{-1}$, respectively G^{-1}.

It is clear that (8) and (9) are related by a congruence transformation depending on P and G, in contrast to (4) and (5). Therefore, considering only one of the $\mathcal{H}_\infty$ conditions for multi-objective control or robust analysis/synthesis purposes suffices. On the other hand, a comparison between the LMI conditions presented in Lemma 3 and those in Lemma 4 reveals that the dimension of the slack variable W is different in each case. This observation, together with the fact that the slack variable G appears differently in the LMIs, explains that the two sets of conditions yield different results when used for multi-objective control or robust analysis/synthesis problems. Since the dimension of W is either equal to the input dimension n_w or the output dimension n_z, the conditions from Lemma 3 and Lemma 4 provide significantly different results when $|n_z - n_w|$ is large, which is confirmed in the next sections.

3. ROBUST STATE FEEDBACK DESIGN

The benefit of considering both the extended $\mathcal{H}_2$ LMIs (4) and (5) to derive robust $\mathcal{H}_2$ state feedback synthesis conditions is demonstrated in this section. An insightful comparison with known LMI approaches for different values of $|n_z - n_w|$ is provided.

Consider the uncertain linear system

$$\begin{cases} x(k+1) = A(\alpha)x(k) + B_w(\alpha)w(k) + B_u(\alpha)u(k), \\ z(k) = C(\alpha)x(k) + D_w(\alpha)w(k) + D_u(\alpha)u(k), \end{cases} \tag{10}$$

with state $x \in \mathbb{R}^{n_x}$, exogenous input $w \in \mathbb{R}^{n_w}$, control input $u \in \mathbb{R}^{n_u}$ and exogenous output $z \in \mathbb{R}^{n_z}$. All the system matrices are assumed to belong to a polytopic domain:

$$\left[\begin{array}{c|c|c} A(\alpha) & B_w(\alpha) & B_u(\alpha) \\ \hline C(\alpha) & D_w(\alpha) & D_u(\alpha) \end{array}\right] = \sum_{i=1}^{N} \alpha_i \left[\begin{array}{c|c|c} A_i & B_{w_i} & B_{u_i} \\ \hline C_i & D_{w_i} & D_{u_i} \end{array}\right]$$

where the vertices are known and α is a time-invariant uncertain parameter vector that takes values in a unit simplex

$$\Lambda_N = \{\xi \in \mathbb{R}^N \mid \sum_{i=1}^{N} \xi_i = 1,\ \xi_i \geq 0,\ i = 1, \ldots, N\}$$

of dimension $N \geq 2$. The objective is to compute a robust state feedback controller K that stabilizes system (10) for all $\alpha \in \Lambda_N$, such that the closed-loop system

$$H(\alpha) : \begin{cases} x(k+1) = (A(\alpha) + B_u(\alpha)K)x(k) + B_w(\alpha)w(k) \\ z(k) = (C(\alpha) + D_u(\alpha)K)x(k) + D_w(\alpha)w(k) \end{cases} \tag{11}$$

satisfies an $\mathcal{H}_2$ performance specification. The $\mathcal{H}_2$ performance of system (11) is defined as the worst-case $\mathcal{H}_2$ performance for all fixed values of $\alpha \in \Lambda_N$.

Based on Lemma 3, the following theorem provides sufficient LMI conditions for the computation of a robust state feedback controller for the system (10), such that the closed-loop system (11) satisfies an $\mathcal{H}_2$ performance specification. See de Oliveira et al. (2002) for equivalent conditions.

Theorem 9. (Robust $\mathcal{H}_2$ State Feedback). $\|H(\alpha)\|_2 < \mu$ if there exist matrices $P_i \in \mathbb{S}^{n_x}$ for $i = 1, \ldots, N$, $W \in \mathbb{S}^{n_z}$, $G \in \mathbb{R}^{n_x \times n_x}$ and $Z \in \mathbb{R}^{n_u \times n_x}$ such that $\mathrm{Tr}\{W\} < \mu^2$ and

$$\begin{bmatrix} G + G^T - P_i & \star & \star \\ (A_i G + B_{u_i} Z)^T & P_i & \star \\ B_{w_i}^T & 0 & I \end{bmatrix} \succ 0, \tag{12}$$

$$\begin{bmatrix} W & \star & \star \\ (C_i G + D_{u_i} Z)^T & P_i & \star \\ D_{w_i}^T & 0 & I \end{bmatrix} \succ 0, \tag{13}$$

for $i = 1, \ldots, N$. The corresponding robustly stabilizing state feedback gain is reconstructed as $K = ZG^{-1}$.

Proof. Assume that the LMI conditions (12) and (13) hold for $i = 1, \ldots, N$. Substitute $K = ZG^{-1}$ and multiply (12) by $\mathrm{diag}\{G^{-1}, G^{-1}, I\}$ on the right and its transpose on the left to obtain

$$\begin{bmatrix} \tilde{G} + \tilde{G}^T - \tilde{P}_i & \star & \star \\ (A_i + B_{u_i} K)^T \tilde{G} & \tilde{P}_i & \star \\ B_{w_i}^T \tilde{G} & 0 & I \end{bmatrix} \succ 0, \quad i = 1, \ldots, N, \tag{14}$$

with $\tilde{G} := G^{-1}$ and $\tilde{P}_i := G^{-T} P_i G^{-1}$ for $i = 1, \ldots, N$. Similarly, postmultiply the LMI (13) by $\mathrm{diag}\{I, G^{-1}, I\}$ and premultiply by its transpose to get

$$\begin{bmatrix} W & \star & \star \\ (C_i + D_{u_i} K)^T \tilde{G} & \tilde{P}_i & \star \\ D_{w_i}^T & 0 & I \end{bmatrix} \succ 0, \quad i = 1, \ldots, N. \tag{15}$$

Multiplying (14) and (15) by α_i, and summing the conditions for $i = 1, \ldots, N$ implies

$$\begin{bmatrix} \tilde{G} + \tilde{G}^T - \tilde{P}(\alpha) & \star & \star \\ (A(\alpha) + B_u(\alpha)K)^T \tilde{G} & \tilde{P}(\alpha) & \star \\ B_w(\alpha)^T \tilde{G} & 0 & I \end{bmatrix} \succ 0,$$

$$\begin{bmatrix} W & \star & \star \\ (C(\alpha) + D_u(\alpha)K)^T \tilde{G} & \tilde{P}(\alpha) & \star \\ D_w(\alpha)^T & 0 & I \end{bmatrix} \succ 0, \quad \forall \alpha \in \Lambda_N.$$

Applying Lemma 3 to the parameter-dependent system (11) finishes the proof.

In analogy with Lemma 4, alternative LMI conditions for robust state feedback $\mathcal{H}_2$ synthesis are presented in the following theorem.

Theorem 10. (Robust $\mathcal{H}_2$ State Feedback). $\|H(\alpha)\|_2 < \mu$ if there exist matrices $P_i \in \mathbb{S}^{n_x}$ for $i = 1, \ldots, N$, $W \in \mathbb{S}^{n_w}$, $G \in \mathbb{R}^{n_x \times n_x}$ and $Z \in \mathbb{R}^{n_u \times n_x}$ such that $\mathrm{Tr}\{W\} < \mu^2$ and

$$\begin{bmatrix} G + G^T - P_i & \star & \star \\ A_i G + B_{u_i} Z & P_i & \star \\ C_i G + D_{u_i} Z & 0 & I \end{bmatrix} \succ 0, \quad \begin{bmatrix} W & \star & \star \\ B_{w_i} & P_i & \star \\ D_{w_i} & 0 & I \end{bmatrix} \succ 0, \tag{16}$$

for $i = 1, \ldots, N$. The corresponding robustly stabilizing state feedback gain is reconstructed as $K = ZG^{-1}$.

Proof. Following similar steps as in the proof of Theorem 9 results in Lemma 4 applied to system (11).

3.1 Numerical Validation

Exhaustive numerical experiments are performed to compare Theorem 9 and Theorem 10. For each case $(n_w, n_z) \in$

Table 1. Number of times that T10 (T10P) performs better than T9 (T9P) and vice versa. The number of synthesized stabilizing state feedback gains is indicated between brackets.

(n_w, n_z)	T10	T9	T10P	T9P
$(1,5)$	96 (100)	4 (97)	96 (100)	4 (84)
$(2,4)$	88 (100)	12 (96)	88 (98)	10 (81)
$(3,3)$	51 (100)	49 (99)	61 (91)	30 (88)
$(4,2)$	30 (99)	69 (99)	52 (93)	41 (81)
$(5,1)$	10 (99)	90 (100)	31 (92)	61 (86)

$\{(1,5),(2,4),(3,3),(4,2),(5,1)\}$, 100 robustly stabilizable systems of the form (10) with $n_x = 4$ and two vertices ($N = 2$) are randomly generated using MATLAB. Subsequently, the following four approaches for robust state feedback design are compared:

- T10: Theorem 10.
- T9: Theorem 9.
- T10P: Theorem 10 with $P_i = P$, $i = 1, \dots, N$ and $G = G^T = P$ (Lyapunov shaping paradigm).
- T9P: Theorem 9 with $P_i = P$, $i = 1, \dots, N$ and $G = G^T = P$ (Lyapunov shaping paradigm).

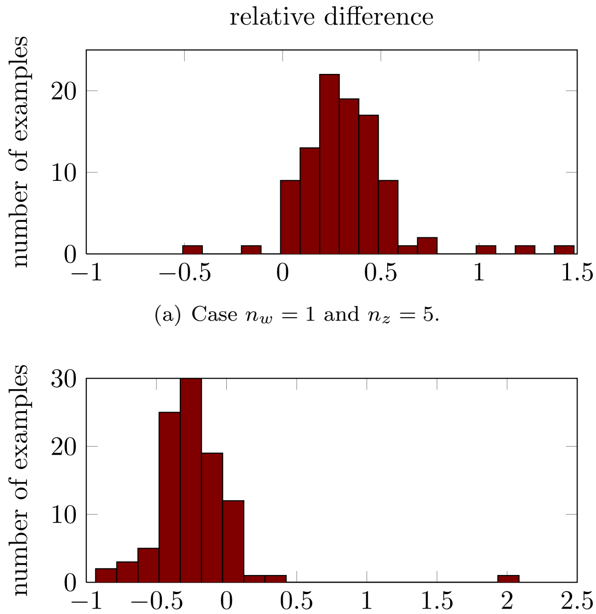
Yalmip (Löfberg (2004)) and SeDuMi (Sturm (1999)) are used to parse and solve the LMIs, respectively.

Table 1 gives an overview of the results. The first column shows the number of inputs n_w and outputs n_z, while the second (third) column shows the number of times that T10 (T9) results in lower $\mathcal{H}_2$ upper bounds than T9 (T10). In a similar fashion, the fourth (fifth) column displays how many times T10P (T9P) yields better results than T9P (T10P). The numbers between brackets indicate how many robustly stabilizing state feedback gains were found for the corresponding case. A clear relation between the results of T10/T9 and the number of inputs/outputs is revealed, confirming our expectations. If the number of outputs is large compared to the number of inputs, T10 usually results in lower $\mathcal{H}_2$ upper bounds, while T9 is beneficial for systems with a high input and a low output dimension. Overall T10 performed better, since T10 outperforms T9 more often than the other way around. Moreover, 498/500 robustly stabilizing controllers were found with T10 opposed to 491/500 with T9. In addition, looking at the cases $(n_w, n_z) = (1,5), (2,4)$ and comparing T9 with T10P, it is seen that T10P outperforms T9 in terms of the number of computed state feedback gains. The histograms in Figures 1(a)-(b) provide some extra insight for the two extremal cases $(n_w, n_z) = (1,5)$ and $(n_w, n_z) = (5,1)$.

Remark 11. While T10 outperforms T9 in some cases, the converse holds as well. Therefore, it is advisable to check both the $\mathcal{H}_2$ conditions T9 and T10, and select the best result.

4. MULTI-OBJECTIVE CONTROL

This section provides extended LMI conditions for multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ full-order dynamic output feedback LTI controller synthesis. The LMIs are derived using a well-known nonlinear transformation of variables, and serve as an extension to the approaches proposed in Scherer et al. (1997); de Oliveira et al. (2002). A comparison with these



(a) Case $n_w = 1$ and $n_z = 5$.

(b) Case $n_w = 5$ and $n_z = 1$.

Fig. 1. Each histogram shows the number of examples (out of 100) that lie in an interval of relative differences $(\mu_{T9} - \mu_{T10})/\min(\mu_{T9}, \mu_{T10})$, where μ_{T9} and μ_{T10} denote the $\mathcal{H}_2$ upper bound resulting from T9, respectively T10.

approaches is performed by means of a multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ controller design.

Consider the finite-dimensional discrete-time LTI system

$$\begin{cases} x(k+1) = Ax(k) + B_w w(k) + B_u u(k), \\ z(k) = C_z x(k) + D_{zw} w(k) + D_{zu} u(k), \\ y(k) = C_y x(k) + D_{yw} w(k). \end{cases} \tag{17}$$

with state $x \in \mathbb{R}^{n_x}$, exogenous input $w \in \mathbb{R}^{n_w}$, control input $u \in \mathbb{R}^{n_u}$, exogenous output $z \in \mathbb{R}^{n_z}$ and measurement output $y \in \mathbb{R}^{n_y}$. Interconnecting (17) with a dynamic output feedback controller

$$\begin{cases} x_c(k+1) = A_c x_c(k) + B_c y(k) \\ u(k) = C_c x_c(k) + D_c y(k) \end{cases} \tag{18}$$

with $x_c \in \mathbb{R}^{n_x}$ results in the closed-loop system

$$H_{cl} : \begin{cases} \tilde{x}(k+1) = \mathcal{A}\tilde{x}(k) + \mathcal{B}w(k), \\ z(k) = \mathcal{C}\tilde{x}(k) + \mathcal{D}w(k). \end{cases} \tag{19}$$

where $\tilde{x} = \begin{bmatrix} x^T & x_c^T \end{bmatrix}^T$ is a closed-loop state vector. The corresponding closed-loop matrices are explicitly calculated as

$$\mathcal{A} = \begin{bmatrix} A + B_u D_c C_y & B_u C_c \\ B_c C_y & A_c \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B_w + B_u D_c D_{yw} \\ B_c D_{yw} \end{bmatrix},$$
$$\mathcal{C} = \begin{bmatrix} C_z + D_{zu} D_c C_y & D_{zu} C_c \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} D_{zw} + D_{zu} D_c D_{yw} \end{bmatrix}. \tag{20}$$

Next, a nonlinear change of variables is described to jointly convexify the set of matrix inequalities

$$\begin{bmatrix} \mathcal{G} + \mathcal{G}^T - \mathcal{P} & \star & \star \\ \mathcal{A}\mathcal{G} & \mathcal{P} & \star \\ \mathcal{C}\mathcal{G} & 0 & I \end{bmatrix} \succ 0, \quad \begin{bmatrix} \mathcal{W} & \star & \star \\ \mathcal{B} & \mathcal{P} & \star \\ \mathcal{D} & 0 & I \end{bmatrix} \succ 0, \tag{21}$$

which is bilinear due to products between the slack variable $\mathcal{G}$ and the closed-loop matrices $\mathcal{A}$ and $\mathcal{C}$, that in turn affinely depend on the controller matrices. Note that the matrix inequalities (21) reduce to LMIs in the case of known system matrices, as in Lemma 4.

4.1 Nonlinear change of variables

First, $\mathcal{G}$ and its inverse are partitioned as

$$\mathcal{G} := \begin{bmatrix} X & ? \\ U & ? \end{bmatrix}, \quad \mathcal{G}^{-1} := \begin{bmatrix} Y^T & ? \\ V^T & ? \end{bmatrix},$$

where ? indicates that the corresponding blocks are unimportant for the change of variables. A transformation matrix similar as in Scherer et al. (1997); de Oliveira et al. (2002) is defined

$$\mathcal{T} := \begin{bmatrix} I & Y^T \\ 0 & V^T \end{bmatrix}.$$

Grouping the controller state-space matrices as

$$K := \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix}$$

and applying the nonlinear change of controller variables

$$\begin{bmatrix} \hat{A}_c & \hat{B}_c \\ \hat{C}_c & \hat{D}_c \end{bmatrix} := \begin{bmatrix} V & YB_u \\ 0 & I \end{bmatrix} K \begin{bmatrix} U & 0 \\ C_yX & I \end{bmatrix} + \begin{bmatrix} YAX & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} P & J \\ J^T & H \end{bmatrix} := \mathcal{T}^T \mathcal{P} \mathcal{T},$$

$$S := YX + VU,$$

the following identities are immediate:

$$\mathcal{T}' \mathcal{A} \mathcal{G} \mathcal{T} = \begin{bmatrix} AX + B_u\hat{C}_c & A + B_u\hat{D}_cC_y \\ \hat{A}_c & YA + \hat{B}_cC_y \end{bmatrix},$$

$$\mathcal{T}'\mathcal{B} = \begin{bmatrix} B_w + B_u\hat{D}_cD_{yw} \\ YB_w + \hat{B}_cD_{yw} \end{bmatrix},$$

$$\mathcal{C}\mathcal{G}\mathcal{T} = \begin{bmatrix} C_zX + D_{zu}\hat{C}_c & C_z + D_{zu}\hat{D}_cC_y \end{bmatrix},$$

$$\mathcal{D} = \begin{bmatrix} D_{zw} + D_{zu}\hat{D}_cD_{yw} \end{bmatrix},$$

$$\mathcal{T}'\mathcal{G}\mathcal{T} = \begin{bmatrix} X & I \\ S & Y \end{bmatrix}.$$

Using these identities, multiplying the first condition in (21) on the right by $\text{diag}\{\mathcal{T}, \mathcal{T}, I\}$ and on the left by its transpose, and the second condition in (21) on the right by $\text{diag}\{I, \mathcal{T}, I\}$ and on the left by its transpose, the extended $\mathcal{H}_2$ performance conditions (21) are rendered convex in terms of the transformed controller variables, which is the context of the following theorem.

Theorem 12. (Full-Order $\mathcal{H}_2$ Control). The system (17) is stabilizable by a dynamic output-feedback controller (18) with a guaranteed upper bound on the closed-loop $\mathcal{H}_2$ performance $\|H_{cl}\|_2 < \mu$ if, and only if, there exist matrices $P, H \in \mathbb{S}^{n_x}$, $\mathcal{W} \in \mathbb{S}^{n_z}$, $J, X, Y, S, \hat{A}_c \in \mathbb{R}^{n_x \times n_x}$, $\hat{B}_c \in \mathbb{R}^{n_x \times n_y}$, $\hat{C}_c \in \mathbb{R}^{n_u \times n_x}$ and $\hat{D}_c \in \mathbb{R}^{n_u \times n_y}$ such that $\text{Tr}\{\mathcal{W}\} < \mu^2$ and following LMIs hold

$$\begin{bmatrix} X + X^T - P & \star & \star & \star & \star \\ I + S - J^T & Y + Y^T - H & \star & \star & \star \\ AX + B_u\hat{C}_c & A + B_u\hat{D}_cC_y & P & \star & \star \\ \hat{A}_c & YA + \hat{B}_cC_y & J^T & H & \star \\ C_zX + D_{zu}\hat{C}_c & C_z + D_{zu}\hat{D}_cC_y & 0 & 0 & I \end{bmatrix} \succ 0, \tag{22}$$

$$\begin{bmatrix} \mathcal{W} & \star & \star & \star \\ B_w + B_u\hat{D}_cD_{yw} & P & \star & \star \\ YB_w + \hat{B}_cD_{yw} & J^T & H & \star \\ D_{zw} + D_{zu}\hat{D}_cD_{yw} & 0 & 0 & I \end{bmatrix} \succ 0. \tag{23}$$

An $\mathcal{H}_2$ dynamic output feedback controller is reconstructed from the LMI solution variables by inverting the nonlinear change of variables:

$$K = \begin{bmatrix} V^{-1} & -V^{-1}YB_u \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{A}_c - YAX & \hat{B}_c \\ \hat{C}_c & \hat{D}_c \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ -C_yXU^{-1} & I \end{bmatrix}.$$

While the extended $\mathcal{H}_2$ characterization of Theorem 12 is of little interest for single-objective control, it offers the possibility to address multi-objective synthesis problems in a less conservative fashion compared to the approaches of (Scherer et al. (1997)) and (de Oliveira et al. (2002)).

Consider the problem of finding a single controller of the form (18) for the system (17) subject to M $\mathcal{H}_2$ and/or $\mathcal{H}_\infty$ performance constraints, and suppose that each of these constraints is imposed on an input-output (I/O) channel $w_j \to z_j$ defined by $w_j = R_jw$ and $z_j = L_jz$, where L_j and R_j are selection matrices. Making the substitutions

$$B_w := B_wR_j, \quad C_z := L_jC_z,$$
$$D_{zw} := L_jD_{zw}R_j, \quad D_{zu} := L_jD_{zu}, \quad D_{yw} := D_{yw}R_j$$

in (17) results in the systems H_j and corresponding closed-loop systems $H_{cl,j}$ for $j = 1, \ldots, M$, each associated with an I/O channel. Imposing an $\mathcal{H}_2$ or $\mathcal{H}_\infty$ constraint on I/O channel j is done by substitution of the corresponding closed-loop matrices in the appropriate synthesis LMIs, see Scherer et al. (1997); de Oliveira et al. (2002) and Theorem 12. Obviously, the LMI variables necessary for reconstruction of a single controller, namely X, Y, S, $\hat{A}_c$, $\hat{B}_c$, $\hat{C}_c$ and $\hat{D}_c$, should be taken constant over all I/O channels. However, using a different Lyapunov matrix for each performance channel may considerably reduce conservatism. To this end, the symmetric matrix variables $\mathcal{P}_j$, $j = 1, \ldots, M$ are used instead of one single symmetric matrix $\mathcal{P}$.

4.2 Numerical Validation

The potential of Theorem 12 in multi-objective $\mathcal{H}_2/\mathcal{H}_\infty$ control design is illustrated by means of a numerical example, see Example 4 in de Oliveira et al. (2002).

Consider the LTI system

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0.5 & 0 \\ 0 & 1 & -0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} w(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k), \\ y(k) &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \end{bmatrix} w(k). \end{aligned} \tag{24}$$

where the elements of input vector w are defined by

$$w(k) = \begin{bmatrix} w_1(k) & w_2(k) & w_3(k) & w_{4x}(k) & w_{4y}(k) \end{bmatrix}^T.$$

The following exogenous outputs are added to the LTI system (24):

$$z_1(k) = [1 \ 0 \ 0]\, x(k), \quad z_2(k) = u(k),$$
$$z_3(k) = [0 \ 1 \ 0]\, x(k), \quad z_4(k) = \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}.$$

Subsequently, an $\mathcal{H}_\infty$ constraint is associated with each of the I/O pairs

$$w_j(k) \to z_j(k), \quad j = 1, 2, 3, \tag{25}$$

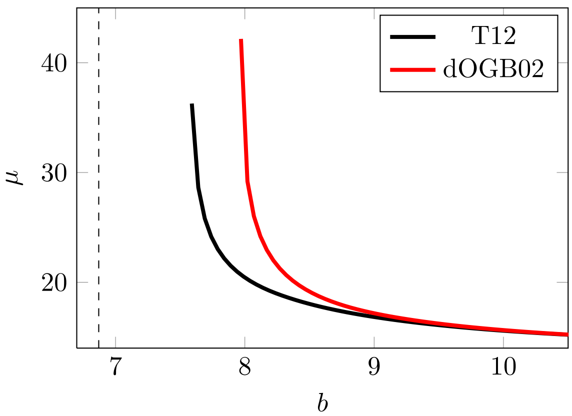


Fig. 2. Trade-off curves comparing the conservatism of the approaches T12 and dOGB02.

and an $\mathcal{H}_2$ performance specification is added to the I/O pair

$$w_4(k) := \begin{bmatrix} w_{4x}(k) \\ w_{4y}(k) \end{bmatrix} \rightarrow z_4(k). \qquad (26)$$

We are interested in the computation of a full-order dynamic output feedback controller (18), such that

$$\|H_{cl,j}\|_\infty < b, \quad j = 1, 2, 3,$$

for some predefined fixed value of $b > 0$, and moreover the $\mathcal{H}_2$ performance $\|H_{cl,4}\|_2$ is minimized. Using the LMI conditions in Theorem 12 and/or Scherer et al. (1997); de Oliveira et al. (2002), this problem is cast as a convex optimization problem, where an upper bound μ on the $\mathcal{H}_2$ performance is minimized. The following three approaches are compared:

- T12: Theorem 12.
- dOGB02: G shaping (de Oliveira et al. (2002)).
- SGC97: Lyapunov shaping (Scherer et al. (1997)).

For all these three approaches, an arbitrary large value b is selected as a starting point, and reduced until SeDuMi does not return a feasible solution anymore. As expected, both T12 and dOGB02 outperform the Lyapunov shaping paradigm, resulting in significantly lower upper bounds μ for any fixed value of b, and lower achievable values of b. However, a more interesting observation is that T12 provides less conservative results than dOGB02 in this case, which is confirmed by the trade-off curves between μ and b in Figure 2. Namely, for any fixed b T12 yields lower performance upper bounds μ compared to dOGB02. Note that theoretically speaking the same minimal value of b should be achieved, which is seen by neglecting the $\mathcal{H}_2$ performance LMI and realizing that there is only one significant $\mathcal{H}_\infty$ LMI. This minimum achievable b equals 6.87, indicated by the vertical dashed line. For $b \rightarrow \infty$ both approaches provide the same results, since the multi-objective problem then tends to a necessary and sufficient single-objective problem.

5. CONCLUSION

New insights into the currently available extended $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance LMIs for discrete-time LTI systems are presented in this paper, and the merits of using two intrinsically different $\mathcal{H}_2$ performance LMIs for robust $\mathcal{H}_2$ state feedback and multi-objective full-order dynamic output feedback control design are numerically demonstrated.

The incorporation of scalar parameters in LMI conditions (see Morais et al. (2013)) to even further reduce conservatism in control problems is currently under investigation.

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