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Connections between Optimal Control Problems and Generalized Solutions of PDEs of the First Order.

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Abstract: We introduced notions of generalized solutions of PDEs of the first order (Hamilton-Jacobi-Bellman equation and a quasi-linear hyperbolic system of the first order). Connections between the solutions and the value functions to optimal control problems are studied. We described properties of the solutions with the help of Cauchy characteristics method.

Keywords: optimal control, Hamilton-Jacobi-Bellman equation, minimax and viscosity solutions, Cauchy method of characteristics, quasi-linear hyperbolic systems of the first order, potential.

1. INTRODUCTION

We introduced notions of generalized solutions of PDEs of the first order (Hamilton-Jacobi-Bellman equation and a quasi-linear hyperbolic system of the first order). Connections between the solutions and the value functions to optimal control problems are studied. We described properties of the solutions with the help of Cauchy characteristics method.

2. THE HAMILTON-JACOBI-BELLMAN EQUATION THEORY

2.1 Statement of the boundary problem

We consider the boundary problem for Hamilton-Jacobi-Bellman equation

$$\varphi_t(t,x) + H(t,x,\varphi_x(t,x)) = 0, \qquad \varphi(T,x) = \sigma(x), \quad (1)$$

$$(t,x) \in [0,T] \times R^n = \Pi_T.$$

Here

$$\varphi_t(t,x) = \partial\varphi(t,x)/\partial t,$$

$$\varphi_x(t,x) = (\partial\varphi(t,x)/\partial x_1, \ldots, \partial\varphi(t,x)/\partial x_n).$$

We consider the boundary problem (1) under the following assumptions about the input data $H(t,x,s)$ and $\sigma(x)$:

A1 the Hamiltonian $H(t,x,s)$ is twice continuously differentiable in all variables, at any $(t,x,s) \in (0,T) \times R^n \times R^n$, the function $s \to H(t,x,s)$ is concave for all $(t,x) \in \Pi_T$;

A2 the boundary function $\sigma(x)$ is continuously differentiable;

A3 the functions $\partial H(t,x,s)/\partial x_i$, $\partial H(t,x,s)/\partial s_j$, $i,j \in \overline{1,n}$, possess sublinear growth in x and s, with a constant $K > 0$.

It is well known (see, for example, Subbotin (1995)) that the classical, smooth solution of problem (1) can be obtained with the help of the Cauchy method of characteristics.

Consider the characteristic system for problem (1)

$$\dot{\tilde{x}} = H_s(t,\tilde{x},\tilde{s}), \quad \dot{\tilde{s}} = -H_x(t,\tilde{x},\tilde{s}), \quad (2)$$

$$\dot{\tilde{z}} = \langle \tilde{s}, H_s(t,\tilde{x},\tilde{s})\rangle - H(t,\tilde{x},\tilde{s}),$$

and the boundary conditions

$$\tilde{x}(T) = \xi,\ \tilde{s}(T) = \sigma_x(\xi),\ \tilde{z}(T) = \sigma(\xi),\ \xi \in R^n. \quad (3)$$

Here

$$H_s(t,\tilde{x},\tilde{s}) = (\partial H/\partial s_1, \ldots, \partial H/\partial s_n),$$

$$H_x(t,\tilde{x},\tilde{s}) = (\partial H/\partial x_1, \ldots, \partial H/\partial x_n),$$

$$\sigma_x(\xi) = (\partial\sigma(\xi)/\partial x_1, \ldots, \partial\sigma(\xi)/\partial x_n),$$

$\xi \in R^n$ is n-dimensional parameter.

Conditions $A1$-$A3$ provide existence, uniqueness and extendability on the interval $[0,T]$ for trajectories $\tilde{x}(\cdot,\xi)$, $\tilde{s}(\cdot,\xi)$, $\tilde{z}(\cdot,\xi)$, of system (2)-(3), for each $\xi \in R^n$. The trajectories of system $(2)-(3)$ are called the characteristics of problem (1).

The Cauchy method of characteristics provides the construction of the classical solution of problem (1) only in domains where the phase components $\tilde{x}(\cdot,\xi)$ of characteristics $(2)-(3)$ are not intersecting.

If the input data $H(t,x,s)$, $\sigma(x)$ are smooth, but nonlinear, then the phase components $\tilde{x}(\cdot,\xi)$ of characteristics (2) – (3) can be intersected close to smooth manifold (3). The problem (1) under assumptions $A1$–$A3$ doesn't have the global classical solution in Π_T, as a rule.

2.2 Generalized solutions of the Hamilton-Jacobi-Bellman equations

Let us remember the following notions of nonsmooth analysis and viability theory (see, for example, Clarke (1983), Subbotin (1995)) to consider notions of generalized solutions to problem (1) in the whole strip Π_T.

Definition 1. A set $\partial^+\varphi(t,x) \in R^{n+1}$ is called *the superdifferential* of the function $\varphi(\cdot,\cdot):\Pi_T \to R$ at the point $(t,x)\in(0,T)\times R^n$ if, for some $\varepsilon>0$, it satisfies the relations

$$\partial^+\varphi(t,x) = \{(\alpha,p)\in R^{n+1}: \quad \forall|\delta t|+|\delta x|\le\varepsilon,$$
$$\varphi(t+\delta t, x+\delta x)-\varphi(t,x)\le \alpha\delta t+\langle p,\delta x\rangle + o(|\delta t|+|\delta x|)\},$$

where $o(|\delta t|+|\delta x|)/(|\delta t|+|\delta x|)\to 0$, as $|\delta t|+|\delta x|\to 0$.

Definition 2. The graph of a continuous function $(t,x)\to\varphi(t,x):\ \Pi_T\to R$ is weakly invariant relative to system (2) – (3) if for any $(t_0,x_0)\in\Pi_T$ there exists $\xi\in R$, such, that solutions $\tilde{x}(\cdot,\xi),\tilde{z}(\cdot,\xi)$ of (2) – (3) satisfy the equalities

$$\tilde{x}(t_0,\xi)=x_0,\ \tilde{z}(t_0,\xi)=\varphi(t_0,x_0);$$
$$\tilde{z}(t,\xi)=\varphi(t,\tilde{x}(t,\xi)),\quad \forall t\in[t_0,T].$$

Consider the following notion of a generalized solution to problem (1) in Π_T.

Definition 3. A continuous function $(t,x)\to\varphi(t,x)$: $\Pi_T \to R$ is called the global generalized solution of problem (1) in Π_T if

• the boundary condition

$$\varphi(T,x)=\sigma(x),\quad \forall x\in R^n$$

is satisfied;

• for all $(t,x)\in(0,T)\times R^n$ the superdifferential $\partial^+\varphi(t,x)$ is not empty and bounded, i.e.

$$\partial^+\varphi(t,x)\ne\emptyset,$$

there exists a continuous function $(0,T)\times R^n\ni(t,x)\to R(t,x)\in(0,\infty)$ such, that

$$||h||\le R(t,x)\ \forall h=(\alpha,s)\in\partial^+\varphi(t,x);$$

• the graph of $\varphi(t,x)$ is weakly invariant relative to system (2) – (3).

The symbol $||h||$ denotes the Euclidean norm of the vector $h\in R^n$.

It is proven (see, Subbotina (2009), Kolpakova (2010)), that definition 3 is equivalent to the well-known definitions of minimax (Subbotin (1995)), and viscosity (Crandall, Lions (1983)) solutions to problem (1). The equivalence and results of the theory of minimax and viscosity solutions to the Hamilton-Jacobi equations imply the following assertion.

Theorem 1. If assumptions $A1$–$A3$ are true, then there exists and unique the function $\varphi(\cdot,\cdot):\Pi_T\mapsto R$ satisfying definition 3 of the global generalized solution to problem (1) in Π_T.

3. AUXILIARY OPTIMAL CONTROL PROBLEMS

We can consider the auxiliary optimal control problems with the Bolza pay-off functional, which are actual to the optimal control theory and applications. See, Bellman (1957), Pontryagin, etc (1962), Krasovskii (1968), Warga (1977), Bardi (1997).

We consider controlling systems of the form

$$\dot{x}(t)=H_p(t,x,p),\quad (t,x)\in\Omega\subset\Pi_T, \tag{4}$$

$$x(t_0)=x_0,\quad (t_0,x_0)\in\Omega.$$

Here Ω is a compact set, such, that

$\Omega\supset\{(t,x(t)):\ t\in[t_0,T],||x(t)||\le||x_0||\exp K(t-t_0)\}$,

the state $x\in R^n$, values of controls p belong to the compact set $P=\{p\in R^m: ||p||\le \max\limits_{(t,x)\in\Omega} R(t,x)\}$.

We put the set $P_{[t_0,T]}$ of admissible open-loop controls equal to the set of all measurable functions $p(\cdot)\colon[t_0,T]\mapsto P$.

We estimate quality of an open-loop control at the initial state $(t_0,x_0)\in[0,T]\times R^n$ with the help of the Bolza pay-off functional:

$$I_{t_0,\ x_0}(p(\cdot))=\sigma(x(T))+$$
$$+\int\limits_{t_0}^{T}\langle p(t),H_p(t,x(t),p(t))\rangle - H(t,x(t),p(t)),dt, \tag{5}$$

where symbol $\langle p,H_p\rangle$ denotes the inner product of vectors p and H_p, $x(\cdot)=x(\cdot;t_0,x_0,p(\cdot))\colon[t_0,T]\mapsto R^n$ is a trajectory of system (4) started at the state $x(t_0)=x_0$ and generated by an admissible control $p(\cdot)\in P_{[t_0,T]}$.

The optimal result $\varphi(t_0,x_0)$ at the initial state $(t_0,x_0)\in\Pi_T$ is defined by the equality:

$$\varphi(t_0,x_0)=\inf_{p(\cdot)\in P_{[t_0,T]}} I_{t_0,\ x_0}(p(\cdot)). \tag{6}$$

The function

$$(t_0,x_0)\mapsto\varphi(t_0,x_0):\ \Pi_T\mapsto R$$

is called the value function of optimal control problem (4)–(6) in strip $\Pi_T\ni(t_0,x_0)$.

The value function takes important place in solving problems (4)–(5). According to (6), the function estimates possibilities of controls at each initial state $(t_0,x_0)\in\Pi_T$. Moreover, the value function is a key element of constructions of optimal open-loop and closed-loop controls. See, for example, Pontryagin, etc (1962), Krasovskii (1968), Subbotin (1995).

The value function is nonsmooth, as a rule. The following assertion is true

Theorem 2. If assumptions $A1$–$A3$ are true, then the function $\varphi(\cdot,\cdot):\Pi_T\mapsto R$ satisfying definition 3 of the global generalized solution to problem (1) in Π_T coincides with the value function of problem (4) – (6) in domain Ω.

3.1 Properties of the generalized solution $\varphi(t,x)$

The following properties of the value function are obtained in the framework of the theory of minimax and viscosity solutions to problem (1) (see, Crandall, Lions (1983), Subbotin (1995), Subbotina (2006), Bardi (1997), Evans (1998)).

Theorem 3. If assumptions $A1$–$A3$ are valid in problem (1), then the generalized solution $\varphi(t,x)$ has the properties:

- the generalized solution $\varphi(t,x)$ is locally Lipshitz continuous;
- for any $(t,x) \in \Pi_T$, the representative formula
$$\varphi(t,x) = \min_{\xi \in R^n} \{\tilde{z}(t,\xi) : \tilde{x}(t,\xi) = x\}, \tag{7}$$
is true;
- for any $(t,x) \in (0,T) \times R^n$, the superdifferential $D^+\varphi(t,x)$ is not empty and has the form
$$D^+\varphi(t,x) = \mathrm{co}\{(H(t,x,\tilde{s}(t,\xi)), \tilde{s}(t,\xi)) : \tilde{x}(t,\xi) = x, \tag{8}$$
$$\varphi(t,x) = \tilde{z}(t,\xi)\}.$$

Here $\tilde{x}(\cdot,\xi), \tilde{s}(\cdot,\xi), \tilde{z}(\cdot,\xi)$ are solutions of the characteristic system (2)–(3).

Definition 4. The set $Q \subset \Pi_T$ is called the singular set of the generalized solution $\varphi(\cdot,\cdot)$ of problem (1) iff it contains all points $(t,x) \in \Pi_T$ where the function $\varphi(\cdot,\cdot)$ is not differentiable.

According to theorem 3 and the Rademacher's theorem, the local Llipshitz continuous value function is differentiable almost everywhere.

Let $Q_{t'} = \{x \in R^n : \ (t',x) \in Q\}, \qquad t' \in [0,T].$

Definition 2–4, and uniqueness of solutions of the Hamiltonian characteristic system (2)–(3), for any parameter $\xi \in R^n$, imply that the singular set Q of the generalized solution $\varphi(\cdot,\cdot)$ has the following properties.

Theorem 4. Let assumptions $A1$–$A3$ be true. A point $x \in Q_{t'}$, iff there exist such parameters
$$\xi_1 \in R^n, \quad \xi_2 \in R^n, \qquad \xi_1 \neq \xi_2,$$
that
$$\tilde{x}(t',\xi_1) = \tilde{x}(t',\xi_2) = x, \ \tilde{z}(t',\xi_1) = \tilde{z}(t',\xi_2) = \varphi(t',x); \tag{9}$$
$$\tilde{s}(t',\xi_1) \neq \tilde{s}(t',\xi_2), \tag{10}$$
$$\tilde{z}(t,\xi_1) = \varphi(t,\tilde{x}(t,\xi_1), \quad \tilde{z}(t,\xi_2) = \varphi(t,\tilde{x}(t,\xi_2), \tag{11}$$
$$\tilde{s}(t,\xi_1) \in \partial_x^+\varphi(t,\tilde{x}(t,\xi_1), \quad \tilde{s}(t,\xi_2) \in \partial_x^+\varphi(t,\tilde{x}(t,\xi_2), \tag{12}$$
for all $t \in [t',T]$.

The symbol $\partial_x^+\varphi(t,x)$ denotes the projection of the set $\partial^+\varphi(t,x)$ to the space $R^n \ni p$, namely,
$$\partial_x^+\varphi(t,x) = \{p \in R^n : \exists\, \alpha \in R \quad (\alpha,p) \in \partial^+\varphi(t,x)\}.$$

4. APPLICATIONS OF THE THEORY OF QUASILINEAR EQUATIONS

4.1 The initial value problem for a system of quasilinear PDEs

Consider the initial value problem for the system of quasilinear equations of the first order
$$\frac{\partial w}{\partial \tau} - H_x(\tau,x,w) = 0, \quad w(0,x) = \nabla\sigma(x), \tag{13}$$
$$\tau \in [0,T], \ x \in R^n.$$
Here vector-function $w : \Pi_T \to R^n$,
$$H_x = (H_{x_1}, \ldots, H_{x_n}),$$
$$\nabla\sigma(x) = (\partial\sigma(x)/\partial x_1, \ldots, \partial\sigma(x)/\partial x_n).$$
The input data of problem (13) can be considered as a result of differentiation in x of the input data for problem (1), and after the time transformation
$$\tau = T - t, \quad t \in [0,T].$$

It is well known, that problem (13) doesn't have a continuous global solution, as a rule.

We introduce the following notion.

Definition 5. A multivalued function $W(\cdot,\cdot) : \ \Pi_T \to 2^{R^{n+1}}$ is called *the global generalized solution* of problem (13) in stripe Π_T, if $W(0,x) = \nabla\sigma(x)$,

B1 $\forall (t,x) \in \Pi_T \ W(t,x) \neq \emptyset, \ W(t,x)$ is compact;
B2 the graph of the map $(t,x) \to W(t,x)$ is closed in $\Pi_T \times R^{n+1}$;
B3 any measurable selector $w(t,x) \in W(t,x)$ satisfy system 13 almost everywhere in Π_T;
B4 for any continuously differentiable $C \subset \Pi_T$ and any measurable selector $w(t,x) \in W(t,x)$ the equality
$$\oint_C H(\tau,x,w(\tau,x))d\tau - \langle w(\tau,x), dx\rangle = 0$$
is true.

The following theorem is proven.

Theorem 5. If assumptions $A1$–$A3$ be true, then the generalized solution w exists.

Let us show a link between the global generalized solution of the problem (13) satisfying definition 5 and the minimax or viscosity solution of problem 1.

Theorem 6. Let w be a measurable selector of the generalized solution W of the problem (13) then the integral
$$I = \int_{0,a}^{t,x} H(\tau,x,w(\tau,x))d\tau - \langle w(\tau,x), dx\rangle \tag{14}$$
doesn't depend of path of integrating $x(t)$, containing point $(0,a)$ and (t,x). And function $v(t,x) = I$ defining in (14) coincides with the minimax or viscosity solution of the problem (1) up to a constant.

Note, that the superdifferential of the minimax solution $D^+u(t,x)$ satisfies the definition 5.

Claim 7. For any point (t,x) there exists at least one $\tilde{x}(\cdot,\xi)$ such that $\tilde{x}(t,\xi) = x$.

The domain of definition for function $w \in BV_{loc}$ is described in Dafrermos (2005):

Theorem 8. If function $w \in BV_{loc}$ then the domain of this function is the union of three, pairwise disjoint, subsets C, J and I with the following properties:

- C is the set of points of approximate continuity of w.

- J is the set of points of approximate jump discontinuity of w. Moreover, J is countably rectifiable, i.e. it is essentially covered by the countable union of C^1 $(k-1)$-dimensional manifolds F_i embedded in R^n.
- I is the set of irregular points of w; its $(k-1)$-dimensional Hausdorff measure is zero.

5. ON STRUCTURE OF THE VALUE FUNCTION.

We have considered the value function $\varphi(t, x)$ of optimal control problem (4)–(6) as the unique minimax and viscosity solution of problem (1) in strip Π_T, according to theorem 1. We have provided a description of the singular set Q of the value function in theorems 3 and 4. The set contains all points (t, x) where the nonempty superdifferential $\partial^+\varphi(t, x)$ is not a singleton.

In the case of n-dimensional state space, we have considered a multivalued function $w(\tau, x)$ defined by $\partial^+\varphi(t, x)$. The singular set Q of the value function $\varphi(t, x)$ in optimal control problem (4)–(6) coincides with the set all points $(t, x) \in \Pi_T$, where the function $w(\tau, x)$ is maultivalued.

Theorem 8 provides the following new properties of the singular set Q.

Theorem 9. If assumptions $A1$-$A3$ are valid in optimal control problem (4)–(6) and the function $w \in BV_{loc}$, then the singular set Q of the value function $\varphi(t, x)$ is covered by the countable union of C^1 $(k-1)$-dimensional manifolds F_i embedded in R^n

6. CONCLUSION

In the paper new notions of generalized solutions of PDEs of the first order are introduced. Existence of the solutions to boundary problems is established.

We have studied connections between the notions and the notion of the value function to the auxiliary optimal control problem. The paper deals with structure of the value function in optimal control problem with the Bolza pay-off functional. We considered the set of points where the value function is not differentiable. The description of the set is important to constructions of optimal feedbacks.

The presented link between generalized solutions to boundary problems for the Hamilton-Jacobi-Bellman equation and the system of quasi-linear PDEs of the first order can be useful to study and construct solutions of optimal control problems and one dimensional conservations laws, too (see Kolpakova (2010)).

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