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# Construction of Lyapunov Functionals for Networks of Coupled Delay Differential and Continuous-Time Difference Equations

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**Abstract:** To address various types of delays including the neutral-type arising in dynamical networks, this paper deals with coupled delay differential and continuous-time difference equations and develops stability and robustness criteria. Subsystems described by differential equations are not required to be input-to-state stable. No assumptions on network topology are made. To tackle networks in such a general formulation, this paper explicitly constructs Lyapunov-type functionals establishing stability and robustness of the overall networks. The construction requires only simple characterizations of subsystems in terms of inequalities with Lyapunov functions and instantaneous norms.

Keywords: Time delay; Nonlinear systems; Large-scale systems; Integral input-to-state stability; Lyapunov methods.

## 1. INTRODUCTION

In many applications, we encounter models of neutral functional differential equations, i.e., differential equations involving delay in the derivative of state variables (Hale and Lunel [1993]). A typical example is the lossless transmission line (Brayton [1967]). Overviews of other examples are given in Hale and Lunel [1993], Niculescu [2001], Rasvan [2006]. To verify stability and robustness of such systems of large size, this paper considers networks of coupled delay differential and continuous-time difference equations. The class of networks includes neutral systems in Hale's form as a special case. In addition, the class this paper considers allows subsystems and their interaction to have delays. Thus, time delay can appear in both state variables and the derivative of state variables of the networks.

Neutral systems are often formulated into equations in Hale's form, and stability has been addressed for such equations (e.g. Hale and Lunel [1993], Kolmanovskii and Myshkis [1999], Niculescu [2001]). For robustness with respect to disturbance, input-to-state stability (ISS) is studied in Pepe [2007] in the case of linear difference operators, and in Pepe et al. [2008] in the case of nonlinear difference operators. The results in Pepe [2007], Pepe et al. [2008] are definition of ISS Lyapunov-Krasovskii functionals that ensure ISS of the system described by coupled delay differential and difference equations, and they provide a framework which is parallel to ISS Lyapunov functions introduced originally to delay-free systems in Sontag and Wang [1995]. Then two questions arise: 1) How can we find such an ISS Lyapunov-Krasovskii functional? 2) Is it possible to extend the idea to integral input-to-state stability (iISS)? This paper answers these two questions for networks.

The notion of iISS describes how robust a system is with respect to disturbance similarly to ISS (Angeli et al. [2000]). In contrast to ISS, the iISS property does not require bounded magnitude of the state even for an input of bounded magnitude. Such unboundedness is often inevitable due to saturation and limitations in applications. For interconnections of delay free-systems, iISS have been studied extensively (e.g. Arcak et al. [2002], Chaillet and Angeli [2008], Panteley and Loría [2001], Ito [2006, 2010] to name a few). Input-to-output stability of coupled delay differential and difference equations has been studied in Karafyllis et al. [2009], Karafyllis and Jiang [2011] without constructing Lyapunov-Krasovskii-type functionals. Although the result is based on a trajectory-based small-gain theorem in the spirit of the ISS argument in Jiang et al. [1994], it allows a transient period during which the solutions do not satisfy input-to-output inequalities.

To address robustness of networks consisting of coupled equations which are not necessarily ISS, this paper pursues explicit construction of iISS Lyapunov functions directly from only information of dissipation inequalities. We impose less stable properties on subsystems than the existing literature. The development includes ISS as a special case.

Throughout this paper, the symbol $|\cdot|$ denotes the Euclidean norm of a real vector in $\mathbb{R}^n$ of a compatible dimension $n$. Let $\mathbb{R}_+ = [0, +\infty)$. For a measurable and essentially bounded function $u : \mathcal{S} \subset \mathbb{R} \to \mathbb{R}^n$, we use $\|u\|_\infty = \operatorname{ess\,sup}_{t\in\mathcal{S}} |u(t)|$. The space of continuous functions mapping $\mathcal{S}$ into $\mathbb{R}^n$ is denoted by $\mathcal{C}_n\mathcal{S}$. By $u_{\mathcal{S}}$ it is meant that $u_{\mathcal{S}}(t) = u(t)$ for all $t \in \mathcal{S}$ and $= 0$ elsewhere. Given a compact set $\mathcal{S} \subset \mathbb{R}$, a function $u : \mathcal{S} \to \mathbb{R}^n$ is said to be piece-wise continuous if it is bounded, right-continuous, continuous except, possibly, at a finite number of points in $\mathcal{S}$. The space of piece-wise continuous functions mapping $\mathcal{S}$ into $\mathbb{R}^n$ is denoted by $\mathcal{PC}_n\mathcal{S}$. A function $u : \mathbb{R}_+ \to \mathbb{R}^n$ is said to be piece-wise continuous if it is piece-wise continuous in $[0, T]$ for any $T \in (0, \infty)$. For a function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$, we write $\omega \in \mathcal{P}$ if it is continuous and satisfies $\omega(0) = 0$, $\omega(s) > 0$ for all $s > 0$. For a function $\omega \in \mathcal{P}$, we write $\omega \in \mathcal{J}$ if it is non-decreasing. A function $\omega \in \mathcal{J}$ is said to be of class $\mathcal{K}$ if it is strictly increasing. A function $\omega \in \mathcal{K}$ is said to be of class $\mathcal{K}_\infty$ if it is unbounded. A function $\beta : \mathbb{R}^2_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if for each fixed $t$ the function $s \mapsto \beta(s, t)$ is of class $\mathcal{K}$ and for each fixed $s$ the function $t \mapsto \beta(s, t)$ is non-increasing and goes to zero as $t \to +\infty$. Composition of functions $\gamma_1, \gamma_2 : \mathbb{R}_+ \to \mathbb{R}_+$ is written as $\gamma_1 \circ \gamma_2$. For brevity, we adopt a nonstandard symbol for repeated composition as $\bigodot_{i=1}^{n} \gamma_i = \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_n$. If $\gamma$ is a class $\mathcal{K}_\infty$ function, its inverse $\gamma^{-1}$ is of class $\mathcal{K}_\infty$. For $\gamma \in \mathcal{K} \setminus \mathcal{K}_\infty$, its inverse $\gamma^{-1}$ is defined on the finite interval $[0, \lim_{\tau\to\infty} \gamma(\tau))$. For $\gamma \in \mathcal{K}$, an operator $\gamma^{\ominus}$: $\overline{\mathbb{R}}_+ := [0, \infty] \to \overline{\mathbb{R}}_+$ is defined as $\gamma^{\ominus}(s) = \sup\{v \in \mathbb{R}_+ : s \geq \gamma(v)\}$. We have $\gamma^{\ominus}(s) = \infty$ for $s \geq \lim_{\tau\to\infty} \gamma(\tau)$, and $\gamma^{\ominus}(s) = \gamma^{-1}(s)$ elsewhere. For $\omega$, $\gamma \in \mathcal{K}$, we have $\omega \circ \gamma^{\ominus}(s) = \lim_{\tau\to\infty} \omega(\tau)$ for $s \geq \lim_{\tau\to\infty} \gamma(\tau)$. The identity $\gamma^{\ominus} = \gamma^{-1} \in \mathcal{K}$ holds if and only if $\gamma \in \mathcal{K}_\infty$. It is stressed that, in the case of $\gamma \in \mathcal{K} \setminus \mathcal{K}_\infty$, we have only $\gamma \circ \gamma^{\ominus}(s) \leq s$ for $s \in \overline{\mathbb{R}}_+$ although $\gamma^{\ominus} \circ \gamma(s) = s$ for $s \in \overline{\mathbb{R}}_+$. For functions $f_i \in \mathcal{K} \cup \{0\}$, $i = 1, 2, ..., m$, let $\#\{f_1, f_2, ..., f_m\}$ denote the number of non-zero functions in $\{f_1, f_2, ..., f_m\}$.

## 2. NETWORK OF COUPLED DIFFERENTIAL AND CONTINUOUS-TIME DIFFERENCE EQUATIONS

Consider

$$\Sigma_i : \; i=1,2,...,n, \quad \begin{aligned} &\dot{x}_i(t) = f_i(x(t), z_i(t), z_i(t - \Delta_i), r_i(t)) && (1)\\ &z_i(t) = g_i(x(t), z_i(t - \Delta_i), r_i(t)) && (2)\\ &\phi_{zi}(0) = g_i(\phi_x(0), \phi_{zi}(-\Delta_i), r_i(0)), && (3)\end{aligned}$$

where $x_i \in \mathbb{R}^{N_i}$, $x(t) = [x_1(t)^T, \ldots, x_n(t)^T]^T \in \mathbb{R}^N$, $\phi_{xi} \in \mathcal{C}_{N_i}[-\Delta, 0]$, $\phi_x = [\phi_{x1}^T, \ldots, \phi_{xn}^T]^T \in \mathcal{C}_N[-\Delta, 0]$, $z_i \in \mathbb{R}^{M_i}$ and $\phi_{zi} \in \mathcal{C}_{M_i}[-\Delta, 0]$ with $\Delta_i \geq 0$ and $\Delta := \max_i \Delta_i \geq 0$. The functions $(\phi_{xi}, \phi_{zi}) \in \mathcal{C}_{N_i}[-\Delta, 0] \times \mathcal{C}_{M_i}[-\Delta, 0]$, $i = 1, 2, ..., n$ are initial conditions [1] fulfilling (3). The piecewise continuous functions $r_i : \mathbb{R}_+ \to \mathbb{R}^{Q_i}$ describe disturbances. Assume that the functions $f_i : \mathbb{R}^{N_i} \times \mathbb{R}^{M_i} \times \mathbb{R}^{M_i} \times \mathbb{R}^{Q_i} \to \mathbb{R}^{N_i}$ and $g_i : \mathbb{R}^{N_i} \times \mathbb{R}^{M_i} \times \mathbb{R}^{Q_i} \to \mathbb{R}^{M_i}$ are locally Lipschitz functions. The vector $z_i$ is an auxiliary variable added to the $x_i$-part of subsystem $i$, and it allows the pair of (1)-(2) to represent a neutral system (Hale and Lunel [1993], Kolmanovskii and Myshkis [1999], Niculescu [2001], Pepe et al. [2008], Karafyllis et al. [2009]). For instance, the neutral system $\dot{x}_i(t) = f_i(x(t), \dot{x}_i(t - \Delta_i))$ with locally Lipschitz function $f_i$ can be rewritten as (1)-(2) by setting $z_i(t) = \dot{x}_i(t)$ and $g_i = f_i$. The overall network $\Sigma$ is

[1] For initial conditions of (1)-(3), $(\phi_{xi}(0), \phi_{zi}) \in \mathbb{R}^{N_i} \times \mathcal{C}_{M_i}[-\Delta, 0]$, $i = 1, 2, ..., n$, are satisfactory. Taking uniformity of expressions throughout the paper into account, $\phi_{xi} \in \mathcal{C}_{N_i}[-\Delta, 0]$ is used for $x_i$.

$$\Sigma : \quad \begin{aligned} &\dot{x}(t) = f(x(t), z(t), v(t), r(t)) && (4)\\ &z(t) = g(x(t), v(t), r(t)) && (5)\\ &\phi_z(0) = g(\phi_x(0), \nu, r(0)) && (6)\\ &\quad v(t) = [z_1(t-\Delta_1), z_2(t-\Delta_2), \ldots, z_n(t-\Delta_n)]^T \\ &\quad \nu = [\phi_{z1}(-\Delta_1), \phi_{z2}(-\Delta_2), \ldots, \phi_{zn}(-\Delta_n)]^T \end{aligned}$$

with suitable maps $f : \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^M \times \mathbb{R}^Q \to \mathbb{R}^N$ and $g : \mathbb{R}^N \times \mathbb{R}^M \times \mathbb{R}^Q \to \mathbb{R}^M$, where $z(t) = [z_1(t)^T, \ldots, z_n(t)^T]^T \in \mathbb{R}^M$ and $r(t) = [r_1(t)^T, \ldots, r_n(t)^T]^T \in \mathbb{R}^Q$. We do not assume that explicit formulas for $f$ and $g$ are known. Instead, we assume the following:

*Assumption 1.* For $i = 1, 2, ..., n$, there exist continously differentiable functions $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ and $\underline{\alpha}_i$, $\overline{\alpha}_i \in \mathcal{K}_\infty$, $\alpha_i \in \mathcal{K}$, $\beta_i, \psi_i, \sigma_{i,j}, \kappa_i \in \mathcal{K} \cup \{0\}$ for $j = 1, 2, ..., n$ such that

$$\underline{\alpha}_i(|a_i|) \leq V_i(a_i) \leq \overline{\alpha}_i(|a_i|), \tag{7}$$

$$\begin{aligned} \frac{\partial V_i}{\partial a_i}(a_i) f_i(a, b_i, c_i, d_i) \leq -\alpha_i(V_i(a_i)) + \beta_i(|b_i|) \\ + \psi_i(|c_i|) + \kappa_i(|d_i|) + \sum_{j=1}^{n} \sigma_{i,j}(V_j(a_j)) \end{aligned} \tag{8}$$

for all $(a_i, b_i, c_i, d_i) \in \mathbb{R}^{N_i} \times \mathbb{R}^{M_i} \times \mathbb{R}^{M_i} \times \mathbb{R}^{Q_i}$, $i = 1, 2, ..., n$, where $a = [a_1^T, \ldots, a_n^T]^T \in \mathbb{R}^N$.

*Assumption 2.* There exist functions $\bar{f}_i, \bar{g}_{i,j}, \bar{\kappa}_i \in \mathcal{K} \cup \{0\}$ for $i, j = 1, 2, ..., n$ such that

$$|g_i(a, c_i, d_i)| \leq \bar{f}_i(|c_i|) + \bar{\kappa}_i(|d_i|) + \sum_{j=1}^{n} \bar{g}_{i,j}(V_j(a_j)) \tag{9}$$

for any $(a_i, c_i, d_i) \in \mathbb{R}^{N_i} \times \mathbb{R}^{M_i} \times \mathbb{R}^{Q_i}$, $i = 1, 2, ..., n$, where $a = [a_1^T, \ldots, a_n^T]^T \in \mathbb{R}^N$.

In the case of $\sigma_{i,i} \neq 0$, Assumption 1 by itself does not guarantee each subsystem to be iISS.

## 3. CONSTRUCTION OF A LYAPUNOV-KRASOVSKII FUNCTIONAL

This section formulates the stability analysis of the network $\Sigma$ into the problem of constructing a Lyapunov-Krasovskii functional in a non-classical sense. For this end, we define a directed graph $G$ associated with the functions

$$G_{i,j} = \#\{\sigma_{i,j},\ \beta_i \circ \bar{g}_{i,j},\ \beta_i \circ \bar{f}_i \circ \bar{g}_{i,j},\ \psi_i \circ \bar{g}_{i,j}\}, \quad i, j = 1, 2, ..., n. \tag{10}$$

The graph is allowed to have loops. A loop is an arc that connects a vertex to itself. Let the vertex set and the arc set be denoted by $\mathcal{V}(G)$ and $\mathcal{A}(G)$. Elements of $\mathcal{V}(G)$ correspond to subsystems $\Sigma_i$, $i = 1, 2, ..., n$. We write merely $i$ instead of $\Sigma_i$. Each element of $\mathcal{A}(G)$ is an ordered pair $(i, j)$ which is directed away from vertex $j$ and directed toward vertex $i$. The pair $(i, j)$ is an element of $\mathcal{A}(G)$ if and only if $G_{i,j} \neq 0$. Without ambiguity, the symbol $\mathcal{V}(G)$ also denotes the set of all singleton graphs contained in $G$. Let $\mathcal{C}(G)$ and $\mathcal{P}(G)$ denote the set of all directed cycle subgraphs and directed path subgraphs, respectively, contained in $G$. Given a directed cycle or a directed path $U$ of length $k$, we employ the notation

$$|U| = k, \quad U = (u(1), u(2), ..., u(k), u(k+1)),$$

where $u(i)$s are all vertices comprising $U$. The starting vertex of the directed path $U$ is $u(k+1)$, and the ending vertex is $u(1)$. If $U$ is a directed cycle, we have $u(1) = u(k+1)$. The cycle subgraph (resp., the path subgraph) consisting of the cycle (resp., the path) $U$ is also denoted by the same symbol $U$. Let $\mathcal{L}(G)$ denote the set of all directed subgraphs consisting of loops in $G$, and we write $|U| = 1$ and $u(1) = u(2)$ for $U \in \mathcal{L}(G)$. We write $|U| = 0$ for $U \in \mathcal{V}(G)$. In the rest of this paper, the term "directed" is omitted in referring to graphs and subgraphs.

For each $i=1, 2, ..., n$, let $H_i = \#\{\bar{f}_i, \bar{g}_{i,1}, ..., \bar{g}_{i,n}, \bar{\kappa}_i\}$. Define

$$\hat{\psi}_i(s) = \beta_i(H_i\bar{f}_i(s)) + \psi_i(s) \tag{11}$$

$$\mathbf{D} = \{i \in \{1, 2, ..., n\} : \hat{\psi}_i \neq 0\}. \tag{12}$$

Consider $J_U, d_i, d_{i,j} \in \mathbb{R}_+$ satisfying

$$1 = d_i \left( D(i) + \sum_{U \in \{W \in \mathcal{C}(G) \cup \mathcal{P}(G) \cup \mathcal{L}(G) \cup \mathcal{V}(G) : \mathcal{V}(W) \ni i\}} J_U \right) \quad \forall i \in \mathcal{V}(G) \tag{13}$$

$$D(i) = \begin{cases} 1, \text{ if } i \in \mathbf{D} \\ 0, \text{ otherwise} \end{cases} \tag{14}$$

$$1 = d_{i,j} \sum_{U \in \{W \in \mathcal{C}(G) \cup \mathcal{P}(G) \cup \mathcal{L}(G) : \mathcal{A}(W) \ni (i,j)\}} J_U, \quad \forall (i,j) \in \mathcal{A}(G). \tag{15}$$

The set of non-zero $J_U$'s fulfilling (13) and (15) defines a covering of the graph $G$ by cycles, loops, paths and isolated vertices. A subgraph $U$ is adopted to cover a part of $G$ if and only if $J_U \neq 0$. Multiple subgraphs $U$ defining coverings can overlap each other. Although the set of subgraphs $U$ covering $G$ is not unique, there always exists such a set of subgraphs. There is also flexibility in choosing the set of real numbers $J_U > 0$. For an arbitrary set of real numbers $J_U > 0$ chosen for each covering, we can always compute $d_i, d_{i,j} > 0$ from (13), (14) and (15).

The next theorem recasts the problem of verifying the stability of $\Sigma$ as the construction of a functional $V$ in a specific form. The construction is possible if a set of inequalities given in terms of functions $\bar{\zeta}_i$ and $\bar{\xi}_{i,j}$ is solved for a set of functions $\lambda_i$, $i = 1, 2, ..., n$.

*Theorem 3.* Consider the network $\Sigma$ satisfying Assumptions 1 and 2. Assume that, for each $i \in \mathbf{D}$, there exist real numbers $\mu_i > 0$ and $p_i > 1$ such that

$$\hat{\psi}_i(H_i\bar{f}_i(s)) \leq \frac{p_i - 1}{p_i e^{\mu_i \Delta_i}} \hat{\psi}_i(s), \ \forall s \in \mathbb{R}_+ \tag{16}$$

$$1 \leq d_{i,j} p_i e^{\mu_i \Delta_i}, \qquad \forall j = 1, 2, ..., n \tag{17}$$

hold. Suppose that there exist continuous functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, ..., n$, such that

$$\lambda_i(s) > 0, \quad \forall s \in (0, \infty), \quad i = 1, 2, ..., n \tag{18}$$

$$\lambda_i \text{ is non-decreasing}, \qquad i = 1, 2, ..., n \tag{19}$$

$$\{\lim_{s\to\infty} \alpha_i(s) = \infty \text{ or } \limsup_{s\to\infty} \lambda_i(s) < \infty\}, \quad i = 1, 2, ..., n \tag{20}$$

$$\sum_{U \in \mathcal{C}(G) \cup \mathcal{P}(G) \cup \mathcal{L}(G)} J_U \sum_{i=1}^{|U|} \{-\bar{\zeta}_{u(i)}(s_{u(i)}) + \bar{\xi}_{u(i),u(i+1)}(s_{u(i+1)})\} \leq 0, \quad \forall s_1, ..., s_n \in \mathbb{R}_+ \tag{21}$$

hold, where, for $i, j = 1, 2, ..., n$,

$$\hat{\alpha}_i(s) = d_i \alpha_i(s), \quad \hat{\sigma}_{i,j}(s) = d_{i,j} \tilde{\sigma}_{i,j}(s) \tag{22}$$

$$1 < \tau_i < c_i \tag{23}$$

$$\bar{\zeta}_i = \frac{\tau_i}{c_i}\left(1 - \frac{1}{\tau_i}\right) \lambda_i(s) \hat{\alpha}_i(s) \tag{24}$$

$$\bar{\xi}_{i,j}(s) = \lambda_i(\hat{\alpha}_i^{\ominus}(\tau_i \hat{\sigma}_{i,j}(s)))\, \hat{\sigma}_{i,j}(s) \tag{25}$$

$$\tilde{\sigma}_{i,j}(s) = \sigma_{i,j}(s) + \beta_i(H_i\bar{g}_{i,j}(s)) + p_i e^{\mu_i \Delta_i} \hat{\psi}_i(H_i\bar{g}_{i,j}(s)). \tag{26}$$

Then the functional $V : \mathbb{R}^N \times \mathcal{PC}_M[-\Delta, 0] \to \mathbb{R}_+$ given by

$$V(\phi_x(0), \phi_z) = \sum_{i=1}^{n} \int_0^{V_i(\phi_{xi}(0))} \lambda_i(s) ds + \sum_{i \in \mathbf{D}} X_i(\phi_{zi}), \tag{27}$$

satisfies

$$D^+V(t) \leq -\sum_{i=1}^{n} \left\{ \delta_i \bar{\zeta}_i(V_i(x_i(t))) + p_i \mu_i \int_{-\Delta_i}^{0} E_i(\tau) \xi_i(|\phi_{zi}(\tau)|) d\tau \right\} + \sum_{i=1}^{n} e_i(|r_i(t)|) \tag{28}$$

along the trajectories of (1)-(3) with some $\delta_i > 0$, $\bar{\zeta}_i \in \mathcal{K}$ and $e_i \in \mathcal{K} \cup \{0\}$, where $X_i : \mathcal{PC}_{M_i}[-\Delta, 0] \to \mathbb{R}_+$ is

$$X_i(\phi_{zi}) = p_i \int_{-\Delta_i}^{0} E_i(\tau) \xi_i(|\phi_{zi}(\tau)|) d\tau, \ E_i(\tau) = e^{\mu_i(\tau + \Delta_i)} \tag{29}$$

$$\xi_i(s) = \lambda_i(\hat{\alpha}_i^{\ominus}(\tau_i \hat{\psi}_i(s)))\, \hat{\psi}_i(s). \tag{30}$$

If $\alpha_i \in \mathcal{K}_\infty$ holds for $i = 1, 2, ..., n$ additionally, we have $\bar{\zeta}_i \in \mathcal{K}_\infty$ for $i = 1, 2, ..., n$.

Here, $D^+V(t)$ denotes the upper right-hand derivative of $V$. Property (20) guarantees that $\bar{\xi}_{i,j}$ and $\xi_i$ are of class $\mathcal{J}$ functions (or zero) defined on the whole $s \in \mathbb{R}_+$. In other words, if (20) is violated, components (1) which are not ISS need to be excluded for $\bar{\xi}_{i,j}$ and $\xi_i$ being well-defined. Recall that the functions $\hat{\alpha}_i$ are not guaranteed to be invertible on the whole $\mathbb{R}_+$.

*Remark 4.* If we restrict $\psi_i$, $\beta_i$, $\bar{f}_i$ and $\lambda_i$ to three linear functions and a constant, respectively, we can use $H_i = 1$ in (11), (16) and (26). It is worth mentioning that the use of constant $\lambda_i$s does not cause any conservativeness if $c_{i,j}\alpha_j \geq \tilde{\sigma}_{i,j}$ holds for some constants $c_{i,j} \geq 0$. This fact can be verified in the same way as the delay free case (Dashkovskiy et al. [2011]). If $\lambda_i$ is a constant for an integer $i \in \{1, 2, ..., n\}$, $D(i) = 0$ can be used in (13).

Assumption (16) requires $\bar{f}_i$ to be at most linear. Since $\bar{f}_i$ creates a self-loop, such a linear requirement is inevitable in seeking stability criteria independent of $\Delta_i$, (Mazenc et al. [2013]). The following replaces the pair of (16) and (17) by a simpler single condition.

*Lemma 5.* Assume that, for each $i \in \mathbf{D}$, there exists a real number $K_i > 1$ such that

$$K_i \hat{\psi}_i(H_i\bar{f}_i(s)) \leq \hat{\psi}_i(s), \ \forall s \in \mathbb{R}_+ \tag{31}$$

holds. Then, for $i \in \mathbf{D}$, there exist real numbers $\mu_i > 0$ and $p_i > 1$ satisfying

$$\frac{p_i e^{\mu_i \Delta_i}}{p_i - 1} \leq K_i, \quad \frac{1}{d_{i,j} e^{\mu_i \Delta_i}} \leq p_i, \quad \forall j = 1, 2, ..., n \tag{32}$$

which imply (16) and (17).

The distinctive difference from the delay-free case which does not need coupling equations (Ito et al. [2013a]) boils down to (31), (30) and (26). The integer $G_{i,j}$ defined in (10) represents the graph topology associated with (26).

*Remark 6.* The functional $V$ constructed explicitly above satisfies properties posed in Pepe et al. [2008] for ISS that employs the difference equation (2) without $r_i$.

*Remark 7.* As done in Mazenc et al. [2013], we can replace $|b_i|$ and $|c_i|$ in (8) and (9) with $W_i(b_i)$ and $W_i(c_i)$, respectively as far as each continuous function $W_i : \mathbb{R}^{M_i} \to \mathbb{R}_+$ admits the existence of $\underline{\alpha}_M$, $\overline{\alpha}_M \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_{Mi}(|b_i|) \le W_i(b_i) \le \overline{\alpha}_{Mi}(|b_i|),\ \forall b_i \in \mathbb{R}^{M_i}. \tag{33}$$

Here, $b_i$ denotes $b_i = g_i(a, c_i, d_i)$ in (9). Theorem 3 and Lemma 5 remain unchanged except for $W_i(\phi_{zi}(\tau))$ replacing $|\phi_{zi}(\tau)|$ in (29).

## 4. A SMALL-GAIN CRITERION

This section shows a formula for $\lambda_i$s achieving the requirements in Theorem 3. A condition under which such $\lambda_i$s exist will be given as a criterion of small-gain form. To obtain the desirable functions $\lambda_i$, the construction of $F_{i,j}$ in the following lemma plays the key role.

*Lemma 8.* (Ito et al. [2013a]) Consider $\hat{\alpha}_i \in \mathcal{K}$, $\hat{\sigma}_{i,j} \in \mathcal{K} \cup \{0\}$, $i, j = 1, 2, ..., n$, satisfying

$$\left\{ \lim_{s\to\infty} \hat{\alpha}_j(s) = \infty \ \text{or} \ \lim_{s\to\infty} \sum_{i=1}^{n} \hat{\sigma}_{i,j}(s) < \infty \right\}, \begin{matrix} j=1,2, \\ ..., n. \end{matrix} \tag{34}$$

Let $\mathcal{CP}(i,j)$ denotes the set of all paths and cycles from vertex $j$ to vertex $i$ of $G$. Suppose that $\tau_i > 0$ for $i = 1, 2, ..., n$. Define for $i, j = 1, 2, ..., n$,

$$F_{i,j}(s) = \max_{U \in \mathcal{CP}(i,j)} \hat{\sigma}_{u(1),u(2)} \circ \bigodot_{i=2}^{|U|} \hat{\alpha}_{u(i)}^{\ominus} \circ \tau_{u(i)} \hat{\sigma}_{u(i),u(i+1)}(s) \quad i \neq j \tag{35}$$

$$F_{i,i}(s) = \hat{\sigma}_{i,i}. \tag{36}$$

Then, we have $F_{i,j} \in \mathcal{J}$ for $i, j = 1, 2, ..., n$.

Assumption (34) guarantees that the functions $F_{i,j}(s)$ do not attain $\infty$ for finite $s \in \mathbb{R}_+$. Note that for an arbitrary pair $(i, j)$, the number of elements in $\mathcal{CP}(i,j)$ is finite. The following theorem gives a formula of the desired $\lambda_i$.

*Theorem 9.* Consider $\hat{\alpha}_i \in \mathcal{K}$, $\hat{\sigma}_{i,j} \in \mathcal{K} \cup \{0\}$, $i, j = 1, 2, ..., n$, satisfying (34). Assume that there exist $c_i > 1$, $i = 1, 2, ..., n$ such that

$$\bigodot_{i=1}^{|U|} \hat{\alpha}_{u(i)}^{\ominus} \circ c_{u(i)} \hat{\sigma}_{u(i),u(i+1)}(s) \le s,\ \forall s \in \mathbb{R}_+ \tag{37}$$

holds for all cycle subgraphs and loop subgraphs $U \in \mathcal{C}(G) \cup \mathcal{L}(G)$. Let $\tau_i$ and $\varphi \ge 0$ be such that (23) and

$$\left(\frac{\tau_i}{c_i}\right)^{\varphi} \le \tau_i - 1, \quad i = 1, 2, ..., n \tag{38}$$

are satisfied. Define $\overline{\lambda}_i \in \mathcal{J}$, $i = 1, 2, ..., n$, by

$$\overline{\lambda}_i(s) = \left[\frac{1}{\tau_i}\hat{\alpha}_i(s)\right]^{\varphi} \prod_{j \in \mathcal{V}(G) - \{i\}} [F_{j,i}(s)]^{\varphi+1}, \tag{39}$$

and let $\nu_i : (0, \infty) \to \mathbb{R}_+$, $i = 1, 2, ..., n$, be continuous functions fulfilling

$$0 < \nu_i(s) < \infty,\ s \in (0, \infty),\ i = 1, 2, ..., n \tag{40}$$

$$\lim_{s\to\infty} \alpha_i(s) = \infty \ \text{or} \ \lim_{s\to\infty} \nu_i(s) < \infty \tag{41}$$

$$\bar{\lambda}_i(s)\nu_i(s) : \text{non-decreasing continuous for } s \in (0, \infty) \tag{42}$$

and

$$\nu_{u(j)} \circ \hat{\alpha}_{u(j)}^{\ominus} \circ \tau_{u(j)} \hat{\sigma}_{u(j),u(j+1)}(s) \le \left(\frac{c_{u(j+1)}}{\tau_{u(j+1)}}\right)^{\varphi} (\tau_{u(j+1)} - 1)\nu_{u(j+1)}(s), \quad \forall s \in (0, \infty),\ j = 1, 2, ..., |U| \tag{43}$$

for all cycle subgraphs $U \in \mathcal{C}(G)$. Then non-decreasing continuous functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1, 2, ..., n$, defined by

$$\lambda_i(s) = \begin{cases} \overline{\lambda}_i(s)\nu_i(s) & , \ s \in (0, \infty) \\ \lim\limits_{s\to 0^+} \overline{\lambda}_i(s)\nu_i(s) & , \ s = 0 \end{cases} \tag{44}$$

achieve (18)-(21).

Note that in the case of $U \in \mathcal{L}(G)$, condition (37) indicates

$$\hat{\alpha}_{u(1)}^{\ominus} \circ c_{u(1)} \hat{\sigma}_{u(1),u(1)}(s) \le s,\ \forall s \in \mathbb{R}_+. \tag{45}$$

If we use only functions in Assumptions 1 and 2 without any intermediate variables, property (34) is identical with

$$\left\{ \lim_{s\to\infty} \alpha_j(s) = \infty \ \text{or} \ \lim_{s\to\infty} \sum_{i=1}^{n} \sigma_{i,j}(s) + \beta_i(\bar{g}_{i,j}(s)) + \beta_i(\bar{f}_i(\bar{g}_{i,j}(s))) + \psi_i(\bar{g}_{i,j}(s)) < \infty \right\}, j = 1, 2, ..., n. \tag{46}$$

Notice that $\hat{\alpha}_1, ..., \hat{\alpha}_n \in \mathcal{K}_\infty$ or equivalently $\alpha_1, ..., \alpha_n \in \mathcal{K}_\infty$ fulfills (34), or equivalently, (46). Theorem 9 is proved by applying the technique in Ito et al. [2013a] to (21) defined with (24) and (25). It is stressed that the left hand side of (37) involving $\hat{\alpha}_i^{\ominus}$ is well-defined for all $s \in \mathbb{R}_+$ due to (34). Notice that there always exist $\tau_i$ and $\varphi \ge 0$ fulfilling (23) and (38). The continuous functions $\nu_i$ satisfying (40)-(43) also always exist (See Remark 11). Thus, Theorems 3 and 9 lead to the following main result:

*Theorem 10.* Consider $\Sigma$ satisfying Assumptions 1 and 2 and (46). Assume that, for each $i \in \mathbf{D}$, there exists a real number $K_i > 1$ satisfying (31). Pick $J_U, d_i, d_{i,j} \in \mathbb{R}_+$ so that (13) and (15) hold. If there exist $c_i > 1$, $i, j = 1, 2, ..., n$ such that (37) holds for all $U \in \mathcal{C}(G) \cup \mathcal{L}(G)$, then the network $\Sigma$ admits a unique solution $(x(t), z(t))$ for all $t \ge 0$ with respect to each initial condition $(\phi_x, \phi_z) \in \mathcal{C}_N[-\Delta, 0] \times \mathcal{C}_M[-\Delta, 0]$, and, furthermore, there exist $\beta_x, \beta_z \in \mathcal{KL}$ and $\gamma_x, \gamma_z, \chi_x, \chi_z \in \mathcal{K}$ such that

$$|x(t)| \le \beta_x(|\phi_x(0)| + \|\phi_z\|_\infty, t) + \chi_x\left(\int_0^t \gamma_x(|r(\tau)|)d\tau\right), \quad \forall t \in \mathbb{R}_+ \tag{47}$$

$$|z(t)| \le \beta_z(|\phi_x(0)| + \|\phi_z\|_\infty, t) + \gamma_z(\|r_{[0,t)}\|_\infty) + \chi_z\left(\int_0^t \gamma_x(|r(\tau)|)d\tau\right), \forall t \in \mathbb{R}_+ \tag{48}$$

are satisfied for any piecewise continuous input $r$. If $\alpha_i \in \mathcal{K}_\infty$ holds for $i = 1, 2, ..., n$ additionally, there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$\left|\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}\right| \le \beta(|\phi_x(0)| + \|\phi_z\|_\infty, t) + \gamma(\|r_{[0,t)}\|_\infty),\ \forall t \in \mathbb{R}_+. \tag{49}$$

The functional establishing the above stability properties is obtained as $V$ in (27). In Pepe et al. [2008], property (49) is called ISS for the system described by the coupled differential and difference equations (1)-(3). Due to Remark 6, Theorem 10 states that the function $V$ in (27) is guaranteed to be an ISS Lyapunov-Krasovskii functional in the sense of Pepe et al. [2008] if $\alpha_i \in \mathcal{K}_\infty$ holds for $i = 1, 2, ..., n$. In the case where $\alpha_i \notin \mathcal{K}_\infty$ holds for some $i$, the $x$-system satisfies the iISS-type property (47)-(48) which is weaker than (49).

*Remark 11.* One of simple choices of continuous functions $\nu_i : (0, \infty) \to \mathbb{R}_+$, $i = 1, 2, ..., n$ fulfilling (40)-(43) is $\nu_1(s) = \nu_2(s) = ... = \nu_n(s) = \text{constant} > 0$. The non-constant choices presented for the delay-free case in Ito et al. [2013a] are also eligible. The flexibility of $\nu_i$ is useful for recursive construction of the functional $V$ when we expand the network afterwards.

*Remark 12.* This paper is a generalization of the result in Ito and Mazenc [2012] developed for a class of neutral systems. In fact, Theorem 9 reduces into the one in Ito and Mazenc [2012] precisely when the network satisfies $z_i = \dot{x}_i$ and $\beta_i = 0$ for $i = 1, 2, ..., n$. Recall that, according to (1) and (2), $\psi_i$ and $\bar{f}_i$ in Assumptions 1 and 2 are functions of $|z_i(t - \Delta_i)|$. The coupled equations (1) and (2) include the class of networks considered in Ito and Mazenc [2012] as a special case. In dealing with neutral-type delays, the choice $z_i \neq \dot{x}_i$ can be used to make some components of the vector $\dot{x}_i$ to be free from constraints (16) and (31). Thus, the pair (1) and (2) is allowed to have neutral-delay terms whose magnitude is not necessarily bounded from above by any linear functions.

## 5. INTERNAL AND COMMUNICATION DELAYS

In order to deal with state delays of subsystems and communication delays between subsystems, consider

$$\Sigma_i : \; i = 1, 2, ..., n, \quad \dot{x}_i(t) = f_i(x(t), v(t), z_i(t), z_i(t - \Delta_i), r_i(t)) \tag{50}$$

$$v(t) = [x_1(t - \Delta_{C,i,1}), x_2(t - \Delta_{C,i,2}), \ldots, x_n(t - \Delta_{C,i,n})]^T \tag{51}$$

$$z_i(t) = g_i(x(t), v(t), z_i(t - \Delta_i), r_i(t)) \tag{52}$$

$$\phi_{zi}(0) = g_i(\phi_x(0), \nu, \phi_{zi}(-\Delta_i), r_i(0)) \tag{53}$$

$$\nu = [\phi_{x1}(-\Delta_{C,i,1}), \phi_{x2}(-\Delta_{C,i,2}) \ldots, \phi_{xn}(-\Delta_{C,i,n})]^T \tag{54}$$

The delays $\Delta_{C,i,j} \geq 0$ are in communication channels, and the delays $\Delta_{C,i,i} \geq 0$ reside in subsystems. Let $\Delta := \max_i\{\Delta_i, \max_j \Delta_{C,i,j}\}$. We replace Assumptions 1 and 2 by the following.

*Assumption 13.* For $i = 1, 2, ..., n$, there exist continously differentiable functions $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ and $\underline{\alpha}_i$, $\overline{\alpha}_i \in \mathcal{K}_\infty$, $\alpha_i \in \mathcal{K}$, $\beta_i, \psi_i, \sigma_{i,j}, \kappa_i \in \mathcal{K} \cup \{0\}$ for $j = 1, 2, ..., n$ such that (7) and

$$\frac{\partial V_i}{\partial a_i}(a_i) f_i(a, h, b_i, c_i, d_i) \leq -\alpha_i(V_i(a_i)) + \beta_i(|b_i|) + \psi_i(|c_i|) + \kappa_i(|d_i|) + \sum_{j=1}^{n} \sigma_{i,j}(V_j(h_j)) \tag{55}$$

for all $(a_i, h_i, b_i, c_i, d_i) \in \mathbb{R}^{N_i} \times \mathbb{R}^{N_i} \times \mathbb{R}^{M_i} \times \mathbb{R}^{M_i} \times \mathbb{R}^{Q_i}$, $i = 1, 2, ..., n$, where $a = [a_1^T, \ldots, a_n^T]^T \in \mathbb{R}^N$ and $h = [h_1^T, \ldots, h_n^T]^T \in \mathbb{R}^N$.

*Assumption 14.* There exist functions $\bar{g}_i \in \mathcal{K}$ and $\bar{f}_i, \bar{g}_{i,j}, \bar{\kappa}_i \in \mathcal{K} \cup \{0\}$ for $i, j = 1, 2, ..., n$ such that

$$|g_i(a, h, c_i, d_i)| \leq \bar{f}_i(|c_i|) + \bar{\kappa}_i(|d_i|) + \bar{g}_i(V_i(a_i)) + \sum_{j=1}^{n} \bar{g}_{i,j}(V_j(h_j)) \tag{56}$$

for any $(a_i, h_i, c_i, d_i) \in \mathbb{R}^{N_i} \times \mathbb{R}^{N_i} \times \mathbb{R}^{M_i} \times \mathbb{R}^{Q_i}$, $i = 1, 2, ..., n$, where $a = [a_1^T, \ldots, a_n^T]^T \in \mathbb{R}^N$ and $h = [h_1^T, ..., h_n^T]^T \in \mathbb{R}^N$.

For each $i = 1, 2, ..., n$, let $H_i = \#\{\bar{f}_i, \bar{g}_i, \bar{g}_{i,1}, \ldots, \bar{g}_{i,n}, \bar{\kappa}_i\}$. For $i, j = 1, 2, ..., n$, introduce the following:

$$\check{g}_{i,j}(s) = \begin{cases} \bar{g}_i(s), & j = i \\ 0, & \text{otherwise} \end{cases}. \tag{57}$$

Define a directed graph $G$ by replacing (10) with

$$G_{i,j} = \#\{\sigma_{i,j},\ \beta_i \circ (\bar{g}_{i,j} + \check{g}_{i,j}),\ \beta_i \circ \bar{f}_i \circ (\bar{g}_{i,j} + \check{g}_{i,j})\ \ \psi_i \circ (\bar{g}_{i,j} + \check{g}_{i,j})\}, \quad i, j = 1, 2, ..., n. \tag{58}$$

Replace (13) by

$$1 = d_i \left( D(i) + B(i) + \sum_{U \in \{W \in \mathcal{C}(G) \cup \mathcal{P}(G) \cup \mathcal{L}(G) \cup \mathcal{V}(G) : \mathcal{V}(W) \ni i\}} J_U \right) \quad \forall i \in \mathcal{V}(G) \tag{59}$$

$$B(i) = \begin{cases} 1, & \text{if } i \in \beta_i(\bar{g}_i) \neq 0 \\ 0, & \text{otherwise} \end{cases} \tag{60}$$

*Theorem 15.* Consider the network (50)-(54) satisfying Assumptions 13 and 14. Assume that, for each $i \in \mathbf{D}$, there exist real numbers $\mu_i > 0$ and $p_i > 1$ such that (16) and (17) hold. Suppose that there exist continuous functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$ and constants $\mu_{C,i,j} > 0$, $i = 1, 2, ..., n$, such that (18)-(21) hold, where, for $i, j = 1, 2, ..., n$, (22)-(25)

$$\tilde{\sigma}_{i,j}(s) = e^{\mu_{C,i,j}\Delta_{C,i,j}}[\sigma_{i,j}(s) + \beta_i(H_i\bar{g}_{i,j}(s)) + p_i e^{\mu_i\Delta_i}\hat{\psi}_i(H_i\bar{g}_{i,j}(s))] + \beta_i(H_i\check{g}_{i,j}(s)) + p_i e^{\mu_i\Delta_i}\hat{\psi}_i(H_i\check{g}_{i,j}(s)). \tag{61}$$

Then the functional $V : \mathcal{C}_N[-\Delta, 0] \times \mathcal{PC}_M[-\Delta, 0] \to \mathbb{R}_+$ given by

$$V(\phi_x, \phi_z) = \sum_{i=1}^{n} \int_0^{V_i(\phi_{xi}(0))} \lambda_i(s)ds + \sum_{i \in \mathbf{D}} X_i(\phi_{zi}) + \sum_{i=1}^{n}\sum_{j=1}^{n} Y_{i,j}(\phi_{xj}) \tag{62}$$

satisfies

$$D^+V(t) \leq -\sum_{i=1}^{n} \delta_i \bar{\zeta}_i(V_i(x_i(t))) - \sum_{i=1}^{n}\sum_{j=1}^{n} \mu_{C,i,j} \int_{-\Delta_{C,i,j}}^{0} E_{C,i,j}(\tau)\theta_{i,j}(V_j(x(t+\tau)))d\tau - \sum_{i=1}^{n} p_i\mu_i \int_{-\Delta_i}^{0} E_i(\tau)\xi_i(|z_i(t+\tau)|)d\tau + \sum_{i=1}^{n} e_i(|r_i(t)|) \tag{63}$$

for the trajectories of (50)-(54) with some $\delta_i > 0$, $\bar{\zeta}_i \in \mathcal{K}$ and $e_i \in \mathcal{K} \cup \{0\}$, where $X_i$, $\xi_i$, $E_i$, $Y_{i,j}$, $\theta_{i,j}$, $E_{C,i,j}$ are given by (29)-(30) and

$$Y_{i,j}(\phi_{xj}) = \int_{-\Delta_{C,i,j}}^{0} E_{C,i,j}(\tau)\theta_{i,j}(V_j(\phi_{xj}(\tau)))d\tau \tag{64}$$

$$\theta_{i,j}(s) = \lambda_i(\hat{\alpha}_i^{\ominus}(\tau_i d_{i,j}(\sigma_{i,j}(s) + \beta_i(H_i\bar{g}_{i,j}(s))))) \cdot (\sigma_{i,j}(s) + \beta(H_i\bar{g}_{i,j}(s))) + p_i e^{\mu_i\Delta_i}\lambda_i(\hat{\alpha}_i^{\ominus}(\tau_i\hat{\psi}_i(H_i\bar{g}_{i,j}(s)))) \cdot \hat{\psi}_i(H_i\bar{g}_{i,j}(s)), \tag{65}$$

$$E_{C,i,j}(\tau) = e^{\mu_{C,i,j}(\tau + \Delta_{C,i,j})}. \tag{66}$$

If $\alpha_i \in \mathcal{K}_\infty$ holds for $i = 1, 2, ..., n$ additionally, we have $\bar{\zeta}_i \in \mathcal{K}_\infty$ for $i = 1, 2, ..., n$.

*Theorem 16.* Consider the network (50)-(54) satisfying Assumptions 13 and 14 and (34). Assume that, for each $i \in \mathbf{D}$, there exists $K_i > 1$ satisfying (31). Pick $J_U, d_i, d_{i,j} \in \mathbb{R}_+$ so that (59) and (15) hold. If there exist $c_i > 1$, $i, j = 1, 2, ..., n$ such that (37) holds for all $U \in \mathcal{C}(G) \cup \mathcal{L}(G)$, then the network (50)-(54) admits a unique solution $(x(t), z(t))$ for all $t \geq 0$ with respect to each initial condition $(\phi_x, \phi_z) \in \mathcal{C}_N[-\Delta, 0] \times \mathcal{C}_M[-\Delta, 0]$, and, furthermore, there exist $\beta_x, \beta_z \in \mathcal{KL}$ and $\gamma_x, \gamma_z, \chi_x, \chi_z \in \mathcal{K}$ such that

$$|x(t)| \leq \beta_z(\|\phi_x\|_\infty + \|\phi_z\|_\infty, t) + \chi_x\left(\int_0^t \gamma_x(|r(\tau)|)d\tau\right), \quad \forall t \in \mathbb{R}_+ \tag{67}$$

$$|z(t)| \leq \beta_z(\|\phi_x\|_\infty + \|\phi_z\|_\infty, t) + \gamma_z(\|r_{[0,t)}\|_\infty) + \chi_z\left(\int_0^t \gamma_x(|r(\tau)|)d\tau\right), \forall t \in \mathbb{R}_+ \tag{68}$$

are satisfied for any piecewise continuous input $r$. If $\alpha_i \in \mathcal{K}_\infty$ holds for $i = 1, 2, ..., n$ additionally, there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$\left|\begin{bmatrix} x(t) \\ z(t) \end{bmatrix}\right| \leq \beta(\|\phi_x\|_\infty + \|\phi_z\|_\infty, t) + \gamma(\|r_{[0,t)}\|_\infty), \ \forall t \in \mathbb{R}_+. \tag{69}$$

In addition to (31), the coupled equations introduce new terms into (61) and (58) which do not appear in the Lyapunov construction developed in Ito et al. [2013b] for delay networks without difference equations.

*Remark 17.* If $\lambda_i$ is restricted to a constant for an integer $i \in \{1, 2, ..., n\}$, $D(i) = B(i) = 0$ can be used in (59).

## 6. CONCLUDING REMARKS

This paper has demonstrated how to construct a Lyapunov-Krasovskii functional for networks of coupled delay differential and continuous-time difference equations. The result reported in Mazenc et al. [2013] for neutral-type delays has been extended to large-scale systems in arbitrary network graph topology. This paper has also focused on state delays (delays of retarded type) and communication delays which are not covered in Mazenc et al. [2013]. To cope with difference equations, this paper has tailored the iISS framework (Ito et al. [2013a]) for delay-free networks described by differential equations. Important technical differences between this paper and the previous results on delay systems have also been highlighted. The differential equation parts of subsystems are not assumed to enjoy an ISS type property, so that this paper covers a broader class of systems than the previous ISS result in Pepe et al. [2008]. In the case where differential equation parts of all subsystems satisfy an ISS-like property, the result of this paper provides us with a way to explicitly compute an ISS Lyapunov-Krasovskii functional defined in Pepe et al. [2008]. Due to space limitation, a result for distributed delays is omitted and will be presented elsewhere.

## REFERENCES

D. Angeli, E.D. Sontag, and Y. Wang. A characterization of integral input-to-state stability. *IEEE Trans. Automat. Contr.*, 45:1082–1097, 2000.

M. Arcak, D. Angeli, and E. Sontag. A unifying integral ISS framework for stability of nonlinear cascades. *SIAM J. Control Optim.*, 40:1888–1904, 2002.

R. Brayton. Nonlinear oscillations in a distributed network. *Quarterly of Applied Math.*, 4:289–301, 1967.

A. Chaillet and D. Angeli. Integral input to state stable systems in cascade. *Systems & Contr. Letters*, 57:519–527, 2008.

S. Dashkovskiy, H. Ito, and F. Wirth. On a small-gain theorem for ISS networks in dissipative Lyapunov form. *European J. Contr.*, 17:357–369, 2011.

J.K. Hale and S.M. Verduyn Lunel. *Introduction to Functional Differential Equations.* Springer, 1993.

H. Ito. State-dependent scaling problems and stability of interconnected iISS and ISS systems. *IEEE Trans. Automat. Contr.*, 51:1626–1643, 2006.

H. Ito. A Lyapunov approach to cascade interconnection of integral input-to-state stable systems. *IEEE Trans. Automat. Contr.*, 55:702–708, 2010.

H. Ito, Z.P. Jiang, S. Dashkovskiy, and B. Rüffer. Robust stability of networks of iISS systems: construction of sum-type Lyapunov functions. *IEEE Trans. Automat. Contr.*, 58:1192–1207, 2013a.

H. Ito, Z.P. Jiang, and P. Pepe. Construction of Lyapunov-Krasovskii functionals for networks of iISS retarded systems in small-gain formulation. *Automatica*, 49: 3246–3257, 2013b.

H. Ito and F. Mazenc. An iISS formulation for establishing robust stability of dynamical networks with neutral, retarded and communication delay. In *Proc. 2012 American Control Conf.*, pages 6394–6399, 2012.

Z.P. Jiang, A.R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Math. Contr. Signals and Syst.*, 7:95–120, 1994.

I. Karafyllis and Z.P. Jiang. *Stability and stabilization of nonlinear systems.* Springer, London, 2011.

I. Karafyllis, P. Pepe, and Z.P. Jiang. Stability results for systems described by coupled retarded functional differential equations and functional difference equations. *Nonlinear Analysis: Theory, Methods & Applications*, 71:3339–3362, 2009.

V. Kolmanovskii and A. Myshkis. *Introduction to the Theory and Applications of Functional Differential Equations.* Kluwer Academic Publishers, Dordrecht, 1999.

F. Mazenc, H. Ito, and P. Pepe. Construction of Lyapunov functionals for coupled differential and continuous time difference equations. In *52nd IEEE Conf. Decision Contr.*, pages 2245–2250, 2013.

S.-I. Niculescu. *Delay effects on stability: a robust control approach.* Springer, Berlin, 2001.

E. Panteley and A. Loría. Growth rate conditions for uniform asymptotic stability of cascaded time-varying systems. *Automatica*, 37:453–460, 2001.

P. Pepe. On Liapunov-Krasovskii functionals under Cara -théodory conditions. *Automatica*, 43:701–706, 2007.

P. Pepe, I. Karafyllis, and Z.P. Jiang. On the Liapunov-Krasovskii methodology for the ISS of systems described by coupled delay differential and difference equations. *Automatica*, 44:2266–2273, 2008.

V. Rasvan. Functional differential equations of lossless propagation and almost linear behavior. In *Plenary Lecture at 6th IFAC Workshop on Time-Delay Systems, IFAC-PapersOnline*, 2006.

E.D. Sontag and Y. Wang. On characterizations of input-to-state stability property. *Systems & Contr. Letters*, 24:351–359, 1995.