Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Robust stability and stabilization of uncertain fractional-order descriptor nonlinear system

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Abstract: This paper considers the robust stability for uncertain fractional-order (FO) descriptor nonlinear systems. A key analysis technique is enabled by proposed a fundamental boundedness lemma, for the first time. It is used for rigorous robust stability analysis of FO systems, especially for Mittag-Leffler stability analysis of FO nonlinear systems. More importantly, how to obtain a more accurate bound is analyzed to reduce conservative. An FO proportional-derivative controller is utilized to normalize the descriptor system. Furthermore, a criterion for stability of the normalized FO nonlinear system is provided by utilizing linear matrix inequality (LMI). Finally, two illustrative examples show the effectiveness of the proposed stability notion.

1. INTRODUCTION

Fractional calculus investigates integrals and derivatives of any orders [1, 2], which is introduced in early 17 century. The stability and stabilization theorems of FO systems have obtained a lot of attention[3, 4]. Many results about FO systems have been obtained based on these theorems, such as ([5]-[11]). In these papers, $V = x^T x$ is selected as the Lyapunov function.

$$\frac{d^{\bar{\beta}}V}{dt^{\bar{\beta}}} = D^{\bar{\beta}}V = (D^{\bar{\beta}}x)^T x + x^T(D^{\bar{\beta}}x) + 2\Upsilon, \tag{1}$$

where $\Upsilon = \sum\limits_{l=1}^{\infty} \frac{\Gamma(1+\bar{\beta})(D^l x)^T (D^{\bar{\beta}-l}x)}{\Gamma(1+l)\Gamma(1-l+\bar{\beta})}$ and $\bar{\beta}$ is an arbitrary finite positive non-integer, $D^{\bar{\beta}}$ is the $\bar{\beta}$th-order fractional derivative operator. To prove the stability of FO systems, the boundedness condition is assumed in these papers

$$\left|\sum_{l=1}^{\infty} \frac{\Gamma(1+\bar{\beta})(D^l x)^T (D^{\bar{\beta}-l}x)}{\Gamma(1+l)\Gamma(1-l+\bar{\beta})}\right| \leq \mu \|x\|, \tag{2}$$

in which $\|\cdot\|$ denotes an arbitrary norm.

However, since $\sum\limits_{l=1}^{\infty} \frac{\Gamma(1+\bar{\beta})(D^l x)^T (D^{\bar{\beta}-l}x)}{\Gamma(1+l)\Gamma(1-l+\bar{\beta})}$ is the sum of an infinite series, the existence of the boundedness condition needs to be verified. Otherwise, the results in the above literatures are questionable, since the main stability analysis technique relies on the boundedness condition (2). In order to investigate the boundedness condition, the problem when $\bar{\beta} \in (0, 1)$ has been discussed in our previous work [12]. Nevertheless, how to obtain a more accurate bound μ is an interesting problem. Furthermore, if $\bar{\beta}$ is not restricted within $(0, 1)$, such as [9], the boundedness condition should be verified for more general FO systems, due to the lack of appropriate mathematical tools.

With this motivation, we will prove the boundedness condition on Υ and establish a fundamental boundedness lemma in this paper. This fundamental boundedness lemma is derived to use for stability analysis of FO systems, especially for Mittag-Leffler stability analysis of FO nonlinear systems.

On the other hand, a descriptor system is presented by combining differential equation with algebraic equation [13]. Intuitively, a descriptor nonlinear system has a stricter exprecession for an extensive class of systems than a state-space system. It has been employed to many areas, such as robotics, power systems, and economic plants [14, 15]. However, there exist limited works [16, 17, 18] about the stability and stabilization of FO singular nonlinear systems, due to the lack of appropriate mathematical tools.

This paper studies the stability and stabilization for FO descriptor nonlinear systems with uncertainty, based on our proposed fundamental boundedness lemma. The central idea is firstly proving a fundamental boundedness lemma, which is utilized to stability analysis of FO nonlinear systems. According to this lemma, the FO PD controller is derived to not only to normalize the descriptor FO system, but also achieve the robust stabilization of the normalized FO system. Furthermore, the novel stability criteria is given via LMIs. Some simulations show the effectiveness of the proposed stability notion.

2. PROBLEM FORMULATIONS

Consider the uncertain FO descriptor nonlinear system

$$\begin{aligned} &ED^{\beta}x(t) = A(t)x(t) + F(t)f(\upsilon(t)) + Bu(t), \\ &\upsilon(t) = Hx(t) \end{aligned} \tag{3}$$

where $x(t) \in R^n, u(t) \in R^m, \upsilon \in R^\eta$ denote the state vector, the control input and the output, $f(\cdot) \in R^l$ represents the vector of the nonlinearities, $E \in R^{n\times n}$ is a singular square matrix, $B \in R^{n\times m}, H \in R^{\eta\times n}$ represent constant known matrices, $A(t), F(t)$ are matrix functions which contain time-varying uncertainties, and β is the fractional commensurate order satisfying $0 < \beta < 1$. Let $A(t) = A+\delta_1(t), F(t) = F+\delta_2(t)$, in which $A \in R^{n\times n}, F \in R^{n\times l}$ are constant known matrices and $\delta_1(t), \delta_2(t)$ are time-varying uncertain of appropriate dimensions. They are assumed as

$$[\delta_1(t)\ \ \delta_2(t)] = DG(t)[N_1\ \ N_2], \tag{4}$$

in which D, N_1, N_2 are constant known matrices, the unknown function $G(t)$ satisfies $\|G(t)\| < 1, \forall t \geq 0$. We assume that $f(\upsilon)\hat{=}[f_1(\upsilon_1),\ f_2(\upsilon_2),\ \cdots,\ f_l(\upsilon_l)]^T$ belongs to sector bound $[b_i, a_i]$ i.e. $b_i \leq \frac{f_i(\upsilon_i)}{\upsilon_i} \leq a_i$. Thus, $f_i(\cdot)$ could be written by using a convex mixture of b_i, a_i

$$f_i(\upsilon_i) = (\Lambda_i^u(\upsilon_i)b_i + \Lambda_i^l(\upsilon_i)a_i)\upsilon_i, \ \ i = 1, 2, \cdots, l, \tag{5}$$

where $\Lambda_i^l(\upsilon_i) = \frac{f_i(\upsilon_i)-b_i\upsilon_i}{(a_i-b_i)\upsilon_i}, \Lambda_i^u(\upsilon_i) = \frac{a_i\upsilon_i-f_i(\upsilon_i)}{(a_i-b_i)\upsilon_i}$.

Since $\Lambda_i^l(\upsilon_i) + \Lambda_i^u(\upsilon_i) = 1, \Lambda_i^u(\upsilon_i) \geq 0$ and $\Lambda_i^l(\upsilon_i) \geq 0$, $f_i(\upsilon_i)$ can be expressed as $f_i(\upsilon_i) = \Lambda_i(\upsilon_i)\upsilon_i$, in which $\Lambda_i(\upsilon_i)$ belongs to the convex hull $Co\{b_i, a_i\}$, with Co denotes the convex hull. Next, define

$$\Lambda = \mathrm{diag}\{\Lambda_1(\upsilon_1(t)), \Lambda_2(\upsilon_2(t)), \cdots, \Lambda_l(\upsilon_l(t))\},$$
$$\Delta_1 = \mathrm{diag}\{b_1, b_2, \cdots, b_l\}, \Delta_2 = \mathrm{diag}\{a_1, a_2, \cdots, a_l\}.$$

Thus, $f(\upsilon)$ can be written as

$$f(\upsilon) = \Lambda\upsilon, \tag{6}$$

where $\Lambda \in Co\{\Delta_1, \Delta_2\}$

In order to derive the main results, the following definition and lemmas are introduced.

Definition 1. [1] Fractional calculus performs an important role in modern sciences.

$$I^{\hat{\alpha}}\zeta(t) = \frac{1}{\Gamma(\hat{\alpha})}\int_0^t (t-\rho)^{\hat{\alpha}-1}\zeta(\tau)d\rho, \tag{7}$$

in which $\zeta(t)$ denotes an any integrable function, $I^{\hat{\alpha}}$ represents the $\hat{\alpha}$th order fractional integral on $[0, t]$, and $\Gamma(\cdot)$ is the Gamma function. Likewise, the Riemann-Liouville definition of the $\tilde{\beta}$th-order derivative is

$$D^{\tilde{\beta}}\zeta(t) = \frac{1}{\Gamma(n-\tilde{\beta})}(\frac{d}{dt})^n \int_0^t \frac{\zeta(\tau)}{(t-\tau)^{1+\tilde{\beta}-n}}d\tau, \tag{8}$$

in which $n - 1 \leq \tilde{\beta} < n$, n is an integer.

Lemma 2. [2] The fractional integration operator $I^{\hat{\alpha}}$ with fractional-order $\hat{\alpha}, (\hat{\alpha} \in C, \mathrm{Re}(\hat{\alpha}) > 0)$, is bounded in $L_p(\hat{a}, \hat{b}), (1 \leq p \leq \infty, -\infty < \hat{a} < \hat{b} < +\infty)$:

$$||I^{\hat{\alpha}}\zeta|| \leq K||\zeta|| \ \ , \quad (K = \frac{(\hat{b}-\hat{a})^{\mathrm{Re}(\hat{\alpha})}}{|\Gamma(\hat{\alpha})|\,\mathrm{Re}(\hat{\alpha})}). \tag{9}$$

Lemma 3. (Fundamental boundedness lemma) Consider the FO nonautonomous system

$$D^{\bar{\beta}}x(t) = h(x, t)$$

with initial condition $x(t_0)$, where $\bar{\beta}$ is an arbitrary finite positive non-integer, $h : \Omega \times [0, +\infty] \to R^n$ is locally Lipschitz in x and piecewise continuous in t on $\Omega \times [0, +\infty]$ and $\Omega \in R^n$ denotes a closed set that includes the origin $x = 0$. Moreover, x_0 represents an equilibrium point (without loss of generality, assume that $x = 0$ is the equilibrium point). Consider a Lyapunov candidate $V(t) = x^Tx$. The $\bar{\beta}$th-order time derivative of $V(t)$ can be expanded as $V^{(\bar{\beta})} = (D^{\bar{\beta}}x)^Tx + x^T(D^{\bar{\beta}}x) + 2\Upsilon$, where $\Upsilon = \sum\limits_{l=1}^{\infty} \frac{\Gamma(1+\bar{\beta})(D^lx)^T(D^{\bar{\beta}-l}x)}{\Gamma(1+l)\Gamma(1-l+\bar{\beta})}$, by according to the Leibniz's rule of differentiation. Then, there exists $\mu > 0$ such that the sum of the infinite terms Υ involving fractional-order derivatives is bounded satisfied as

$$\left|\sum_{l=1}^{\infty} \frac{\Gamma(1+\bar{\beta})(D^lx)^T(D^{\bar{\beta}-l}x)}{\Gamma(1+l)\Gamma(1-l+\bar{\beta})}\right| \leq \mu\,\|x\|. \tag{10}$$

Proof. First, one has

$$\left|\sum_{l=1}^{\infty} \frac{\Gamma(1+\bar{\beta})}{\Gamma(1+l)\Gamma(1-l+\bar{\beta})}(D^lx)^T(D^{\bar{\beta}-l}x)\right|$$
$$\leq \sum_{l=1}^{\infty} \frac{\Gamma(1+\bar{\beta})}{\Gamma(1+l)|\Gamma(1-l+\bar{\beta})|}\left\|D^lx\right\|\left\|D^{\bar{\beta}-l}x\right\|. \tag{11}$$

Since $D^lx, (l = 1, 2, 3, \cdots)$ exist, D^lx are continuous. Moreover, one has that $D^lx, (l = 1, 2, 3, \cdots)$ are bounded since Ω is the closed set. Thus, there exists M such that $\left\|D^lx\right\| \leq M, (l = 1, 2, 3, \cdots)$.

Since $\bar{\beta}$ is a positive non-integer, there exists an integer $\hat{N}$ such that $\hat{N} - 1 < \bar{\beta} < \hat{N}$. $D^{\bar{\beta}-l}x, (l = 1, 2, 3, \cdots)$ can be divided two parts $D^{\bar{\beta}-l}x, (l = 1, 2, \cdots, \hat{N} - 1)$ and $D^{\bar{\beta}-l}x, (l = \hat{N}, \hat{N} + 1, \hat{N} + 2, \cdots)$. Firstly, for $D^{\bar{\beta}-l}x, (l = 1, 2, \cdots, \hat{N} - 1)$, one has that $\left\|D^{-l}y\right\| = \left\|I^ly\right\| \leq \bar{K}_{\max}\|y\|, (l = 1, 2, \cdots, \hat{N} - 1)$ in which $\bar{K}_{\max} > 0$, according to Lemma 2. Meanwhile, there exists $L > 0$ such that$\|h(x, t)\| \leq L\|x\|$, because that h is Lipschitz in x on Ω. Hence, one has $\left\|D^{\bar{\beta}-l}x\right\| \leq \bar{K}_{\max}L\|x\|$. For $D^{\bar{\beta}-l}x, (l = \hat{N}, \hat{N} + 1, \hat{N} + 2, \cdots)$, according to Lemma 2, there exists $K_{\max} > 0$ such that $\left\|D^{\beta-l}x\right\| \leq K_{\max}\|x\|, (l = \hat{N}, \hat{N} + 1, \hat{N}+2, \cdots)$. Let $K = \max\{\bar{K}_{\max}L, K_{\max}\}$, therefore, one has $\left\|D^{\bar{\beta}-l}x\right\| \leq K\|x\|, (l = 1, 2, 3, \cdots)$.

Furthermore, there exist no complexor z satisfying $\Gamma(z) = 0)$ [13]. Thus, $1/\Gamma(z)$ should be an entire function, with zeros at $z = 0, -1, -2, \cdots$. Hence, one has $0 < L_{\min} \leq |\Gamma(1 - \bar{\beta} + l)|$ for $l = 1, 2, 3, \cdots$, where $L_{\min} > 0$.

Because $\frac{\Gamma(l)}{\Gamma(l+1)} = \frac{1}{l}, (l = 1, 2, 3, \cdots)$, the infinite series $\sum\limits_{l=1}^{\infty} \frac{1}{\Gamma(1+l)}$ is convergence. Therefore, there exists $\tilde{H} > 0$ such that $0 < \sum\limits_{l=1}^{\infty} \frac{1}{\Gamma(1+l)} < \tilde{H}$.

From to the above analysis, the following inequality can be derived

$$\left|\sum_{l=1}^{\infty} \frac{\Gamma(1+\bar{\beta})(D^lx)^T(D^{\bar{\beta}-l}x)}{\Gamma(1+l)\Gamma(1-l+\bar{\beta})}\right| \leq \mu\,\|x\|, \tag{12}$$

where $\mu = \frac{\Gamma(1+\bar{\beta})MK\tilde{H}}{L_{\min}}$. This completes the proof.

Remark 4. In order to investigate the boundedness condition, the problem when $\bar{\beta} \in (0,1)$ has been discussed in our previous work [12]. Based on this work [12], the boundedness condition on Υ should be derived for more general FO nonlinear systems. The above lemma is proposed to solve this issue in its full generality.

Lemma 5. ([20]) For constant matrices Ξ_1, Ξ_2, Ξ_3, in which $\Xi_1 = \Xi_1^T, \Xi_2 = \Xi_2^T > 0$, then $\Xi_1 + \Xi_3^T \Xi_2^{-1} \Xi_3 < 0$ if and only if

$$\begin{bmatrix} \Xi_1 & \Xi_3^T \\ \Xi_3 & -\Xi_2 \end{bmatrix} < 0, \quad \text{or} \quad \begin{bmatrix} -\Xi_2 & \Xi_3 \\ \Xi_3^T & \Xi_1 \end{bmatrix} < 0. \tag{13}$$

Lemma 6. ([20]) For any matrix M_1 and M_2 of compatible dimensions and any scalar $\varsigma > 0$, one has $M_1^T M_2 + M_2^T M_1 \le \varsigma M_1^T M_1 + (1/\varsigma) M_2^T M_2$.

Lemma 7. [18] Assume that $\Omega \in C^{n\times n}$ is a complex matrix. Then Ω is nonsingular if and only if a nonsingular matrix $\Psi \in C^{n\times n}$ satisfies

$$\Omega\Psi + \bar{\Psi}\bar{\Omega}^T < 0, \tag{14}$$

where $\bar{\Omega}$ is the complex conjugate of Ω.

Lemma 8. [3] Assume that $x = 0$ is an equilibrium point of $D^\beta x(t) = h(x,t)$. If there exists a Lyapunov candidate $V(t,x)$ satisfying

$$\begin{aligned} &\tilde{\alpha}_1 \|x\|^{\tilde{a}} \le V(t,x) \le \tilde{\alpha}_2 \|x\|^{\tilde{a}\tilde{b}}, \\ &D^\beta V(t,x) \le -\tilde{\alpha}_3 \|x\|^{\tilde{a}\tilde{b}}, \end{aligned} \tag{15}$$

in which $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3, \tilde{a}, \tilde{b} > 0$. Then $x = 0$ is Mittag-Leffler stable.

3. ROBUST STABILITY AND STABILIZATION

3.1 Normalization

The technology of converting descriptor systems into normal systems is named a regularization or normalization [13]. An FO PD controller for the uncertain FO descriptor nonlinear system (3) is proposed to normalize the system

$$u(t) = K_d D^\beta x(t) + K_p x(t), \tag{16}$$

where K_d, K_p are gain matrices. They should guarantee the system (3) can be normalized. Furthermore, the corresponding normalized plant with the controller can be asymptotically stable. Adding the controller (16) into the system (3), one has

$$\begin{aligned} (E - BK_d)D^\beta x(t) = &(A + BK_p + \delta_1(t))x(t) \\ &+(F + \delta_2(t))f(v(t)). \end{aligned} \tag{17}$$

Next, we consider how to normalize the FO descriptor nonlinear system (3) by choosing the gain matrix K_d. Let us consider the system (17). The following proposition provides the existence condition of normalizable, by selecting K_d in the LMI formulation.

Proposition 9. The system (3) is normalizable if and only if the LMI

$$E\Theta - BW + \Theta^T E^T - W^T B^T < 0, \tag{18}$$

exists, in which Θ is a nonsingular matrix and W satisfying $K_d = W\Theta^{-1}$.

It is obvious that the system (3) is normalizable iff there exists K_d such that $(E - BK_d)$ is nonsingular. According to Lemma 7, one has $(E - BK_d)$ is nonsingular if and only if

$$(E - BK_d)\Theta + \Theta^T (E - BK_d)^T < 0, \tag{19}$$

in which Θ is a nonsingular matrix. The inequality (19) equals to the LMI (18). In the following, the system (17) is assumed to be normalizable. Moreover, K_d can be solved from the result of Proposition 9. Next, the design of the gain matrices in the FO PD controller will be investigated in order to ensure the stabilization of the corresponding normalized system.

3.2 Robust stability and stabilization analysis

From Proposition 9, the normalized system is derived

$$\begin{aligned} D^\beta x(t) = &E_1(A + BK_p + \delta_1(t))x(t) \\ &+E_1(F + \delta_2(t))f(v(t)), \end{aligned} \tag{20}$$

where $E_1 = (E - BK_d)^{-1}$. In the following, the stability analysis of the normalized FO nonlinear system will be investigated.

Theorem 10. Assume that the uncertain FO descriptor nonlinear system (3) is normalizable, then there exist gain matrices K_d, K_p such that the corresponding normalized system (19) under the FO PD control (16) is asymptotically stable, if there exist any matrices $M_i, (i = 1,2,3)$ with appropriate dimensions and positive scalars $\varepsilon_1, \varepsilon_2, \varepsilon_3$ satisfying:

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & \Xi_{13} & \Xi_{14} & \Xi_{15} & 0 & 0 & 0 & 0 & \Xi_{1r} \\ * & \Xi_{22} & \Xi_{23} & 0 & 0 & \Xi_{26} & 0 & 0 & \Xi_{29} & 0 \\ * & * & \Xi_{33} & \Xi_{34} & 0 & 0 & \Xi_{37} & \Xi_{38} & 0 & 0 \\ * & * & * & \Xi_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \Xi_{55} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Xi_{66} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Xi_{77} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Xi_{88} & 0 & 0 \\ * & * & * & * & * & * & * & * & \Xi_{99} & 0 \\ * & * & * & * & * & * & * & * & * & \Xi_{rr} \end{bmatrix} < 0, \tag{21}$$

where

$\Xi_{11} = (E_1 A + E_1 BK_p)^T + E_1 A + E_1 BK_p + 2\mu I$ $+ (M_1 \Lambda H)^T + M_1 \Lambda H + \varepsilon_1 N_1^T N_1, \Xi_{12} = (M_2 \Lambda H)^T,$
$\Xi_{13} = E_1 F - M_1 + (M_3 \Lambda H)^T, \Xi_{14} = N_1^T, \Xi_{15} = E_1 D,$
$\Xi_{1r} = E_1 D, \Xi_{22} = -\varepsilon_3 I, \Xi_{23} = -M_2, \Xi_{26} = \Xi_{29} = E_1 D,$
$\Xi_{33} = -M_3 - M_3^T + \varepsilon_2 N_2^T N_2, \Xi_{34} = N_2^T, \Xi_{37} = E_1 D,$
$\Xi_{38} = E_1 D, \Xi_{44} = -(\varepsilon_3 I)^{-1}, \Xi_{55} = -\varepsilon_1 I, \Xi_{66} = -\varepsilon_1 I,$
$\Xi_{77} = -\varepsilon_1 I, \Xi_{88} = -\varepsilon_2 I, \Xi_{99} = -\varepsilon_2 I, \Xi_{rr} = -\varepsilon_2 I.$

Proof. First, define $\xi(t) = [x^T(t) \;\; p^T(t) \;\; f^T(v(t))]^T$. Consider $V(t) = x^T x$. The βth-order time derivative of V is

$$D^\beta V = (D^\beta x)^T x + x^T (D^\beta x) + 2\Upsilon, \tag{22}$$

where $\Upsilon = \sum_{l=1}^{\infty} \frac{\Gamma(1+\beta)(D^l x)^T (D^{\beta - l} x)}{\Gamma(1+l)\Gamma(1-l+\beta)}$.

From Lemma 3, one has

$$D^\beta V \le \xi^T(t) \begin{bmatrix} \Omega_{11} & 0 & E_1(F + \delta_2) \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} \xi(t). \tag{23}$$

where $\Omega_{11} = (E_1 A + E_1 BK_p)^T + E_1 A + E_1 BK_p + 2\mu I + E_1\delta_1(t) + \delta_1^T(t) E_1^T$. Considering the uncertainty δ_1 in the above equality (23), from Lemma 6, one has

$$\begin{bmatrix} E_1DG(t)N_1+(E_1DG(t)N_1)^T & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}$$

$$= \Sigma \begin{bmatrix} G(t) & 0 & 0 \\ 0 & G(t) & 0 \\ 0 & 0 & G(t) \end{bmatrix} \begin{bmatrix} N_1 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} N_1^T & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} \begin{bmatrix} G^T(t) & 0 & 0 \\ 0 & G^T(t) & 0 \\ 0 & 0 & G^T(t) \end{bmatrix} \Sigma^T$$

$$\leq \varepsilon_1 \begin{bmatrix} N_1^T N_1 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} + 1/\varepsilon_1 \Sigma\Sigma^T, \quad (24)$$

in which $\Sigma = \begin{bmatrix} E_1D & 0 & 0 \\ 0 & E_1D & 0 \\ 0 & 0 & E_1D \end{bmatrix}$.

Similarly, considering the uncertainty δ_2, one has

$$\begin{bmatrix} 0 & 0 & E_1DG(t)N_2 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & E_1D \\ 0 & E_1D & 0 \\ E_1D & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & G(t) \\ 0 & G(t) & 0 \\ G(t) & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & N_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ N_2^T & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & G^T(t) \\ 0 & G^T(t) & 0 \\ G^T(t) & 0 & 0 \end{bmatrix} \Pi$$

$$\leq \varepsilon_2 \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & N_2^T N_2 \end{bmatrix} + \frac{1}{\varepsilon_2} \begin{bmatrix} 0 & 0 & E_1D \\ 0 & E_1D & 0 \\ E_1D & 0 & 0 \end{bmatrix} \Pi, \quad (25)$$

in which $\Pi = \begin{bmatrix} 0 & 0 & D^TE_1^T \\ 0 & D^TE_1^T & 0 \\ D^TE_1^T & 0 & 0 \end{bmatrix}$.

Utilizing the convex property of $f(v)$ in (23), one has

$$f(v) = \Lambda Hx. \quad (26)$$

In order to derive a less conservative stability criterion, the zero equation is used

$$2\xi^T(t) \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} [\Lambda H \quad 0 \quad -I]\xi(t) = \xi^T(t)\Pi\xi(t) = 0. \quad (27)$$

where

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} \\ * & \Pi_{22} & \Pi_{23} \\ * & * & \Pi_{33} \end{bmatrix},$$

with $\Pi_{11} = (M_1\Lambda H)^T + M_1\Lambda H, \Pi_{12} = (M_2\Lambda H)^T, \Pi_{13} = -M_1 + (M_3\Lambda H)^T, \Pi_{22} = 0, \Pi_{23} = -M_2, \Pi_{33} = -M_3 - M_3^T$. On the other hand, define $p(t) = G(t)(N_1x(t) + N_2f(v(t))$. Therefore, the inequality can be obtained

$$p^T(t)p(t) \leq \xi^T(t)N^TN\xi(t), \quad (28)$$

where $N = [N_1 \ 0 \ N_2]$. Using the S-procedure [21], suppose there exists $\varepsilon_3 > 0$ satisfying

$$\begin{bmatrix} \bar{\Omega}_{11} & 0 & E_1F \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix} + \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} \\ * & \Pi_{22} & \Pi_{23} \\ * & * & \Pi_{33} \end{bmatrix} + \varepsilon_1 \begin{bmatrix} N_1^TN_1 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{bmatrix}$$

$$+\Sigma\Sigma^T + \varepsilon_2 \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & N_2^TN_2 \end{bmatrix} + \frac{1}{\varepsilon_2} \begin{bmatrix} 0 & 0 & E_1D \\ 0 & E_1D & 0 \\ E_1D & 0 & 0 \end{bmatrix} \Pi$$

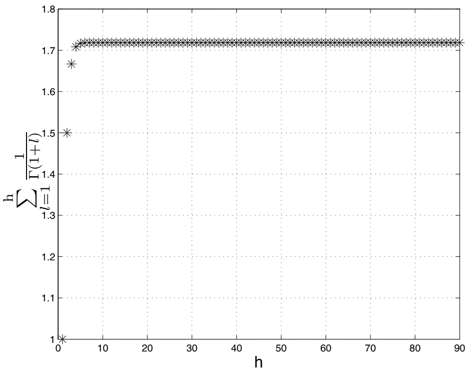

Fig. 1. The value of $\sum_{l=1}^{h} \frac{1}{\Gamma(1+l)}$.

$$+\varepsilon_3 N^TN + \begin{bmatrix} 0 & 0 & 0 \\ * & -\varepsilon_3 I & 0 \\ * & * & 0 \end{bmatrix} < 0, \quad (29)$$

in which $\bar{\Omega}_{11} = (E_1A + E_1BK_p)^T + E_1A + E_1BK_p + 2\mu I$. Then, based on Lemma 8, we have $\dot{V} < 0$. By Lemma 5, the inequality (29) is satisfied if the inequality (21) holds. Hence, it guarantees the normalized system asymptotic stability.

3.3 Obtaining a accurate bound and reduce conservatism

For the stability and stabilization of the FO nonlinear system (19), the key analysis method is enabled by Lemma 3. Obtaining a more accurate bound μ can help to reduce conservatism. The following remark is presented to show how to derive a more accurate bound in the fundamental boundedness lemma.

Remark 11. The fundamental boundedness lemma has been proven in Lemma 3. Furthermore, how to obtain a more accurate bound μ can be solved by analyzing of the upper bound $\tilde{H} > 0$. From the definition of Gamma function, one has $\Gamma(1 + l) = l!$. Therefore,

$$\sum_{l=1}^{\infty} \frac{1}{\Gamma(1+l)} = 1 + \sum_{l=1}^{\infty} \frac{1}{(l+1)!}.$$

Set $\tilde{a}_l = \dfrac{1}{(l+1)!}, \tilde{b}_l = \dfrac{1}{(l+1)l}$. Obviously, $0 < \tilde{a}_l \leq \tilde{b}_l, (l = 1, 2, 3, \cdots)$. Now, one can calculate that

$$\sum_{l=1}^{n} \frac{1}{(l+1)l} = \sum_{l=1}^{n} (\frac{1}{l} - \frac{1}{l+1}) = 1 - \frac{1}{n+1} \to 1, (n \to \infty).$$

So, $\sum_{l=1}^{\infty} \tilde{b}_l \leq 1$. So $\sum_{l=1}^{\infty} \frac{1}{\Gamma(1+l)} \leq 1 + \sum_{l=1}^{\infty} \tilde{b}_l = 2$. ($\sum_{l=1}^{h} \frac{1}{\Gamma(1+l)}$ *is depicted in Fig. 1. It shows* $\sum_{l=1}^{\infty} \frac{1}{\Gamma(1+l)} < 2$.) Thus, one can set $\tilde{H} = 2$. Hence, one can obtain $\mu = \frac{2\Gamma(1+\bar{\beta})MK}{L_{\min}}$.

Remark 12. In our previous work [12], the boundedness property for $\bar{\beta} \in (0, 1)$ can be used in ([5, 6, 7, 8, 9, 11]). However, other references (such as [10]) use the boundedness condition $\left| \sum_{l=1}^{\infty} \frac{\Gamma(1+\bar{\beta})(D^lx)^T(D^{\bar{\beta}-l}x)}{\Gamma(1+l)\Gamma(1-l+\bar{\beta})} \right| \leq \mu \|x\|, \bar{\beta} \in$

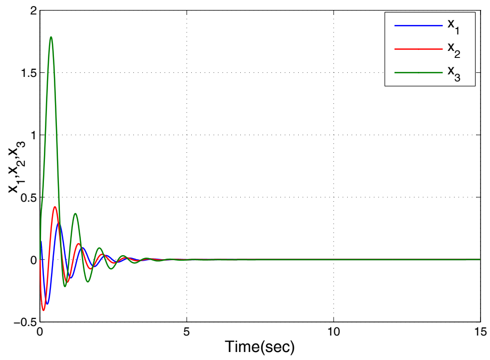

Fig. 2. Time responses of x_1, x_2, x_3 for the FO descriptor nonlinear system under the FO PD controller.

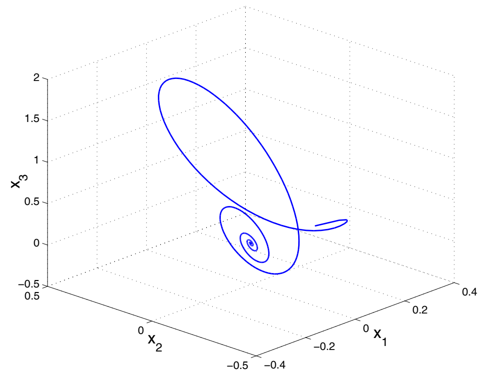

Fig. 3. The state trajectory of the FO descriptor nonlinear system under the FO PD controller.

$(1, 2)$, which cannot be verified by the results in [12]. Based on Theorem 10, the boundedness condition on Υ can be used for more general FO systems.

4. SIMULATION RESULT

4.1 Example 1.

Consider the FO descriptor nonlinear system shown in (3) with the following parameters

$$E = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -\bar{a}m_1 & a & 0 \\ 1 & -1 & 1 \\ 0 & -\bar{b} & 0 \end{bmatrix}, \delta_1(t) = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{bmatrix},$$
$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, F = \begin{bmatrix} -\bar{a}(\varpi_0 - \varpi_1) \\ 0 \\ 0 \end{bmatrix}, H = [1 \ 0 \ 0], \quad (30)$$

with $f(x_1) = 0.5(\varpi_0 - \varpi_1)(|x_1 + \varrho| - |x_1 - \varrho|), \epsilon = 0.01\sin(0.5t)$ and $\beta = 0.9, \bar{a} = 9, \bar{b} = 14.28, \varrho = 1, \varpi_0 = -(1/7), \varpi_1 = (2/7)$. The nonlinear function $f(x_1)$ belongs to the sector bound $b_1 = 0, a_1 = 10$. The uncertain $\delta_1 = DG(t)N_1$ is expressed as

$$D = N_1 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, G(t) = \sin(0.5t) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (31)$$

The purpose of designing an FO PD control law is to guarantee the stable of the resulting normalized FO system. Let $K_d = [0 \ \ 0 \ \ -1]$. It is obvious that the system with the above parameters (30) can be normalized by the gain K_d. Then, in order to stable the system via the FO PD controller, let us solve the problem given in Theorem 10. By computing the LMI (21), one has $K_p = [-12.4681 \ 2.4812 \ -0.7860]$.

By applying the controller, the time responses of x_1, x_2, x_3 for the FO descriptor nonlinear system are drawn in Fig. 2 with $x(0) = [0.11 \ \ -0.17 \ \ 0.22]^T$. It means that x_1, x_2, x_3 converge to zero. Fig. 3 illustrates the state trajectory of the system under the FO PD controller. The time response of the controller u is depicted in Fig. 4.

4.2 Example 2.

Consider the FO descriptor system shown in (3) with the following parameters

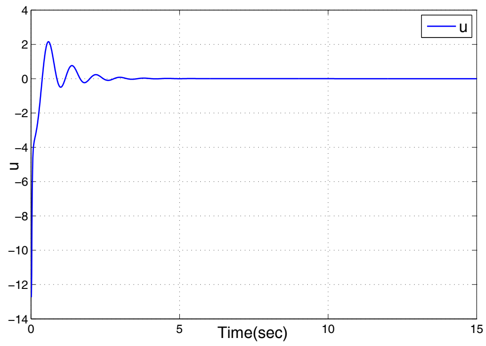

Fig. 4. The time response of the controller u.

$$E = \begin{bmatrix} 1 & 0 & 0.5 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 2.4 & 0.2 & 1.2 \\ -2 & -1.5 & -4 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix},$$
$$\delta_1 = \begin{bmatrix} 0.01\sin(0.2t) & 0 & 0 \\ 0 & 0.01\sin(0.2t) & 0 \\ 0 & 0 & 0.01\sin(0.2t) \end{bmatrix}. \quad (32)$$

In order to normalize the above FO descriptor system

$$\Theta = \begin{bmatrix} -0.1328 & 1.5656 & -1.0313 \\ 1.5656 & 0.6657 & 0.7753 \\ -1.0313 & 0.7753 & -0.4713 \end{bmatrix},$$
$$W = \begin{bmatrix} -1.3845 & 0.5131 & 0.2415 \\ 0.6015 & -0.3557 & -1.4225 \end{bmatrix}, \quad (33)$$

the gain matrix K_d is given by

$$K_d = \begin{bmatrix} -0.8754 & 0.4099 & 2.0775 \\ 1.6701 & -1.2761 & -2.7355 \end{bmatrix}. \quad (34)$$

Hence, the FO descriptor system with the parameters (32) under the FO PD controller can be normalized. Then, to stable the system via the controller, let us solve the problem given in Theorem 10. By calcuting the LMI (21), one obtains

$$K_p = \begin{bmatrix} -1.8017 & -2.0010 & -0.1002 \\ 3.8547 & -1.9850 & 2.8851 \end{bmatrix}. \quad (35)$$

Fig. 5-6 show the simulation results for the selected system under the FO PD controller. By applying the FO PD controller, the time responses x_1, x_2, x_3 for the normalized

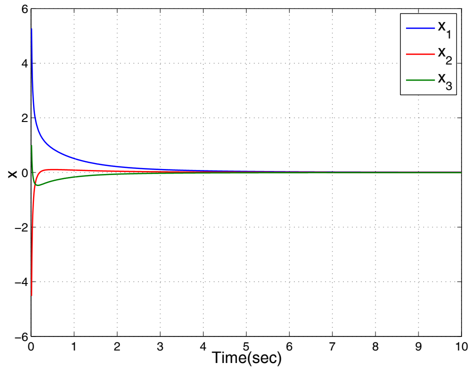

Fig. 5. Time responses of x_1, x_2, x_3 for the FO descriptor nonlinear system under the FO PD controller.

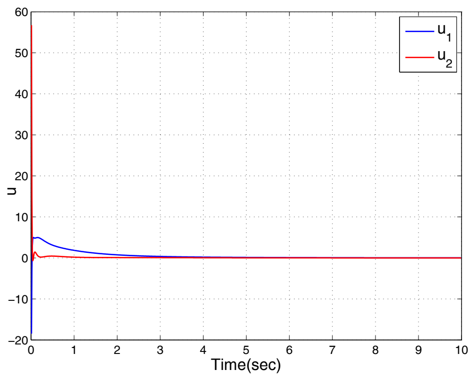

Fig. 6. The time responses of the controller u.

FO system (43) are shown in Fig. 5 with $x(0) = [5.2 \ \ -4.5 \ \ 1]^T$. This shows that x_1, x_2, x_3 to zero. Fig. 6 shows the time responses of the FO PD control input $u = [u_1 \ \ u_2]^T$. They support the effectiveness of the proposed stability notion.

5. CONCLUSION

This paper has investigated the stability and stabilization of uncertain FO nonlinear system under the FO PD controller. The key issue is to prove the boundedness property of $\sum_{l=1}^{\infty} \frac{\Gamma(1+\beta)(D^l x)^T (D^{\beta-l} x)}{\Gamma(1+l)\Gamma(1-l+\beta)}$. In addition, how to reach a more accurate bound μ and reduce conservatism has been analyzed. With the help of the fundamental boundedness lemma, the FO PD controller has been applied to ensure the regulation of the FO descriptor system. Furthermore, it has also guaranteed stability and stabilization of the normalized FO nonlinear system. Two examples have been used to show the effectiveness of the proposed stability notion.

REFERENCES

[1] I. Podlubny. *Fractional Differential Equations.* New York: Academic Press, 1999.
[2] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional Integrals and Derivatives: Theory and Applications.* Gordon and Breach Science Publishers, Switzerland, 1993.
[3] Y. Li, Y. Q. Chen, and I. Podlubny. Mittag-Leffler stability of fractional order nonlinear dynamic systems. *Automatica*, 45, pages 1965–1969, 2009.
[4] J. G. Lu and G. Chen. Robust stability and stabilization of fractional-order interval system: an LMI approach. *IEEE Trans. Autom. Control*, 54, pages 1294–1299, 2009.
[5] M. Ö. Efe. Fractional fuzzy adaptive sliding-mode control of a 2-DOF direct-drive robot arm. *IEEE Trans. Syst., Man, Cybern.*, 38, pages 1561–1570, 2008.
[6] M. Ö. Efe. Adaline based robust control in robotics: a Riemann-Liouville fractional differintegration based learning scheme. *Soft Comput.*, 13, pages 23–29, 2009.
[7] M. Ö. Efe. Fractional Order Systems in Industrial Automation Survey. *IEEE Trans. Ind. Inf.*, 7, pages 582-591, 2011.
[8] M. P. Aghababa. Comments on Adaptive synchronization of fractional-order chaotic systems via a single driving variable[Nonlinear Dyn.(2011), doi: 10.1007/s11071-011-9944-2]. *Nonlinear Dynamics*, 66, pages 839-842, 2011.
[9] S. Dadras and H. R. Momeni. LMI-Based Fractional-Order Surface Design for Integral Sliding-Mode Control of Uncertain Fractional-Order Nonlinear systems. in *the 4th IFAC Workshop on Fractional Differentiation and Its Applications*, pages 18-20, 2010.
[10] S. Dadras and H. R. Momeni. Fractional sliding mode observer design for a class of uncertain fractional order nonlinear systems. in *Proc. 50th IEEE Conf. Decision Control*, pages 6925-6930, 2011.
[11] D. M. Senejohnny and H. Delavari. Active sliding observer scheme based fractional chaos synchronization *Commun Nonlinear Sci. Numer. Simulat.*, 17, pages 4373–4383, 2012.
[12] C. Yin, Y. Q. Chen, and S. M. Zhong. LMI based design of a sliding mode controller for a class of under uncertain fractional-order nonlinear systems. in *Proc. 2013 Amer. Control Conf.*, pages 6526–6531, 2013.
[13] L. Dai. *Lecture notes in control and information sciences: vol. 118. Singular control systems.* New York: Springer-Verlag, 1989.
[14] Q. L. Han. A descriptor system approach to robust stability of uncertain neutral systems with discrete and distributed delays. *Automatica*, 40, pages 1791–1796, 2004.
[15] K. Tanaka, H. Ohtake, and H. O. Wang. Hamilton-Jacobi equation for descriptor systems. *IEEE Trans. Fuzzy Syst.*, 15, pages 333–341, 2007.
[16] C. Bai and J. Fang. The existence of a positive solution for a singular coupled system of nonlinear fractional differential equations. *Applied Math. Comp.*, 150, pages 611–621, 2004.
[17] M. S. Tavazoei and M. Haeri. Rational approximations in the simulation and implementation of fractional-order dynamics: A descriptor system approach. *Automatica*, 46, pages 94–100, 2010.
[18] I. N. Doyea, M. Darouachb, M. Zasadzinski, and N. Radhy. Robust stabilization of uncertain descriptor fractional-order systems. *Automatica*, 49, pages 1907–1913, 2013.
[19] E. Artin. *The Gamma Function.* Hot, Rinehart and Winston, NewYork, 1964.
[20] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM, Philadelphia, PA, 1994.
[21] C. Yin, S. M. Zhong, and W. Chen. Robust H_∞ control for uncertain Lur'e systems with sector and slope restricted nonlinearities by PD state feedback. *Nonlinear Analysis Real World Applications*, 12, pages 501-512, 2011.