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# Lyapunov-based Boundary Control for a MIMO Counter-Propagating Raman Amplifier

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**Abstract:**
This paper considers a boundary feedback control problem for a MIMO counter-propagating Raman amplifier. The system is modeled as a set of coupled semilinear hyperbolic partial differential equations with Lotka-Volterra type nonlinearity. The system is linearized about the steady-state solution, and a boundary controller is designed based on a Lyapunov functional whose time derivative is made strictly negative by an appropriate choice of boundary conditions. As a result, exponential convergence to the steady-state solution is shown in the $L^2$-norm. The results are extended to the nonlinear system under a key assumption.

*Keywords:* Partial differential equations, optical amplifiers, boundary control, lyapunov function, nonlinear control

## 1. INTRODUCTION

It is not always ideal to apply finite dimensional approximation techniques to an underlying infinite dimensional dynamical system (Lasiecka (1987)). Hyperbolic systems, for example, have properties such as finite speed of propagation that require higher order approximations if one decides to take a finite dimensional approach as in Dower and Farrell (2006). Boundary control operating directly on infinite dimensional models has successfully been done using Lyapunov-based techniques (see Pavel and Chang (2012), Coron et al. (2004), Vazquez et al. (2011)).

In this paper, we consider a boundary control problem for $N$ coupled, first-order, semilinear hyperbolic partial differential equations of space and time. The nonlinear reaction terms are of the Lotka-Volterra type which can be used to model a number of different phenomena, but our particular interest is that of stimulated Raman scattering used for optical amplification (Dower and Farrell (2006)). In Pavel and Chang (2012), static and dynamic boundary controllers were designed using a Lyapunov functional whose time derivative is made strictly negative along solutions of the closed loop system. The paper showed that if the $H^1$-norm of initial conditions is chosen sufficiently small, then exponential (asymptotic) convergence to the desired steady state solution is guaranteed in the $L^2$ ($C^0$) norm. However, only the simplest 2x2 co-propagating setup was considered. It is well known that the counter-propagating configuration is preferred because amplitude fluctuations in the pump power are averaged out in the Raman process (Headley and Agrawal (2005)). Also, a typical setup has several pumps and signals. Therefore we are primarily concerned with the $N$x$N$ counter-propagating setup in which the pump(s) and signal(s) propagate in opposite directions. Our contribution is *developing a Lyapunov-based boundary control approach to treat the $N$x$N$ fully actuated counter-propagating case*. Boundary control using this method is currently not known to exist for the *counter-propagating* case. In this paper, we see challenges arise when searching for subsets invariant in the $C^0$ norm. For the co-propagating system, sublevel sets of a Lyapunov function solved this problem. However, this Lyapunov function fails for the counter-propagating system.

The paper carries out analysis in a Sobolev space framework as in Vazquez et al. (2011). The paper is organized as follows. Section 2 presents the model and boundary control problem along with existence and uniqueness of solutions. Section 3 presents $L^2$ convergence results based on the system linearized about the steady-state solution. Section 4 presents $L^2$ convergence results for the nonlinear system under a key assumption, and Section 5 presents simulation results.

The following notation is used. Let $\Omega := (0,1), \overline{\Omega} = [0,1], \partial\Omega = \{0,1\}$. Let $L^2(\Omega;\mathbb{R}^n)$ denote the Lebesgue space of $\mathbb{R}^n$-valued square-integrable functions on $\Omega$, and use $\mathcal{L}^2(\Omega) := L^2(\Omega;\mathbb{R}^n)$ as a compact notation for this space of vector-valued functions. $\mathcal{L}^2(\Omega)$ is a Hilbert space with the inner product denoted $< \cdot, \cdot >$ and the $L^2$-norm denoted by $\|\cdot\|_{\mathcal{L}^2}$

$$\|u\|_{\mathcal{L}^2} := \int_0^1 \|u(z)\|^2 \, dz$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. Let $\mathcal{L}^\infty(\Omega)$ denote the Lebesgue space of $\mathbb{R}^n$-valued measurable functions on $\Omega$ that are essentially bounded with the norm

$$\|u\|_{\mathcal{L}^\infty} := ess \sup_{z\in\Omega} |u(z)|_\infty < \infty$$

and $C^0(\overline{\Omega})$ the space of $\mathbb{R}^n$-valued continuous functions on $\overline{\Omega}$ with the norm

$$\|u\|_\infty := \max_{z\in\overline{\Omega}} |u(z)|_\infty$$

where $|u|_\infty = \max_i |u_i|$ in $\mathbb{R}^n$. For continuous functions the $\mathcal{L}^\infty(\Omega)$-norm is equal to the $C^0(\overline{\Omega})$-norm. For $k \in \mathbb{Z}_+$ we denote the usual Sobolev space by

$$\mathcal{H}^k(\Omega) = \left\{ u \in \mathcal{L}^2(\Omega) | \frac{d^n u}{dz^n} \in \mathcal{L}^2(\Omega), \forall n \leq k \right\}$$

equipped with the usual $\mathcal{H}^k$-norm

$$\|u\|_{\mathcal{H}^k} := \sum_{n=0}^{k} \|\frac{d^n u}{dz^n}\|_{\mathcal{L}^2}$$

## 2. MODEL AND PROBLEM FORMULATION

We consider a counter-propagating Raman amplifier model with $n$ signals and $n$ pumps. Let $N := 2n$. Define the vector of pumps and signals as

$$\mathbf{p}(t,z) := [\, p_1(t,z) \;\ldots\; p_n(t,z) \; p_{n+1}(t,z) \;\ldots\; p_N(t,z) \,]^T$$

where $p_i(t,z) \geq 0$ is the $i^{th}$ signal power if $i \in \{1,\ldots,n\}$ or the $(i-n)^{th}$ pump power if $i \in \{n+1,\ldots,N\}$. These powers propagate with characteristic speeds $\lambda_i$ along the length of the amplifier $\Omega = (0,1)$. Without loss of generality we may assume that $0 < \lambda_1 < \cdots < \lambda_N$. Consider the following $N$x$N$ first-order semilinear hyperbolic system which models this *counter-propagating* configuration

$$\begin{aligned} \frac{\partial p_i(t,z)}{\partial t} &= -\lambda_i \frac{\partial p_i(t,z)}{\partial z} - p_i + \sum_{j=n+1}^{N} p_i p_j, \\ i &= 1,\ldots,n \\ \frac{\partial p_i(t,z)}{\partial t} &= \;\;\lambda_i \frac{\partial p_i(t,z)}{\partial z} - p_i - \sum_{j=1}^{n} p_i p_j, \\ i &= n+1,\ldots,N \end{aligned} \tag{1}$$

$t \geq 0, z \in \Omega$ with initial conditions

$$p_i(0,z) = p_{i0}(z), \quad i = 1,\ldots,N, \quad z \in \overline{\Omega} \tag{2}$$

Note in (1) the opposite signs for pumps and signals. Also note that we use the non-depleted pump approximation as in Dower and Farrell (2006) by neglecting coupling terms between the pumps and signals themselves. Typically we maintain a constant input signal $p_i(t,0) = \bar{u}_{is}$ $\forall i \in \{1,\ldots,n\}$ and a desired output signal (set point) $p_i(t,1) = y_{id}$ $\forall i \in \{1,\ldots,n\}$ by manipulating only the pumps at their input $(z = 1)$. This gives rise to the following boundary conditions that hold $\forall t \geq 0$ for system (1)

$$\begin{aligned} p_i(t,0) &= \bar{u}_{is}, & i &\in \{1,\ldots,n\} \\ p_i(t,1) &= u_{ic}(t), & i &\in \{n+1,\ldots,N\} \end{aligned} \tag{3}$$

where $u_{ic}(t)$ are the control actions to be designed based on the signal power at the output $(z = 1)$. For constant control actions $u_{ic}(t) = \bar{u}_{ic}$, let

$$\bar{\mathbf{p}}(z) := [\, \bar{p}_1(z) \;\ldots\; \bar{p}_n(z) \; \bar{p}_{n+1}(z) \;\ldots\; \bar{p}_N(z) \,]^T$$

denote the steady-state solution that satisfies the set point $(\bar{p}_1(1),\ldots,\bar{p}_n(1)) = (y_{1d},\ldots,y_{nd})$, along with boundary conditions

$$\begin{aligned} \bar{p}_i(0) &= \bar{u}_{is}, & i &\in \{1,\ldots,n\} \\ \bar{p}_i(1) &= \bar{u}_{ic}, & i &\in \{n+1,\ldots,N\} \end{aligned} \tag{4}$$

Let $\mathbf{y}_d \in \mathbb{R}^n$ denote the vector of desired output signals, $\mathbf{p}_+(t,z) \in \mathbb{R}^n$ as the vector of signals and $\mathbf{p}_-(t,z) \in \mathbb{R}^n$ as the vector of pumps. We then have the following block diagram (Fig. 1) for the closed loop system (1)-(3)

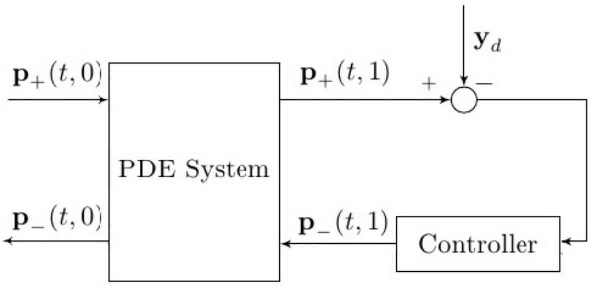


Fig. 1. Boundary controlled system

The following assumption is used

*Assumption 1.* There exists a unique steady-state solution $\bar{\mathbf{p}}(z) \in \mathcal{H}^2(\Omega), \bar{p}_i(z) > 0$ $\forall i \in \{1,\ldots,N\}, z \in \overline{\Omega}$ that satisfies the set point $\bar{p}_i(1) = y_{id}$ $\forall i \in \{1,\ldots,n\}$ and satisfying boundary conditions (4).

The steady-state solution gives rise to the following the set of coupled ordinary differential equations

$$\begin{aligned} \lambda_i \frac{d\bar{p}_i(z)}{dz} &= -\bar{p}_i + \sum_{j=n+1}^{N} \bar{p}_i \bar{p}_j, \quad i = 1,\ldots,n \\ -\lambda_i \frac{d\bar{p}_i(z)}{dz} &= \;-\bar{p}_i - \sum_{j=1}^{n} \bar{p}_i \bar{p}_j, \quad i = n+1,\ldots,N \end{aligned} \tag{5}$$

$z \in \Omega$ with boundary conditions (4). The following change of variables is made

$$w(t,z) := \mathbf{p}(t,z) - \bar{\mathbf{p}}(z) \tag{6}$$

We represent the signal and pump deviations propagating in the positive and negative directions by $w_+(t,z) \in \mathbb{R}^n$ and $w_-(t,z) \in \mathbb{R}^n$ respectively . Using (1), (5), (6) the closed-loop system is written in terms of $w(t,z)$ as

$$\frac{\partial}{\partial t}w(t,z) + a\frac{\partial}{\partial z}w(t,z) + b(z)w(t,z) = F(w(t,z)), \quad (7)$$

$t \geq 0, z \in \Omega$ where

$$a := \operatorname{diag}\{\lambda_1 ... \lambda_n, -\lambda_{n+1}, ..., -\lambda_N\}$$

$$b(z) := \begin{bmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{bmatrix}$$

$$b_{11}(z) := \begin{bmatrix} 1 - \sum_{j=n+1}^{N} \bar{w}_j(z) & 0_{1\text{x}(n-1)} \\ 0_{(n-1)\text{x}1} & \ddots \end{bmatrix}$$

$$b_{12}(z) := \begin{bmatrix} -\bar{w}_1(z) & \dots & -\bar{w}_1(z) \\ \vdots & & \vdots \\ -\bar{w}_n(z) & \dots & -\bar{w}_n(z) \end{bmatrix}$$

$$b_{21}(z) := \begin{bmatrix} \bar{w}_{n+1}(z) & \dots & \bar{w}_{n+1}(z) \\ \vdots & & \vdots \\ \bar{w}_N(z) & \dots & \bar{w}_N(z) \end{bmatrix}$$

$$b_{22}(z) := \begin{bmatrix} 1 + \sum_{j=1}^{n} \bar{w}_j(z) & 0_{1\text{x}(n-1)} \\ 0_{(n-1)\text{x}1} & \ddots \end{bmatrix}$$

and $\bar{w}_i(z) := \bar{p}_i(z)$, $a, b(z) \in \mathbb{R}^{N\text{x}N}$. $F(w(t,z))$ is specifically a column vector in $\mathbb{R}^N$ with entries $F_i(w(t,z))$ defined as $F_i(w(t,z)) = \sum_{j=n+1}^{N} w_i w_j$ for $i \in \{1, \dots, n\}$ and $F_i(w(t,z)) = -\sum_{j=1}^{n} w_i w_j$ for $i \in \{n+1, \dots, N\}$. With boundary conditions from (3) and (4) we have

$$\begin{bmatrix} w_+(t,0) \\ w_-(t,1) \end{bmatrix} = G \begin{bmatrix} w_+(t,1) \\ w_-(t,0) \end{bmatrix}, \quad t \geq 0 \quad (8)$$

where the map to be designed $G : \mathbb{R}^N \to \mathbb{R}^N$ is of class $C^2$ and vanishes at 0. These boundary conditions are of the same form as in Coron et al. (2008). We see through Lyapunov analysis in Section III that a natural choice for boundary conditions is of the form

$$\begin{bmatrix} w_+(t,0) \\ w_-(t,1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix} \begin{bmatrix} w_+(t,1) \\ w_-(t,0) \end{bmatrix}, \quad t \geq 0 \quad (9)$$

where $G := \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix}$, $K := \operatorname{diag}\{k_1, \dots, k_n\}$. We have the following initial conditions from (2)

$$w(0,z) = \begin{bmatrix} p_{10}(z) - \bar{p}_1(z) \\ \vdots \\ p_{N0}(z) - \bar{p}_N(z) \end{bmatrix} := w_0(z) \quad (10)$$

### *2.1 Linearized Model*

In this section we present the linearized model for the counter-propagating system. In Section 3 we apply Lyapunov analysis to this model. It is easy to verify that the right hand side of (7) vanishes after linearization and thus gives rise to the following system

$$\frac{\partial}{\partial t}w(t,z) + a\frac{\partial}{\partial z}w(t,z) + b(z)w(t,z) = 0, \ t \geq 0, z \in \Omega \quad (11)$$

with boundary conditions (9) and initial conditions (10). The following are initial-boundary compatibility conditions for (9)-(11)

$$\begin{bmatrix} w_+^0(0) \\ w_-^0(1) \end{bmatrix} = \begin{bmatrix} 0 \\ Kw_+^0(1) \end{bmatrix} \quad (12)$$

$$\frac{\partial}{\partial_t}(Kw_+^0(1)) + a\frac{\partial}{\partial_z}(Kw_+^0(1)) + b(1)Kw_+^0(1) = 0 \quad (13)$$

In the next subsection we discuss existence and uniqueness of classical solutions for the mixed initial-boundary value problem (9)-(11).

### *2.2 Existence and Uniqueness of Solutions*

For the *co-propagating* system considered in Pavel and Chang (2012), existence and uniqueness of classical solutions of the closed loop system follows from the results of Pavel (2009), Pavel (2013). It is shown that if a uniform apriori bound on the $C^0(\overline{\Omega})$-norm of the local in time classical solution exists, then the $\mathcal{H}^1$-norm cannot blow up in finite time and global existence is guaranteed. The authors solved this problem by finding subsets that are invariant in the $C^0(\overline{\Omega})$-norm using sublevel sets of a Lyapunov function. As a result, asymptotic properties of the solution could be studied and an entropy-like Lyapunov function was constructed to guarantee stability in both the $\mathcal{L}^2(\Omega)$ and $C^0(\overline{\Omega})$ norms.

Therefore, the next natural step is to derive a uniform apriori bound on the $C^0(\overline{\Omega})$-norm of the local in time classical solution for the *counter-propagating* system. Unfortunately, the Lyapunov function used for the co-propagating system fails for the counter-propagating system. The opposite signs in front of the spatial differential operators do not allow a differential inequality comparison as was elegantly done for the co-propagating case in Pavel and Chang (2012).

However, some work has been done for hyperbolic systems of conservation laws in $\mathcal{H}^2$ spaces (see Coron et al. (2004), Vazquez et al. (2011), Coron et al. (2008)). These systems are similar to (7) in the sense that there is a counter-propagating configuration. In Coron et al. (2008), it is shown that a unique maximal classical solution exists provided that the $\mathcal{H}^2$-norm of the initial condition is sufficiently small. The solution mapping is contained in a subset that is uniformly bounded in the $L^\infty$-norm. Many of the existence and uniqueness proofs for such systems deal with $\mathbb{R}$ (see Lax et al. (1973), Majda (1984)) but Coron et al. (2008) gives a detailed adaption of these proofs to boundary conditions of the same form as in (9). Although only the *homogenous* case for systems of conservation laws is considered, the results can be extended to systems with nonzero right hand side $h(u)$ provided $h$ is of class $C^2$ and vanishes at zero. We therefore claim

If initial-boundary compatibility conditions (12), (13) are satisfied then there exists a sufficiently small $\delta > 0$ such that if $\|w_0\|_{\mathcal{H}^2} < \delta$ then the Cauchy problem (9)-(11) has a unique maximal classical solution

$$w(t) \in C^0([0,T), \mathcal{H}^2(\Omega))$$

with $T \in [0, +\infty]$.

We directly apply the existence and uniqueness results from Coron et al. (2008) to the linearized closed loop system. In Coron et al. (2008), the stability analysis is done for homogeneous systems of conservation laws. Our linearized system is the inhomogeneous case. Therefore, it is worth it to explicitly carry out the stability analysis, which is done in the next section. From this, clear $\mathcal{L}^2$ stability results are derived in terms of the system parameters $(b(z), \lambda_1, K)$.

*Remark 2.* Note that this existence and uniqueness result could also be applied to the *nonlinear* counter-propagating system (7), (9), (10). However, as we see in Section 4, this case is treated using $\mathcal{H}^1$ existence and uniqueness results from Pavel (2009), Pavel (2013).

## 3. LYAPUNOV ANALYSIS

In this section we construct a Lyapunov functional for the linearized counter-propagating closed loop system. The Lyapunov functional is instrumental in showing exponential stability in the $\mathcal{L}^2$-norm. Consider the following quadratic-like Lyapunov candidate function for the linearized counter-propagating system (9)-(11)

$$V(w(t)) := \frac{1}{2}\int_0^1 \sum_{i=1}^{n} w_i^2 e^{-\mu z}\,dz + \frac{1}{2}\int_0^1 \sum_{i=n+1}^{N} w_i^2 e^{+\mu z}\,dz \tag{14}$$

where we define $w_i := w_i(t,z)$ for simplicity of notation. This type of function has been used in various other works involving the stability of hyperbolic systems (see Coron et al. (2004), Vazquez et al. (2011), Diagne et al. (2012)). This function is instrumental in the proof of the theorem stated below:

*Theorem 3.* For system (9)-(11), there exists constants $c, \sigma, \delta > 0$ and matrix $K \in \mathbb{R}^{nxn}$ such that if $\|w_0\|_{\mathcal{H}^2} < \delta$ then

$$\|w(t)\|_{\mathcal{L}^2} \le c e^{-\sigma t}\|w_0\|_{\mathcal{L}^2},\ \forall t \in [0, T]$$

where $T > 0$ is the duration of the unique maximal classical solution of the closed loop system.

We begin by introducing the following two lemmas that assist in proving Theorem 3.

*Lemma 4.* The following holds $\forall \mu > 0$

$$e^{-\mu}\|w(t)\|^2_{\mathcal{L}^2} \le V(w(t)) \le e^{\mu}\|w(t)\|^2_{\mathcal{L}^2} \tag{15}$$

**Proof.** The proof is simple, and follows from the definitions of the $\mathcal{L}^2$-norm and $V(w(t))$

$$e^{-\mu}\|w(t)\|^2_{\mathcal{L}^2} = \int_0^1 e^{-\mu}\sum_{i=1}^{N} w_i^2\,dz \le V(w(t)) \tag{16}$$

and it is also clear that

$$V(w(t)) \le \int_0^1 e^{\mu}\sum_{i=1}^{N} w_i^2\,dz = e^{\mu}\|w(t)\|^2_{\mathcal{L}^2} \tag{17}$$

□

*Lemma 5.* For sufficiently small $\|w_0\|_{\mathcal{H}^2}$ and $\forall \mu > 0$, $\forall \theta \in (0,1)$ such that if $\lambda_1 \ge \frac{nK_b}{(1-\theta)}\left(\frac{1}{\mu} + \frac{e^{2\mu}}{\mu}\right)$ and $\|\sum_{j=n+1}^{N} \bar{w}_j\|_\infty < 1$ then the following holds for the closed loop system (9)-(11) for $\forall t \in [0, T]$

$$\begin{aligned}\dot{V}(w(t)) \le{}& -\mu\lambda_1\theta V(w(t)) \\ &+ \sum_{i=1}^{n}\left(\frac{\lambda_{n+i}}{2}k_i^2 w_i^2(t,1)e^{\mu} - \frac{\lambda_i}{2}w_i^2(t,1)e^{-\mu}\right)\end{aligned} \tag{18}$$

where $K_b := \|b_{ij}(z)\|_\infty < \infty$

**Proof.** Evaluating the time derivative of (14) along solutions of the closed loop system yields

$$\begin{aligned}\dot{V}(w(t)) ={}& \int_0^1 \sum_{i=1}^{n} w_i\left(-\lambda_i\frac{\partial w_i}{\partial z} - b_{ii}(z)w_i\right)e^{-\mu z}\,dz \\ &+ \int_0^1\left(\sum_{i=1}^{n}\sum_{j=n+1}^{N}\bar{w}_i w_i w_j\right)e^{-\mu z}\,dz \\ &+ \int_0^1 \sum_{i=n+1}^{N} w_i\left(\lambda_i\frac{\partial w_i}{\partial z} - b_{ii}(z)w_i\right)e^{\mu z}\,dz \\ &- \int_0^1\left(\sum_{i=n+1}^{N}\sum_{j=1}^{n}\bar{w}_i w_i w_j\right)e^{\mu z}\,dz\end{aligned} \tag{19}$$

Define the following:

$$\mathbf{v}_{\mathrm{s}}(w_+(t,z)) := \sum_{i=1}^{n}\frac{\lambda_i}{2}w_i^2,\quad \mathbf{v}_{\mathrm{p}}(w_-(t,z)) := \sum_{i=n+1}^{N}\frac{\lambda_i}{2}w_i^2 \tag{20}$$

Using (19), (20) and the fact that $b_{ii}(z) > 0\ \forall i, \forall z \in \overline{\Omega}$ (since by assumption we have $\| \sum_{j=n+1}^{N} \bar{w}_j \|_\infty < 1$) we can write the following estimate for $\dot{V}(w(t))$

$$\begin{aligned} \dot{V}(w(t)) \leq & \int_0^1 -\frac{d\mathbf{v}_s(w_+(t,z))}{dz} e^{-\mu z}\, dz \\ & + \int_0^1 \left( \sum_{i=1}^{n} \sum_{j=n+1}^{N} \bar{w}_i w_i w_j \right) e^{-\mu z}\, dz \\ & + \int_0^1 \frac{d\mathbf{v}_p(w_-(t,z))}{dz} e^{\mu z}\, dz \\ & - \int_0^1 \left( \sum_{i=n+1}^{N} \sum_{j=1}^{n} \bar{w}_i w_i w_j \right) e^{\mu z}\, dz \end{aligned} \tag{21}$$

We now derive estimates for the fourth term in (21)

$$\left| \sum_{i=n+1}^{N} \sum_{j=1}^{n} \bar{w}_i w_i w_j \right| \leq K_b \sum_{i=n+1}^{N} \sum_{j=1}^{n} |w_i w_j| \tag{22}$$

$$\leq \frac{K_b}{2} \sum_{i=n+1}^{N} \sum_{j=1}^{n} \left( w_i^2 + w_j^2 \right) \tag{23}$$

$$\leq \frac{nK_b}{2} \sum_{i=1}^{N} w_i^2 \tag{24}$$

where in (22) we used the fact that $\bar{w}_i \leq K_b\ \forall i, \forall z \in \overline{\Omega}$ from (7) and the definition of $K_b$. We used Young's inequality in (23). An equivalent estimate holds for the second term in (21). After integrating by parts the terms involving $\mathbf{v}_s$ and $\mathbf{v}_p$ and using boundary conditions from (9) together with estimate (24) we write the following estimate for $\dot{V}(w(t))$

$$\begin{aligned} \dot{V}(w(t)) \leq & -\mu\lambda_1 V(w(t)) \\ & + \sum_{i=1}^{n} \left( \frac{\lambda_{n+i}}{2} k_i^2 w_i^2(t,1) e^{\mu} - \frac{\lambda_i}{2} w_i^2(t,1) e^{-\mu} \right) \\ & + \int_0^1 \frac{nK_b}{2} \sum_{i=1}^{N} w_i^2 \left( e^{-\mu z} + e^{\mu z} \right) dz \end{aligned} \tag{25}$$

It is easy to verify that $\max_{z \in \overline{\Omega}} \left( e^{-\mu z} + e^{\mu z} \right) = e^{-\mu} + e^{\mu}$ so that (25) becomes

$$\begin{aligned} \dot{V}(w(t)) \leq & -\mu\lambda_1 V(w(t)) \\ & + \sum_{i=1}^{n} \left( \frac{\lambda_{n+i}}{2} k_i^2 w_i^2(t,1) e^{\mu} - \frac{\lambda_i}{2} w_i^2(t,1) e^{-\mu} \right) \\ & + \frac{nK_b}{2} \left( e^{-\mu} + e^{\mu} \right) \|w(t)\|_{\mathcal{L}^2}^2 \end{aligned} \tag{26}$$

We equivalently rewrite (26) as follows for $\forall \theta \in (0,1)$

$$\begin{aligned} \dot{V}(w(t)) \leq & -\mu(1-\theta)\lambda_1 V(w(t)) - \mu\theta\lambda_1 V(w(t)) \\ & + \sum_{i=1}^{n} \left( \frac{\lambda_{n+i}}{2} k_i^2 w_i^2(t,1) e^{\mu} - \frac{\lambda_i}{2} w_i^2(t,1) e^{-\mu} \right) \\ & + \frac{nK_b}{2} \left( e^{-\mu} + e^{\mu} \right) \|w(t)\|_{\mathcal{L}^2}^2 \end{aligned} \tag{27}$$

Note that by assumption in the Lemma 5 statement

$$\lambda_1 \geq \frac{nK_b}{(1-\theta)} \left( \frac{1}{\mu} + \frac{e^{2\mu}}{\mu} \right) := \frac{nK_b}{(1-\theta)} g(\mu) \tag{28}$$

Multiplying both sides of (28) by $\frac{1}{2}\mu(1-\theta)e^{-\mu}\|w(t)\|_{\mathcal{L}^2}$ we have

$$\frac{e^{-\mu}}{2} \|w(t)\|_{\mathcal{L}^2} \mu\lambda_1(1-\theta) \geq \frac{nK_b}{2} (e^{-\mu} + e^{\mu}) \|w(t)\|_{\mathcal{L}^2} \tag{29}$$

and using the left hand side of Lemma 4 we have

$$\mu\lambda_1(1-\theta)V(w(t)) \geq \frac{nK_b}{2} \left( e^{-\mu} + e^{\mu} \right) \|w(t)\|_{\mathcal{L}^2} \tag{30}$$

hence the last term in (27) is dominated and we have

$$\begin{aligned} \dot{V}(w(t)) \leq & -\mu\theta\lambda_1 V(w(t)) \\ & + \sum_{i=1}^{n} \left( \frac{\lambda_{n+i}}{2} k_i^2 w_i^2(t,1) e^{\mu} - \frac{\lambda_i}{2} w_i^2(t,1) e^{-\mu} \right) \end{aligned} \tag{31}$$

which completes the proof of Lemma 5. □

*Remark 6.* It may first appear that (28) imposes a minimum speed requirement on the slowest signal. However, from a practical perspective this condition should always be satisfied for an appropriate range of $\mu$. One can verify that $\min_{\mu \in (0,\infty)} g(\mu) \approx 7.18$ occurs at $\mu_{\min} \approx 0.639$. Since the $\lambda_i$ are generally of much higher order, and $K_b$ on much lower order, we see that (28) does not impose a strict practical constraint for an appropriate value of $\theta$.

### *3.1 Proof of Theorem 3*

We now prove the main result for the linearized Cauchy problem (9)-(11). We begin by showing that the following holds for an appropriate choice of $K$ and $\mu$.

$$\dot{V}(w(t)) \leq -\mu\lambda_1\theta V(w(t)),\ \forall t \in [0,T] \tag{32}$$

It suffices to use Lemma 5 and choose constants $k_i$ so that

$$\frac{\lambda_{n+i}}{2} k_i^2 w_i^2(t,1) e^{\mu} - \frac{\lambda_i}{2} w_i^2(t,1) e^{-\mu} < 0 \tag{33}$$

and we see (32) holds provided that

$$k_i^2 < \frac{\lambda_i}{\lambda_{n+i} e^{2\mu}} \tag{34}$$

After applying Gronwall's inequality (Robinson (2001)) to (32) along with Lemma 4 we have

$$\frac{e^{-\mu}}{2}\|w(t)\|_{\mathcal{L}^2} \leq V(w(t)) \leq V(w_0)e^{-\mu\theta\lambda_1 t} \leq \frac{e^{\mu}e^{-\mu\theta\lambda_1 t}}{2}\|w_0\|_{\mathcal{L}^2} \tag{35}$$

$\forall t \in [0,T]$. Thus, by (35) we see that $\|w(t)\|_{\mathcal{L}^2} \leq ce^{-\sigma t}\|w_0\|_{\mathcal{L}^2}$ $\forall t \in [0,T]$ with $c = e^{2\mu}$ and $\sigma = \theta\mu\lambda_1$. This completes the proof of Theorem 3. □

*Remark 7.* By Theorem 3 we conclude that the deviated closed loop system (9)-(11) is exponentially stable $\forall t \in [0,T]$ in the $\mathcal{L}^2$-norm for any given $T \in (0,\infty)$ provided $\|w_0\|_{\mathcal{H}^2}$ is sufficiently small. Hence, the pumps and signals approach the desired steady-state solution $\bar{\mathbf{p}}(z)$ in the $\mathcal{L}^2$ sense.

*Remark 8.* Note the trade off between $\theta$ and the minimum speed requirement for the slowest signal $\lambda_1$. A larger value of $\theta$ will result in faster decay to the steady-state solution (in the $\mathcal{L}^2$ sense), but it also strengthens the constraint on $\lambda_1$ in (28).

## 4. EXTENSION OF RESULTS TO THE NONLINEAR SYSTEM

We *briefly* explain how the results for the linearized system (9)-(11) can be extended to the nonlinear system (7), (9), (10) under a key assumption. This assumption is needed to derive estimates on a Lyapunov functional and apply existence and uniqueness results in Pavel (2013). First we recall an entropy-like function used in Pavel and Chang (2012) for the co-propagating system. For any given real number $\bar{u} > 0$ let $v : I = (-\bar{u}, \infty) \to \mathbb{R}_+$

$$v(u) := u - \bar{u}_i \ln\left(1 + \frac{u}{\bar{u}}\right) \tag{36}$$

Then (36) has the following properties: $v(0) = 0$, positive definite on $I\setminus\{0\}$, and radially unbounded. Also, $\forall \bar{a}, \bar{b} > 0$ such that $1/\bar{b} < \bar{u} < 1/\bar{a}$ , the following holds on the domain $\mathcal{R} := \{1/\bar{b} - \bar{u} \leq u \leq 1/\bar{a} - \bar{u}\}$

$$\frac{1}{2}\bar{a}|u|^2 \leq v(u) \leq \frac{1}{2}\bar{b}|u|^2 \tag{37}$$

Using this entropy function we introduce a new Lyapunov functional for the nonlinear *counter-propagating* system

$$V(w(t)) := \int_0^1 \sum_{i=1}^{n} v_i(w_i(t,z))e^{-\mu z}\, dz + \int_0^1 \sum_{i=n+1}^{N} v_i(w_i(t,z))e^{+\mu z}\, dz \tag{38}$$

where

$$v_i(w_i(t,z)) := w_i(t,z) - \bar{w}_i(z) \ln\left(1 + \frac{w_i(t,z)}{\bar{w}_i(z)}\right)$$

Unlike the quadratic-like Lyapunov functional considered in (14), the entropy-like Lyapunov functional (38) matches the structure of the Lotka-Volterra nonlinearity in (7).

Global $\mathcal{L}^2$ exponential stability can be obtained by following the same methodology in Pavel and Chang (2012) for the counter-propagating system under the following assumption

*Assumption 9.* Let $\mathcal{H}^1_{bc}(\Omega)$ denote the usual Sobolev space with embedded initial-boundary compatibility conidition. Then there exists constants $a_1, b_0, \bar{a}, \bar{b}, \delta > 0$ such that if $\|w_0\|_{\mathcal{H}^1} < \delta$ then $w(t) \in \underline{\mathcal{D}}_1$ $\forall t \geq 0$ where

$$\underline{\mathcal{D}}_1 := \left\{u \in \mathcal{H}^1_{bc}(\Omega) \mid \|u\|_\infty \leq \min\{d, d_1\}\right\} \tag{39}$$

$$d = \min\left\{(1/\bar{a}) - \|u\|_\infty, \bar{u}_m - (1/\bar{b})\right\} \tag{40}$$

$$d_1 < \min\left\{\bar{u}_m, (1/a_1) - \max_{i\in\{1,\ldots,n\}} \bar{u}_i(1)\right\} \tag{41}$$

and $\forall i \in \{1,\ldots,n\}$

$$(1/a_1) > \bar{w}_i(1),\ (1/b_0) < \bar{w}_{n+i}(1) \tag{42}$$

where $\bar{u}_m := \min_i \min_{z\in\overline{\Omega}} \bar{u}_i(z)$, $i \in \{1,\ldots,N\}$. Note that under this assumption we inherently assume a uniform bound on the $C^0(\overline{\Omega})$-norm of the solution. This implies global in time existence and uniqueness by Pavel (2013). Also note that $\underline{\mathcal{D}}_1 \subset \mathcal{D}_1 \cap \mathcal{D}$ where $\mathcal{D}_1, \mathcal{D}$ are defined analogously to (39). $w(t) \in \mathcal{D}_1$ $\forall t \geq 0$ allows us to use (37) on the terms that appear after an integration by parts in the time derivative estimate of (38). $w(t) \in \mathcal{D}$ $\forall t \geq 0$ allows us to use 2), Lemma 2 in Pavel and Chang (2012) which is analogous to (15). The main difference now is that the Lotka-Volterra terms do not cancel in the time derivative estimate for (38) as they did in Pavel and Chang (2012). This problem is solved by an application of Young's inequality as in (23) and we arrive at a similar (but practical) constraint like (28) on $\lambda_1$. The following constraints on the $k_i$ ($i \in \{1,\ldots,n\}$) appear in the Lyapunov analysis

$$-\frac{\bar{w}_{n+i}(1) - 1/b_0}{1/a_1 - \bar{w}_i(1)} < k_i < \frac{\bar{w}_{n+i}(1) - 1/b_0}{\bar{w}_i(1)} \tag{43}$$

$$k_i^2 < \frac{a_1 e^{-\mu}\lambda_i}{b_0\lambda_{n+i}} \tag{44}$$

Note the similarity between (44) and (34). The extra condition (43) comes from using (37) after an integration by parts which was not required in the linearized case.

*Remark 10.* Note that Assumption 9 could be eliminated if one can show how to choose initial conditions $w_0$ so that the solution $w(t)$ remains in subsets invariant in the $C^0(\overline{\Omega})$-norm. This has two important consequences. First, global in time existence and uniqueness is guaranteed by Pavel (2013). Second, we may choose $w_0$ so that $w(t)$ remains in $\underline{\mathcal{D}}_1$. For the co-propagating system in Pavel and Chang (2012), Assumption 9 is not required. Instead, subsets invariant in the $C^0(\overline{\Omega})$-norm are found using a scalar comparison lemma for differential inequalities with an entropy-like Lyapunov function. However, this method fails for the counter-propagating configuration, hence the requirement for Assumption 9.

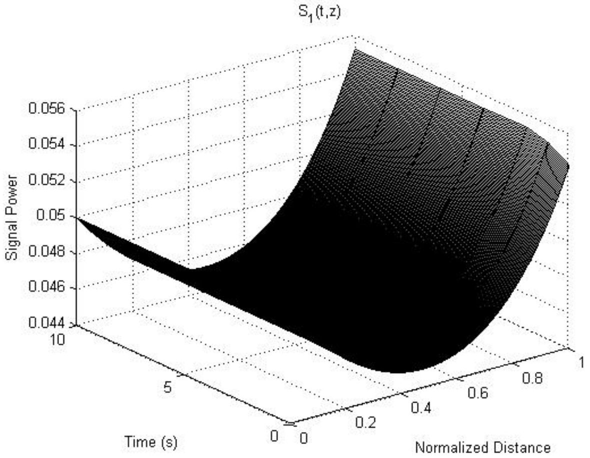


Fig. 2. Signal 1 evolution

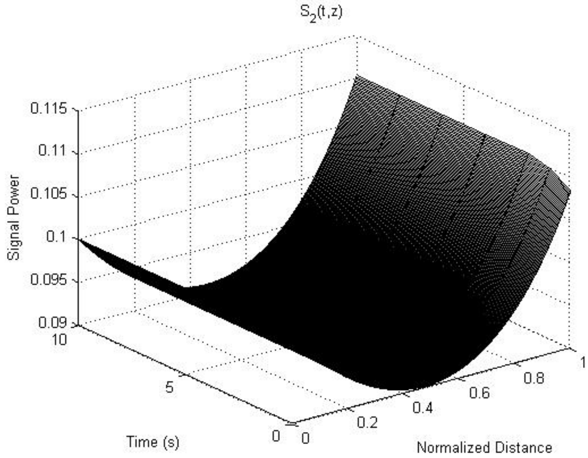


Fig. 3. Signal 2 evolution

## 5. SIMULATION RESULTS

We simulate the 4x4 counter-propagating case using the *nonlinear* model. The simulation was done in MATLAB using a hyperbolic partial differential equation solver (see Shampine (2005)). The closed loop Cauchy problem (1)-(3) was solved using a central finite-difference method with a time step of $10^{-4}$ and spatial discretization step of $2 \cdot 10^{-3}$. The open loop system (1), (2) was first simulated to generate a steady-state with

$$\bar{s}_1(1) = 0.0542,\ \bar{p}_1(1) = 0.8350 \tag{45}$$
$$\bar{s}_2(1) = 0.1083,\ \bar{p}_2(1) = 0.95 \tag{46}$$

where $s_i(t, z)$ and $p_i(t, z)$ are the signals and pumps respectively. At $t = 0$, a 10% increase in reference was added for the signals at $z = 1$. The following proportional controller is used to to track this change for 10 seconds with $k_1 = k_2 = -2.3$ (see Fig. 2, 3, 4, 5).

$$p_1(t, 1) = k_1(s_1(t, 1) - y_{1d}) + \bar{u}_{1c} \tag{47}$$
$$p_2(t, 1) = k_2(s_2(t, 1) - y_{2d}) + \bar{u}_{2c} \tag{48}$$

and tracking was achieved with 7.5% error.

*Remark 11.* One can verify that the conditions (42)-(44), are satisfied for $k_1 = k_2 = -2.3$. Using $a_1 = 9.22, b_0 = 1.5$ along with (45),(46) we see that (42) is satisfied since for $i = 1, 2$ we have $1/b_0 < \bar{p}_i(1)$ and $1/a_0 > \bar{s}_i(1)$. Condition (43) gives rise to the lower bounds $k_1 > -3.1$ and $k_2 > -3.8 \cdot 10^3$, which are satisfied. The simulation was done with $\frac{\lambda_i}{\lambda_{n+i}} \approx 1$ and it is easy to verify (44) is satisfied for $\mu = 0.1$.

*Remark 12.* Future work will address designing a dynamic boundary controller with integral action for the counter-propagating system (1)-(3). Such a controller was designed in Pavel and Chang (2012) and performed with much higher accuracy than the proportional controller.

## 6. CONCLUSION

In this paper, we considered a coupled, semilinear hyperbolic PDE model for a normalized $N$x$N$ counter-propagating Raman amplifier and designed a boundary controller. After a change of coordinates, existence and uniqueness results for classical solutions of the linearized closed loop system were used in $\mathcal{H}^2$ spaces. Boundary conditions were chosen appropriately in order to ensure the decay of a quadratic-like Lyapunov functional along solutions to the closed loop system. As a result, an ex-

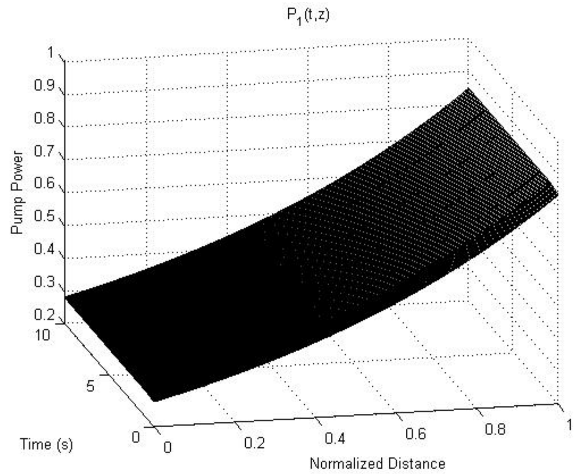


Fig. 4. Pump 1 evolution

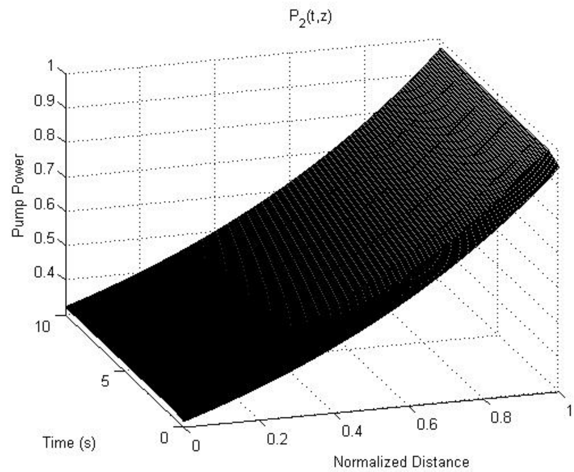


Fig. 5. Pump 2 evolution

ponential stability result was obtained in the $\mathcal{L}^2$-norm. Based on a key assumption, these results were extended to the nonlinear system following the same methodology in Pavel and Chang (2012). Future work will address under-actuated, non-normalized cases, as well as finding subsets invariant in the $C^0(\overline{\Omega})$-norm to remove Assumption 9 for the nonlinear counter-propagating system.

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