Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Stability analysis of nonlinear models via exact piecewise Takagi-Sugeno models ⋆

Temoatzin González, Miguel Bernal, Raymundo Márquez [*]

[*] *Sonora Institute of Technology, 5 de febrero 818 Sur, Col. Centro, C.P. 85000. Cd. Obregón, Mexico (email: miguel.bernal@itson.edu.mx)*

Abstract: This paper introduces a novel approach for stability analysis of nonlinear models based on an exact piecewise Takagi-Sugeno representation. The idea comes from the fact that larger modeling regions may lead to stability conditions which are harder to meet than those for smaller ones. Therefore, instead of applying sector nonlinearity to a single compact set of the state space as it is usually done, different exact Takagi-Sugeno representations are obtained for different compacts (partitions) of the state-space. Due to the piecewise nature of the proposed model, further relaxation of stability conditions is earned by using piecewise Lyapunov functions instead of common quadratic ones. The contribution is illustrated using examples which show the improvement over existing methods.

1. INTRODUCTION

Takagi-Sugeno (TS) models first appeared in Takagi and Sugeno (1985) as a bridge between traditional system structures and rule-based fuzzy representations. Since then, they have been the subject of an intensive research because they can be used to recast nonlinear models as convex ones whose structure allows some linear tools to be easily adapted through the direct Lyapunov method (Tanaka and Wang, 2001; Lendek et al., 2010). On one hand, TS models can be obtained from a nonlinear one by a convex nonlinear blending of linearizations, though this approach yields an approximate representation of the original one (Johansen et al., 2000). On the other hand, the well-established sector nonlinearity methodology presented in Taniguchi et al. (2001) has been successfully used to exactly rewrite nonlinear models as TS ones in a compact set of the state space: the model nonlinearities are captured in membership functions (MFs) which hold the convex-sum property. In this way, TS models can be easily used to investigate the stability and perform stabilization of nonlinear models via a quadratic Lyapunov function; most of the results thus obtained are expressed as linear matrix inequalities (LMIs), which can be efficiently implemented in commercially available software (Boyd et al., 1994).

Despite its exactness, the convex structure in a TS model has some drawbacks: the MFs should be removed in order to obtain LMI conditions which are therefore only sufficient (Tuan et al., 2001; Liu and Zhang, 2003; Sala and Ariño, 2007); if a quadratic Lyapunov function is used, stability of a TS model is established through the more general class of linear parameter varying (LPV) models, which makes the analysis conservative (Johansson et al., 1999; Blanco et al., 2001); moreover, a TS model representation of a nonlinear one is not unique, so there may be more conservative TS representations than others for stability's sake (Hori et al., 2002). Diverse Lyapunov functions have been proposed in the literature to overcome the conservatism of quadratic solutions for TS models: one direction has been concerned with nonquadratic Lyapunov functions, better known as fuzzy ones, which reproduce the convex structure of the TS model in the Lyapunov function (Tanaka et al., 2003; Guerra and Vermeiren, 2004; Rhee and Won, 2006; Bernal and Guerra, 2010; Guerra and Bernal, 2012); a second one has proposed piecewise Lyapunov functions (PWLFs) for TS models where not all their linear components are simultaneously activated, thus inducing state-space partitions (Johansson et al., 1999; Bernal et al., 2009).

When stability of a continuous-time nonlinear model is investigated via TS models, it is important to exactly represent it: this implies the use of the sector nonlinearity approach which produces a TS model with all its linear consequents simultaneously activated, thus making impossible to apply the piecewise analysis in Johansson et al. (1999). Difficulties arising from the use of nonquadratic Lyapunov functions are tantamount to those in the piecewise case since the time derivatives of the MFs are hard to deal with (Guerra and Bernal, 2009). A question naturally arises: is there any way to use a piecewise methodology for stability analysis of nonlinear models via exact TS representations? And moreover, is there any advantage in doing so? A first answer has been provided in Hori et al. (2002) and pursued in Ohtake et al. (2003) by mixing the piecewise methodology with fuzzy Lyapunov functions, which leads to very involved conditions. This paper provides a simpler answer by extending the relaxed stability analysis first appeared in Johansson et al. (1999) to TS models which exactly represent nonlinear systems through the sector nonlinearity approach; therefore, relaxation is twofold: on one hand, it uses a more general class of Lyapunov functions; on the other hand, it exactly represents a nonlinear model in a piecewise manner. It is shown that the proposed methodology can deal in a simpler yet more effective way with problems traditionally considered under the fuzzy nonquadratic approach.

⋆ This work has been supported by the Project Ciencia Básica SEP-CONACYT 168406 and the Project PROMEP ITSON-CA-18 Fortalecimiento.

This report is organized as follows: section 2 establishes notation, properties, and the sector nonlinearity approach which allows an exact TS model representation to be obtained from a nonlinear one; section 3 defines the piecewise Takagi-Sugeno representation (PWTS) of a nonlinear model on which the piecewise analysis in Johansson et al. (1999) can be straightforwardly applied (in fact, any method guaranteeing the continuity of the PWLF can be applied as will be shown); section 4 provides some examples which point out the advantages of the proposed methodology over a number of involved non-quadratic fuzzy approaches; the paper concludes in section 5 gathering some remarks and discussing future work.

2. DEFINITIONS AND NOTATIONS

Consider the following autonomous nonlinear model:

$$\dot{x}(t) = f(x(t)) \tag{1}$$

with $x(t) \in \mathbb{R}^n$ being the state vector and $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ being a smooth nonlinear vector field, which is assumed to be bounded in a compact set $\mathcal{C} \supset 0$ of the state space.

Sector nonlinearity methodology: The standard methodology to obtain an exact TS representation of a nonlinear model (1) is usually referred to as the sector nonlinearity approach (Taniguchi et al., 2001). The methodology begins by rewriting the vector field as $f(x(t)) = F(x(t))x(t)$, with $F(x(t)) \in \mathbb{R}^{n\times n}$. If $F(x(t))$ has p non-constant terms $z_1(x(t)), z_2(x(t)), \ldots, z_p(x(t))$, (which means that there are p nonlinearities in $f(x(t))$), they are all captured in the following weighting functions:

$$w_0^j(\cdot) = \frac{\overline{z}_j - z_j(\cdot)}{\overline{z}_j - \underline{z}_j}, \quad w_1^j(\cdot) = 1 - w_0^j(\cdot), \tag{2}$$

with $z_j(\cdot) \in \left[\underline{z}_j, \overline{z}_j\right]$, $\underline{z}_j = \min_{x(t)\in\mathcal{C}} z_j(\cdot)$, $\overline{z}_j = \max_{x(t)\in\mathcal{C}} z_j(\cdot)$, $j \in \{1, 2, \ldots, p\}$. As a remain of the rule-based origin of the TS models, $z(\cdot) = \left[\, z_1(\cdot)\ z_2(\cdot)\ \ldots\ z_p(\cdot)\,\right]^T$ is known as the premise or scheduling vector. Notice that the fact that a TS representation of a nonlinear model is not unique comes from this stage, since the premise vector can be chosen in different ways.

Once the weights have been constructed, the membership functions (MFs) which provide the TS model with a convex structure are defined as:

$$h_i(z(t)) = \prod_{j=1}^{p} w_{i_j}^j(z_j), \tag{3}$$

with $i \in \{1, 2, \ldots, r\}$, $i = i_p \times 2^{p-1} + \ldots + i_2 \times 2 + i_1 + 1$, $r = 2^p$, $i_j \in \{0, 1\}$. These functions hold the *convex sum property*, i.e., $\sum_{i=1}^{r} h_i(\cdot) = 1$, $0 \leq h_i(\cdot) \leq 1$, in $\mathcal{C}$.

Then, the nonlinear model (1) is rewritten in the following TS form:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right) A_i x(t), \tag{4}$$

with $A_i = F\left(x(t)\right)\big|_{h_i=1}$. Sector nonlinearity guarantees that TS model (4) is an exact representation of the nonlinear one (1) in $\mathcal{C}$; it is *not* an approximation. Convex sums of matrix expressions Υ_i, $i \in \{1, 2, \ldots, r\}$, are usually written shortly as $\Upsilon_z = \sum_{i=1}^{r} h_i\left(z(t)\right)\Upsilon_i$. Under this notation, TS model (4) is equivalent to $\dot{x}(t) = A_z x(t)$.

Should a matrix expression be involved with symbols "$>$" and "$<$", they will stand for positive and negative-definiteness, respectively; "$\succ$" and "$\prec$" will stand for element-wise positive and negative, respectively. Arguments will be omitted when convenient.

3. CONTRIBUTION

Let $\mathcal{C}_k \subset \mathcal{C}$, $k \in \{1, 2, \ldots, q\}$ be a collection of compact subsets partitioning the modeling compact $\mathcal{C}$ of nonlinear model (1). The key idea behind the proposed approach is to apply the sector nonlinearity technique to each compact subset $\mathcal{C}_k$, as to obtain q different TS representations (4) of the nonlinear model (1). Each of these TS representations will preserve all the information in (1) since they all rewrite the original system like in (4), but the polytope associated with each convex representation may vary since it depends on the extreme values of the non-constant terms of $F(x(t))$ in $\mathcal{C}_k$; these variations give room to improvement as will be proven in the next section.

Using the sector nonlinearity approach to represent nonlinear model (1) in q compacts yet preserving its unity, requires a proper notation. To this end, consider the non-constant terms of $F(x(t))$ in $\mathcal{C}_k$ as $z_j^k(x(t)) \in \left[\underline{z}_j^k, \overline{z}_j^k\right]$. As shown in the previous section for a single compact set, weights can be constructed taking into account these bounds as

$$w_0^{jk}(\cdot) = \frac{\overline{z}_j^k - z_j^k(\cdot)}{\overline{z}_j^k - \underline{z}_j^k}, \quad w_1^{jk}(\cdot) = 1 - w_0^{jk}(\cdot). \tag{5}$$

Consequently, r MFs for each of the q compact subsets $\mathcal{C}_k$ can be defined as

$$h_i^k(z(t)) = \prod_{j=1}^{p} w_{i_j}^{jk}(z_j), \tag{6}$$

with $i \in \{1, 2, \ldots, r\}$, $i = i_p \times 2^{p-1} + \ldots + i_2 \times 2 + i_1 + 1$, $r = 2^p$, $i_j \in \{0, 1\}$, $k \in \{1, 2, \ldots, q\}$. For each compact subset $\mathcal{C}_k$, the MFs thus defined hold the convex sum property, i.e., $\sum_{i=1}^{r} h_i^k(\,\cdot\,) = 1$, $0 \leq h_i^k(\,\cdot\,) \leq 1$, $k \in \{1, 2, \ldots, q\}$. Notice that the number of non-constant terms captured by the MFs h_i^k does not vary from one compact subset to another. Thus, nonlinear model (1) can be written as the following PWTS model:

$$\begin{aligned} \dot{x}(t) &= \bigcup_{k=1}^{q} A_z^k x(t) \\ &= \begin{cases} \sum_{i=1}^{r} h_i^1\left(z(t)\right) A_i^1 x(t) = A_z^1 x(t), & x(t) \in \mathcal{C}_1 \\ \sum_{i=1}^{r} h_i^2\left(z(t)\right) A_i^2 x(t) = A_z^2 x(t), & x(t) \in \mathcal{C}_2 \\ \vdots \\ \sum_{i=1}^{r} h_i^q\left(z(t)\right) A_i^q x(t) = A_z^q x(t), & x(t) \in \mathcal{C}_q \end{cases} \end{aligned} \tag{7}$$

with $A_i^k = F\left(x(x)\right)\big|_{h_i^k=1}$, $i \in \{1, 2, \ldots, r\}$, $k \in \{1, 2, \ldots, q\}$.

It is important to stress that the original nonlinear model (1) is *equivalent* to the PWTS model (7), just in the same way it is equivalent to the single TS model (4), i.e.,

$\dot{x}(t) = F\big(x(t)\big)\,x(t) = \bigcup_{k=1}^{q} A_z^k x(t)$. On the other hand, note that this does not imply that the matrices in each submodel are the same, i.e., in general $A_i^k \neq A_i^l$, $\forall k \neq l$.

The nature of the PWTS representation makes PWLF candidates suitable for stability analysis. PWLFs should be continuous for continuous-time systems, i.e.:

$$V\big(x(t)\big) = V_k\big(x(t)\big),\; x(t) \in \mathcal{C}_k, \tag{8}$$

with $V_k\big(x(t)\big) > 0$ for $x(t) \neq 0$, $k \in \{1, 2, \ldots, q\}$, and $V_k\big(x(t)\big) = V_l\big(x(t)\big)$, $\forall x(t) \in (\mathcal{C}_k \cap \mathcal{C}_k)$. Continuity of the PWLF candidate may be hard to meet if the partition does not translate into specific LMI conditions along with the structure of the PWLF. Some works simply assume continuity (Hori et al., 2002; Ohtake et al., 2003) while others induce it explicitly via polyhedral partitions (Johansson et al., 1999). A first result will be presented using the latter approach: it is a direct coupling of hyperplane-induced (linear) partitions for the PWTS representation above. A second result will step up the degree of the boundaries from linear to quadratic ones.

3.1 Linear boundaries

Notation in Johansson et al. (1999) will be partly adopted. To this end, let K_0 be the set of indexes of those compact subsets $\mathcal{C}_k$ that include the origin; otherwise, the indexes belong to the set K_1. Considering the following notation for those compacts which do not include the origin:

$$\bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix},\; \bar{A}_i^k = \begin{bmatrix} A_i^k & 0 \\ 0 & 0 \end{bmatrix},$$

with $i \in \{1, 2, \ldots, r\}$, $k \in K_1$, the PWTS model in (7) can be further rewritten as:

$$\begin{aligned} \dot{x}(t) &= \bigcup_{k \in K_0} A_z^k x(t), \\ \dot{\bar{x}}(t) &= \bigcup_{k \in K_1} \bar{A}_z^k x(t). \end{aligned} \tag{9}$$

If a polyhedral partition of the state-space is performed, i.e., if the compact subsets $\mathcal{C}_k$, $k \in \{1, 2, \ldots, q\}$ are all polyhedral, the partition boundaries are all affine linear functions of the state x or its extension $\bar{x}$. In other words, this means there exists matrices $\bar{E} = [\,E_k \;\; e_k\,]$ with $e_k = 0$ for $k \in K_0$, satisfying $\bar{E}_k \bar{x} \succeq 0$, $x \in \mathcal{C}_k$, $k \in \{1, \ldots, q\}$. Then, the following result directly arises from applying the piecewise stability analysis in Johansson et al. (1999) to the PWTS representation above:

Theorem 1. If there exist symmetric matrices T, $U_k \succeq 0$, and $W_{ki} \succeq 0$ such that

$$\left.\begin{aligned} &P_k = F_k^T T F_k,\; P_k - E_k^T U_k E_k > 0, \\ &\left(A_i^k\right)^T P_k + P_k A_i^k + E_k^T W_{ki} E_k < 0, \end{aligned}\right\} k \in K_0, \tag{10}$$

$$\left.\begin{aligned} &\bar{P}_k = \bar{F}_k^T T \bar{F}_k,\; \bar{P}_k - \bar{E}_k^T U_k \bar{E}_k > 0, \\ &\left(\bar{A}_i^k\right)^T \bar{P}_k + \bar{P}_k \bar{A}_i^k + \bar{E}_k^T W_{ki} \bar{E}_k < 0, \end{aligned}\right\} k \in K_1, \tag{11}$$

for $i \in \{1, 2, \ldots, r\}$, where $\bar{F} = [\,F_k \;\; f_k\,]$ with $f_k = 0$ for $k \in K_0$, satisfying $\bar{F}_k \bar{x} = \bar{F}_l \bar{x}$ for $x \in (\mathcal{C}_k \cap \mathcal{C}_l)$, $k, l \in \{1, 2, \ldots, q\}$; then $x(t)$ tends to zero exponentially for every continuous differentiable piecewise trajectory in $\mathcal{C} = \bigcup_{k=1}^{q} \mathcal{C}_k$ satisfying (9). The corresponding piecewise Lyapunov function (PWLF) is given by:

$$V(x) = \begin{cases} x^T P_k x, & x \in \mathcal{C}_k \;\; k \in K_0 \\ \begin{bmatrix} x \\ 1 \end{bmatrix}^T \bar{P}_k \begin{bmatrix} x \\ 1 \end{bmatrix}, & x \in \mathcal{C}_k,\; k \in K_1 \end{cases} \tag{12}$$

Proof. It follows directly from Theorem 1. in Johansson et al. (1999), provided that r matrices A_i^k pr $\bar{A}_i^k$ are always defined for each region $\mathcal{C}_k$, thus explaining the inclusion of index set $i \in \{1, \ldots, r\}$ in (10) and (11). □

Remark 1. Since the PWTS model (7) is equivalent to the original nonlinear one (1), stability conclusions derived from Theorem 1 are also valid for the latter. This represents a substantial difference with respect to the former work in Johansson et al. (1999), where only TS models with operating and interpolation regimes are considered, since these models are usually approximations of nonlinear ones obtained by linearization.

Remark 2. A systematic way to construct non-unique matrices $\bar{E}_k$, $\bar{F}_k$, $k \in \{1, 2, \ldots, q\}$ is described in Johansson et al. (1999); the interested reader is referred to this report for details.

3.2 Quadratic boundaries

The use of polyhedral partitions (and consequently of hyperplane boundaries) for piecewise stability analysis has been conditioned by the need of systematic (possibly LMI) procedures to construct continuous PWLF candidates. Nevertheless, for PWTS models, it is natural to consider partitions based on the nonlinearities of the model, since they are already bounded and can be further divided in sub-levels. Should polyhedral partitions be inadequate, higher-degree boundaries such as quadratic ones might be used instead. A quadratic boundary is described as:

$$x^T Q x = c \iff \bar{x}^T \bar{Q} \bar{x} = 0, \tag{13}$$

where x and $\bar{x}$ are state- and extended-state vectors as defined above, Q and $\bar{Q}$ are matrices of adequate size, and $c \in \mathbb{R}$. This contrasts with the polyhedral partition whose boundaries can be expressed as $\bar{E}_k \bar{x} = 0$ for a partition $\mathcal{C}_k$, with $\bar{E}_k$ defined as above. Let $\bar{x}^T \bar{Q}_1 \bar{x} = 0$, $\bar{x}^T \bar{Q}_2 \bar{x} = 0$,..., $\bar{x}^T \bar{Q}_q \bar{x} = 0$, be q quadratic boundaries (13) creating a partition of the modeling compact $\mathcal{C}$ on which the nonlinear model (1) is represented. Then, the induced compact subsets $\mathcal{C}_k$ partitioning $\mathcal{C}$ are:

$$\mathcal{C}_k = x : \left\{ \begin{array}{c} (-1)^{d_1^k} \bar{x}^T \bar{Q}_1 \bar{x} \geq 0 \\ (-1)^{d_2^k} \bar{x}^T \bar{Q}_2 \bar{x} \geq 0 \\ \vdots \\ (-1)^{d_q^k} \bar{x}^T \bar{Q}_q \bar{x} \geq 0 \end{array} \right\}, \tag{14}$$

with $k \in \{1, \ldots, 2^q\}$, $k = 1 + d_1^k + d_2^k(2) + \cdots + d_q^k(2)^{q-1}$ providing all possible sign combinations in the above inequalities. Note that some of the induced subsets $\mathcal{C}_k$ may be empty depending on the state-space dimensions and the number of boundaries. Note also that the boundary $\bar{x}^T \bar{Q}_j \bar{x} = 0$ between contiguous subsets $\mathcal{C}_k$ and $\mathcal{C}_l$ holds the following relation: $\mathcal{C}_k \cap \mathcal{C}_l = \left\{x : \bar{x}^T \bar{Q}_j \bar{x} = 0\right\}$.

Consider a PWLF candidate as in (12), with $\bar{P}_k$ = block-diag $[P_k \; 0]$ for those regions $\mathcal{C}_k$ with $k \in K_0$ (i.e., regions containing the origin); continuity of these functions in some border $\bar{x}^T \bar{Q}_j \bar{x} = 0$ between contiguous subsets $\mathcal{C}_k$ and $\mathcal{C}_l$ is guaranteed if

$$
\begin{aligned}
&\bar{x}^T \bar{P}_k \bar{x} = \bar{x}^T \bar{P}_l \bar{x},\ \forall x \in (\mathcal{C}_k \cap \mathcal{C}_l)\,,\\
&\iff \bar{x}^T \bar{P}_k \bar{x} = \bar{x}^T \bar{P}_l \bar{x},\ \bar{x}^T \bar{Q}_j \bar{x} = 0,\\
&\iff \bar{x}^T \bar{P}_k \bar{x} - \bar{x}^T \bar{P}_l \bar{x} = 0 = \bar{x}^T \left(\bar{Q}_j + \bar{Q}_j^T\right) \bar{x},\\
&\iff \bar{P}_k - \bar{P}_l = \bar{Q}_j + \bar{Q}_j^T. \qquad (15)
\end{aligned}
$$

This condition implies there is no need of matrices $\bar{F}_k$ for guaranteeing continuity of the PWLF candidate in the quadratic-boundary case. The second result can now be stated:

Theorem 2. If there exist symmetric matrices $\bar{P}_k =$ block-diag $[P_k\ 0]$, $P_k = P_k^T$ for $k \in K_0$, $\bar{P}_k = \bar{P}_k^T$ for $k \in K_1$, and constants $\epsilon_j > 0$, such that

$$
\bar{P}_k - \bar{P}_l = \bar{Q}_j + \bar{Q}_j^T, \qquad (16)
$$

$$
\bar{P}_k - \sum_{j=1}^{q} (-1)^{d_i^k} \epsilon_j \left(\bar{Q}_j + \bar{Q}_j^T\right) > 0, \qquad (17)
$$

$$
\left(\bar{A}_i^k\right)^T \bar{P}_k + \bar{P}_k \bar{A}_i^k + \sum_{j=1}^{q} (-1)^{d_i^k} \epsilon_j \left(\bar{Q}_j + \bar{Q}_j^T\right) < 0, (18)
$$

hold for $i \in \{1, 2, \ldots, r\}$, $j \in \{1, 2, \ldots, q\}$, $k \in (K_0 \cup K_1)$, $k = 1 + d_1^k + d_2^k(2) + \cdots + d_q^k(2)^{q-1}$, $l : \mathcal{C}_k \cap \mathcal{C}_l = \left\{x : \bar{x}^T \bar{Q}_j \bar{x} = 0\right\}$; then $x(t)$ tends to zero exponentially for every continuous differentiable piecewise trajectory in $\mathcal{C} = \bigcup_{k=1}^{q} \mathcal{C}_k$ satisfying (9). The corresponding piecewise Lyapunov function (PWLF) is given by (12).

Proof. Condition (16) guarantees continuity of the PWLF candidate (12), since it arises from the development leading to (15). On the other hand, since a partition $\mathcal{C}_k$ is specified from its boundaries as in (14), it is clear that:

$$
x^T \left(\sum_{j=1}^{q} (-1)^{d_i^k} \left(\bar{Q}_j + \bar{Q}_j^T\right)\right) x > 0, \qquad x \in \mathcal{C}_k,
$$

which means that each term in the sum above multiplied by a constant $\epsilon_j > 0$ can further relax the standard piecewise conditions by means of the S-procedure. This leads to LMIs (17) and (18), thus concluding the proof. □

Remark 3. Conditions (10) and (11) in theorem 1 as well as (16), (17), and (18) in theorem 2 are LMIs, which means they are efficiently solved via convex optimization techniques (Boyd et al., 1994).

4. EXAMPLES

Example 1. Consider the following nonlinear model in Tanaka et al. (2003):

$$
\dot{x}(t) = \begin{bmatrix} -3.5 - 1.5\sin x_1 & -4 \\ 9.5 - 10.5\sin x_1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \qquad (19)
$$

There is one non-constant term in the system matrix of (19); it is naturally bounded over all the state space by $\underline{z} \le \sin x_1 \le \overline{z}$ with $\overline{z} = 1$ and $\underline{z} = -1$. Rewriting the previous model in the TS form through sector nonlinearity approach for $\mathcal{C} = \mathbb{R}^2$, we obtain:

$$
\dot{x}(t) = A_z x(t) = \sum_{i=1}^{2} h_i\left(x(t)\right) A_i x(t), \qquad (20)
$$

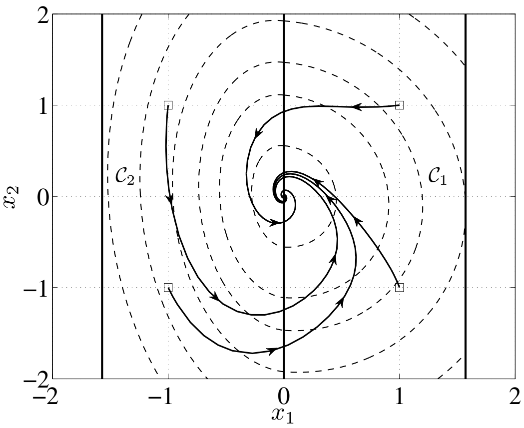

Fig. 1. Level surface of the computed Lyapunov function.

where $A_1 = \begin{bmatrix} -3.5 - 1.5\underline{z} & -4 \\ 9.5 - 10.5\underline{z} & -2 \end{bmatrix}$, $A_2 = \begin{bmatrix} -3.5 - 1.5\overline{z} & -4 \\ 9.5 - 10.5\overline{z} & -2 \end{bmatrix}$, $h_1 = \dfrac{\overline{z} - \sin x_1}{\overline{z} - \underline{z}}$, $h_2 = 1 - h_1(x_1)$.

Conditions for quadratic stability fail, though simulations suggest that the system is stable (Tanaka et al., 2003); moreover, nonquadratic analysis via fuzzy Lyapunov functions is proved to be only local and involves handling the time-derivatives of the membership functions (Tanaka et al., 2003; Guerra and Bernal, 2009). Thus, the proposed approach comes at hand.

To this end, the compact set $\mathcal{C}$ is divided into subsets $\mathcal{C}_1 = \left\{x : 0 \le x_1 \le \frac{\pi}{2}\right\}$ and $\mathcal{C}_2 = \left\{x : -\frac{\pi}{2} \le x_1 \le 0\right\}$, both of which contain the origin; then, applying the sector nonlinearity technique in each subset, the following PWTS model is obtained:

$$
\dot{x}(t) = \begin{cases} A_z^1 x(t) = \displaystyle\sum_{i=1}^{2} h_i^1(x) A_z^1 x(t), & x(t) \in \mathcal{C}_1 \\ A_z^2 x(t) = \displaystyle\sum_{i=1}^{2} h_i^2(x) A_z^2 x(t), & x(t) \in \mathcal{C}_2 \end{cases} \qquad (21)
$$

where $h_1^1 = 1 - \sin x_1$, $h_2^1 = 1 - h_1^1$, $h_1^2 = \sin x_1$, $h_2^2 = 1 - h_1^2$, $A_1^1 = \begin{bmatrix} -3.5 & -4 \\ 9.5 & -2 \end{bmatrix}$, $A_2^1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}$, $A_1^2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}$, $A_2^2 = \begin{bmatrix} -3.5 & -4 \\ 9.5 & -2 \end{bmatrix}$.

The following matrices F_k and E_k, $k \in \{1, 2\}$ can be constructed via the PWLTOOL (Hedlund and Johansson, 1999):

$$
F_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, F_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, E_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, E_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.
$$

Conditions (10) in theorem 1 were found feasible. The following matrices were obtained for the PWLF (12):

$$
P_1 = \begin{bmatrix} 9.7663 & 0.6145 \\ 0.6145 & 6.4884 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 26.0256 & 1.03 \\ 1.03 & 6.4884 \end{bmatrix}. \quad (22)
$$

The level surfaces of the computed PWLF are shown in Fig. 1.

Example 2. Consider the following model of a nonlinear spring-mass-damper system (Ohtake et al., 2003):

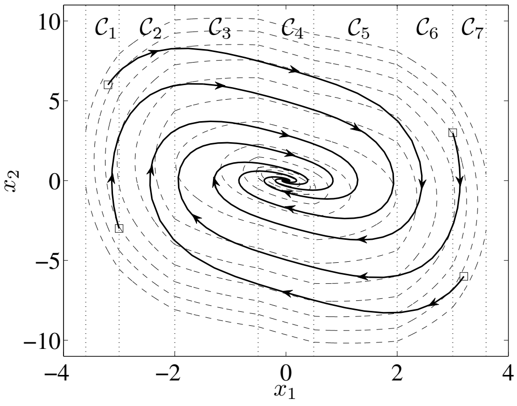

Fig. 2. Level surface of the computed Lyapunov function.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{l}{m}\left(1 + x_1^2(t)\right) & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{23}$$

where $l = 1$, $m = 1$, $c = 1$, and the state is assumed to lie in the compact set $\mathcal{C} = \{x : |x_1(t)| \leq d\}$. This model can be rewritten in the PWTS form (7) for $d = 3.6$ via $q = 7$ partitions given by $\mathcal{C}_1 = \{x : -3.6 \leq x_1 \leq -3\}$, $\mathcal{C}_2 = \{x : -3 \leq x_1 \leq -2\}$, $\mathcal{C}_3 = \{x : -2 \leq x_1 \leq -0.5\}$, $\mathcal{C}_4 = \{x : -0.5 \leq x_1 \leq 0.5\}$, $\mathcal{C}_5 = \{x : 0.5 \leq x_1 \leq 2\}$, $\mathcal{C}_6 = \{x : 1 \leq x_1 \leq 3\}$, and $\mathcal{C}_7 = \{x : 3 \leq x_1 \leq 3.6\}$; there is only one nonlinearity, which means that $r = 2$. Therefore, in each partition, the local TS representation is given by:

$$\dot{x}(t) = \sum_{i=1}^{2} h_i^k\left(z(t)\right) A_i^k x(t), \tag{24}$$

where $A_1^k = \begin{bmatrix} 0 & 1 \\ -\frac{l}{m}\left(1 + \underline{z}\right) & -1 \end{bmatrix}$, $A_2^k = \begin{bmatrix} 0 & 1 \\ -\frac{l}{m}\left(1 + \overline{z}\right) & -1 \end{bmatrix}$, $h_1^k(x_1) = \dfrac{\overline{z} - x_1^2}{\overline{z} - \underline{z}}$, $h_2^k(x_1) = 1 - h_1^k(x_1)$, $k \in \{1, \ldots, 7\}$, $\overline{z} = \max\limits_{x_1 \in \mathcal{C}_k} \left(x_1^2\right)$, $\underline{z} = \min\limits_{x_1 \in \mathcal{C}_k} \left(x_1^2\right)$.

In Ohtake et al. (2003) is reported that quadratic stability test fails for a single-compact TS representation of model (23) in $\mathcal{C}$ with $d > 1.74$, while nonquadratic stability test also fails for $d > 2.43$ when a fuzzy Lyapunov function as in Tanaka et al. (2001) is employed. On the other hand, the PWTS representation in (24) altogether with theorem 1 allows establishing stability for $d = 3.6$, which outperforms previously reported stability tests.

Due to space limitations, only the level surfaces of the computed PWLF are shown in Fig. 2 along with four system trajectories.

Example 3. Consider the following nonlinear model:

$$\dot{x}(t) = \begin{bmatrix} a + b\left(x_1^2 + x_2^2\right) & -3 - \left(x_1^2 + x_2^2\right) \\ c - d\left(x_1^2 + x_2^2\right) & -2.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{25}$$

with $a = 1.8002$, $b = -0.9906$, $c = 7.1617$, and $d = 4.1468$. Taking $z\left(x(t)\right) = x_1^2 + x_2^2$, this model can be rewritten in the PWTS form (9) for $\mathcal{C} = \left\{x : x_1^2 + x_2^2 \leq 0.64^2\right\}$ via $q = 2$ partitions given by $\mathcal{C}_1 = \left\{x : x_1^2 + x_2^2 \leq 0.4^2\right\}$ and $\mathcal{C}_2 = \left\{x : 0.4^2 \leq x_1^2 + x_2^2 \leq 0.64^2\right\}$. Therefore, the PWTS model is given by:

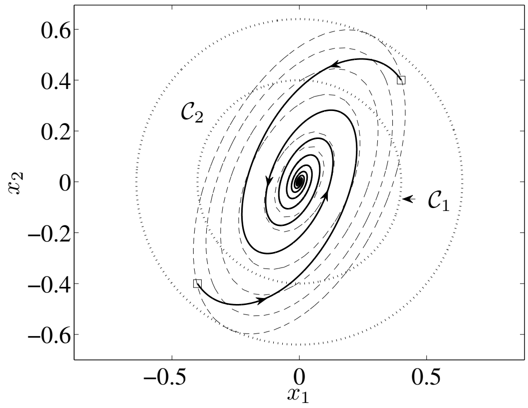

Fig. 3. Level surface of the computed Lyapunov function.

$$\dot{x}(t) = \begin{cases} A_z^1 x(t) = \sum\limits_{i=1}^{2} h_i^1(x) A_z^1 x(t), & x(t) \in \mathcal{C}_1 \\ A_z^2 x(t) = \sum\limits_{i=1}^{2} h_i^2(x) A_z^2 x(t), & x(t) \in \mathcal{C}_2, \end{cases} \tag{26}$$

where $h_1^1 = \dfrac{0.4^2 - \left(x_1^2 + x_2^2\right)}{0.4^2}$, $h_1^2 = \dfrac{0.64^2 - \left(x_1^2 + x_2^2\right)}{0.64^2 - 0.4^2}$, $h_2^1 = 1 - h_1^1$, $h_2^2 = 1 - h_1^2$, $A_1^1 = \begin{bmatrix} 1.8002 & -3 \\ 7.1617 & -2.5 \end{bmatrix}$, $A_2^1 = \begin{bmatrix} 1.6417 & -3.16 \\ 6.4982 & -2.5 \end{bmatrix}$, $A_1^2 = \begin{bmatrix} 1.6417 & -3.16 \\ 6.4982 & -2.5 \end{bmatrix}$, $A_2^2 = \begin{bmatrix} 1.3945 & -3.4096 \\ 5.4632 & -2.5 \end{bmatrix}$.

The following matrices $\overline{Q}_j$,$k \in \{1, 2\}$ describe the quadratic boundaries between $\mathcal{C}_1$ and $\mathcal{C}_2$:

$$\overline{Q}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -0.4^2 \end{bmatrix}, \quad \overline{Q}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -0.64^2 \end{bmatrix}.$$

Conditions in theorem 2 were found feasible with $\epsilon_1 = 1$ and $\epsilon_2 = 0$. The following matrices were obtained for the PWLF (12):

$$P_1 = \begin{bmatrix} 655.5033 & -208.4659 \\ -208.4659 & 318.7870 \end{bmatrix},$$
$$P_2 = \begin{bmatrix} 653.5033 & -208.4659 & 0 \\ -208.4659 & 316.7870 & 0 \\ 0 & 0 & 0.32 \end{bmatrix}.$$

In Fig. 3 the level surfaces of the computed PWLF are shown along with two trajectories.

5. CONCLUSION

This paper introduced a novel exact piecewise Takagi-Sugeno representation of nonlinear systems which proved to be useful for piecewise stability analysis. Besides the polyhedral partition, the use of quadratic boundaries has shown its ability to relax quadratic as well as nonquadratic limitations. Future work is twofold: a deeper study of higher-degree boundaries for relaxation of stability conditions (which necessarily have to be written as LMIs to be useful) and piecewise stabilization (a task which has often lead to BMI conditions in the past).

REFERENCES

Bernal, M. and Guerra, T.M. (2010). Generalized non-quadratic stability of continuous-time Takagi-Sugeno models. *IEEE Transactions on Fuzzy Systems*, 18(4), 815–822.

Bernal, M., Guerra, T.M., and Kruszewski, A. (2009). A membership-function-dependent approach for stability analysis and controller synthesis of Takagi-Sugeno models. *Fuzzy Sets and Systems*, 160(19), 2776–2795.

Blanco, Y., Perruqueti, W., and Borne, P. (2001). Non-quadratic stability of non linear systems in the Takagi-Sugeno form. In *Proc. of the European Control Conference 2001*. Porto, Portugal.

Boyd, S., Ghaoui, L.E., Feron, E., and Belakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory*, volume 15. SIAM: Studies In Applied Mathematics, Philadelphia, USA.

Guerra, T. and Bernal, M. (2009). A way to escape from the quadratic framework. In *Proc. FUZZ-IEEE Conference*, 784–789. Jeju, Korea.

Guerra, T. and Bernal, M. (2012). Strategies to exploit non-quadratic local stability analysis. *International Journal of Fuzzy Systems*, 13, 372–379.

Guerra, T. and Vermeiren, L. (2004). LMI-based relaxed non-quadratic stabilization conditions for nonlinear systems in Takagi-Sugeno's form. *Automatica*, 40(5), 823–829.

Hedlund, S. and Johansson, M. (1999). PWLTLOOL: A matlab toolbox for analysis of piecewise linear systems.

Hori, T., Tanaka, K., and Wang, H. (2002). A piecewise takagi-sugeno fuzzy model construction and relaxation of stability conditions. In *Proceedings of the 41st IEEE Conference on Decision and Control*, volume 2, 2149–2150.

Johansen, T., Shorten, R., and Murray-Smith, R. (2000). On the interpretation and identification of dynamic Takagi-Sugeno fuzzy models. *IEEE Transactions on Fuzzy Systems*, 8(3), 297–313.

Johansson, M., Rantzer, A., and Arzen, K. (1999). Piecewise quadratic stability of fuzzy systems. *IEEE Transactions on Fuzzy Systems*, 7(6), 713–722.

Lendek, Z., Guerra, T., Babuska, R., and De-Schutter, B. (2010). *Stability Analysis and Nonlinear Observer Design Using Takagi-Sugeno Fuzzy Models*. Springer-Verlag, Netherlands.

Liu, X. and Zhang, Q. (2003). New approaches to controller designs based on fuzzy observers for t-s fuzzy systems via LMI. *Automatica*, 39(9), 1571–1582.

Ohtake, H., Tanaka, K., and Wang, H. (2003). Piecewise nonlinear control. In *42nd IEEE Conference on Decision and Control, 2003*, volume 5, 4735–4740.

Rhee, B. and Won, S. (2006). A new fuzzy Lyapunov function approach for a Takagi-Sugeno fuzzy control system design. *Fuzzy Sets and Systems*, 157(9), 1211–1228.

Sala, A. and Ariño, C. (2007). Asymptotically necessary and sufficient conditions for stability and performance in fuzzy control: Applications of Polya's theorem. *Fuzzy Sets and Systems*, 158(24), 2671–2686.

Takagi, T. and Sugeno, M. (1985). Fuzzy identification of systems and its application to modeling and control. *IEEE Transactions on System Man and Cybernetics*, 15(1), 116–132.

Tanaka, K., Hori, T., and Wang, H. (2001). A fuzzy Lyapunov approach to fuzzy control system design. Washington, USA.

Tanaka, K., Hori, T., and Wang, H. (2003). A multiple Lyapunov function approach to stabilization of fuzzy control systems. *IEEE Transactions on Fuzzy Systems*, 11(4), 582–589.

Tanaka, K. and Wang, H. (2001). *Fuzzy Control Systems Design and Analysis. A linear matrix inequality approach*. John Wiley & Sons, New York.

Taniguchi, T., Tanaka, K., and Wang, H. (2001). Model construction, rule reduction and robust compensation for generalized form of Takagi-Sugeno fuzzy systems. *IEEE Transactions on Fuzzy Systems*, 9(2), 525–537.

Tuan, H., Apkarian, P., Narikiyo, T., and Yamamoto, Y. (2001). Parameterized linear matrix inequality techniques in fuzzy control system design. *IEEE Transactions on Fuzzy Systems*, 9(2), 324–332.