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Extension of PID to fractional orders controllers: a frequency-domain tutorial presentation

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Abstract: This paper presents how common PID controllers have been generalized to fractional order PID controllers and how the additional tuning parameters can be used to meet more requirements. It is shown that the first generation CRONE control-system design methodology is able to provide robust fractional order PID for uncertain gain perturbed plants.

Keywords: Fractional order PID, CRONE control-system design, robust control.

1. INTRODUCTION

The first definitions of differentiation (or integration) with fractional or non-integer orders (Oldham 1974, Samko 1993, Miller 1993) were given by Leibniz and Euler at the end of the 17th and during the 18th century. In the 19th century many mathematicians generalized these definitions: Laplace, Lacroix, Fourier, Liouville, Abel, Hargreave, Riemann etc. In 1869 Sonin extended the Cauchy integral to fractional integration orders and came up with the Riemann-Liouville definition which can be found using operational calculus.
Let $y(t)$ be the order n derivative of the causal signal $x(t)$:

$$y(t) = x^{(n)}(t) = D^n x(t), \tag{1}$$

with $n \in \mathrm{C}$ and where D is the differentiation operator. If the real part of n is negative, then $y(t)$ is the order n integral of $x(t)$. From (1), neglecting the initial conditions of $x(t)$, the transfer function of the operator D^n is:

$$D^n(s) = {L\{y(t)\}} \big/ {L\{x(t)\}} = s^n , \tag{2}$$

where s is the Laplace variable. Many authors proposed applications of the fractional order operator and more precisely to automatic control. First ones are Tustin (1958), Manabe (1960, 1961) and Oustaloup (1981). Even Bode (1945) proposed to design controllers that behave as fractional order controllers. While the CRONE methodology (Oustaloup 1991a, 1995, 1999, Lanusse 1994) has been developed for the robustness purpose, many authors (Podlubny 1999a-b, Chen 2004, Petras 1999, Nataraj 2007, Monje 2010, etc.) proposed the design of fractional order PID controllers. This paper deals with this class of controller. It presents fractional order PID controllers and compares them with the first generation CRONE controller.

2. GENERALISATION OF THE ACADEMIC PID CONTROLLER

2.1 Parallel PI$^\lambda$D$^\mu$ controller

Within the common unity-feedback configuration (Fig. 1), the well-known academic PID controller is defined by:

$$C(s) = C_\mathrm{P} + \frac{C_\mathrm{I}}{s} + C_\mathrm{D} s , \tag{3}$$

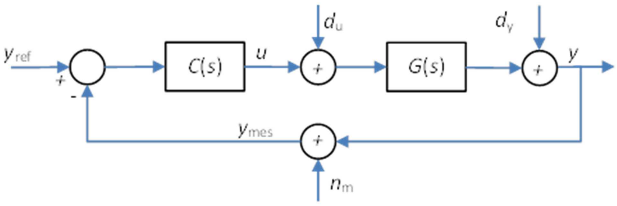

Fig. 1. Unity-feedback configuration.

Replacing s by fractional powers of s generalizes this academic PID controller. Thus, the parallel PID (3) becomes a PI$^\lambda$D$^\mu$ defined by:

$$C(s) = C_\mathrm{P} + \frac{C_\mathrm{I}}{s^\lambda} + C_\mathrm{D} s^\mu , \tag{4}$$

With two more parameters λ and μ, this controller permits a more flexible design of the control-system. Some works (Monje 2004, Cervera 2006, Valerio 2012) dedicated to this class of controllers explain how to use this flexibility to solve control problems even if often, only the integral or derivative part is used and thus only a PI$^\lambda$ or PD$^\mu$ is designed. As for the integer order PID controller, to avoid that the *CS* control sensitivity function tends towards infinity, the fractional order PID controller needs to be modified:

$$C(s) = C_\mathrm{P} + \frac{C_\mathrm{I}}{s^\lambda} + \frac{C_\mathrm{D} s^\mu}{1+\tau_\mathrm{F} s^\gamma} . \tag{5}$$

Depending on the value of γ, $C(s)$ is biproper for $\gamma = \mu$ and strictly proper for $\gamma > \mu$. Fig. 2 shows a Bode diagram of $C(s)$ for $\gamma = \mu$. The corner frequencies in Fig. 2 are defined by:

$$\omega_\mathrm{I} = \left(\frac{C_\mathrm{I}}{C_\mathrm{P}}\right)^{1/\lambda},\ \omega_\mathrm{D} = \left(\frac{C_\mathrm{P}}{C_\mathrm{D} + C_\mathrm{P}\tau_\mathrm{F}}\right)^{1/\mu} \text{ and } \omega_\mathrm{F} = \left(\frac{1}{\tau_\mathrm{F}}\right)^{1/\gamma} . \tag{6}$$

As five design parameters can be tuned, five (or more) different requirements can be taken into account. They can be chosen among:

- a given phase margin
- a given gain margin
- a given open-loop gain crossover frequency ω_{cg}
- a flat open-loop phase around ω_{cg} (for plants with gain perturbation)
- a given controller high-frequency gain
- no steady-state error
- given gains at given frequencies of the four closed loop sensitivity functions.

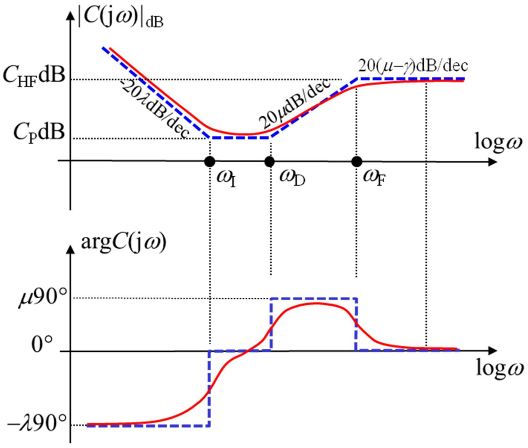

Fig. 2. Frequency response of a biproper $PI^{\lambda}D^{\mu}$ controller ($\gamma = \mu$).

2.2 Academic parallel $PI^{\lambda}D^{\mu}$ controller drawbacks

If the fractional PID controller (5) is used to solve a control problem, three major drawbacks can arise.

1. Although it is interesting to use a fractional differentiation transfer function for a lead effect and to shape the open-loop phase, the fractional integrator transfer function has a weak interest. Indeed, for an input disturbance of the form $1/s^k$, a fractional integrator of order $k+\lambda$ (with $k \in \mathbb{N}$ and $0 < \lambda < 1$) is no more efficient (even less) for steady-state error cancellation than a simpler integer integrator of order $k+1$.
2. It would require a really resource consuming discrete-time implementation of the fractional order λ integrator to avoid its band-limitation at low-frequencies. Moreover, this mandatory band-limitation for the implementation leads to controller that behaves like an integer integrator.
3. If γ differs from μ, it would also require a really resource consuming discrete-time approximation or an infinite order equivalent continuous-time model to implement the high-frequency behavior of the order μ derivative part.

Let note that as their gains tend towards 0, even if the fractional order integrator and derivative operators are band-limited for the controller implementation, respectively at high and low frequencies, it could be achieved without modifying the global controller behavior. To avoid the third drawback, it could be useful to choose γ such that $\gamma - \mu = N$ ($N \in \mathbb{N}$) with $N = 0$ for a biproper controller or $N > 0$ for a strictly proper controller. The first and second drawbacks can be avoided by replacing the fractional order integrator by a fractionally band-limited integer order (Maamri 2010) that can be approximated with a low order rational transfer function. Thus, for $0 < \lambda < 1$, the fractional $PI^{\lambda}D^{\mu}$ becomes

$$C(s) = C_P + C'_I \frac{1+\left(\dfrac{s}{\omega'_I}\right)^{1-\lambda}}{s} + \frac{C_D s^{\mu}}{1+\tau_F s^{N+\mu}}, \qquad (7)$$

where C'_I equals $\omega'^{\lambda-1}_I C_I$ to ensure the same ω_I corner frequency as in (6). Fig. 3 shows how the Bode diagram of this modified parallel fractional controller looks like. It is quite similar to the Bode diagram in fig. 2.

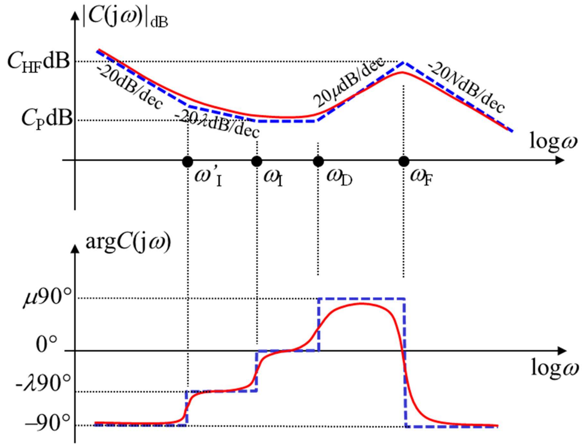

Fig. 3. Frequency response of a strictly proper $PI^{\lambda}D^{\mu}$ controller with $N = 1$

3. SERIES FRACTIONAL ORDER PID CONTROLLER

3.1 Definition

An easy to tune series PID is often used rather than the parallel one defined by (1). It is defined by:

$$C(s) = C_0 \left(1+\frac{\omega_I}{s}\right)\left(\frac{1+s/\omega_1}{1+s/\omega_2}\right)^{N_D} \frac{1}{1+s/\omega_F}. \qquad (8)$$

Each term being band-limited and ensuring one of the controller behaviors: proportional, integral, derivative and filtering. The integer power N_D is used to permit both a large derivative action and the reduction of the high-frequency gain of the controller. The low-pass filter has been added to make the controller strictly proper and thus to ensure a decreasing gain of the control sensitivity function.

In the same way, in order to simplify its design and implementation, without loss of performance, it is possible to choose a fractional order PID controller defined by:

$$C(s) = C_0 \left(1+\frac{\omega_I}{s}\right)^{n_I} \left(\frac{1+\dfrac{s}{\omega_1}}{1+\dfrac{s}{\omega_2}}\right)^{\mu} \frac{1}{\left(1+\dfrac{s}{\omega_F}\right)^{n_F}}, \qquad (9)$$

with $\omega_I \le \omega_1 \le \omega_2 \le \omega_F$ and $n_I \in \mathbb{N}$, $\mu \in \mathbb{R}$, $n_F \in \mathbb{N}$. Fig. 4 presents a Bode diagram of this $PID^{\mu}(F)$ controller. One more time, the Bode diagram in fig. 4 looks like the Bode diagram in fig. 2 and 3, thus permits to conclude that controllers (9), (7) and (5) will have a similar contribution in the important frequency range [ω_I, ω_F]. Concerning the controller tuning, the following algorithm can be used.

- Let us consider the control diagram of Fig. 1 in which, at low frequencies ($\omega < \omega_I$), the reference input signal spectrum is defined by $Y_{ref}(s) \approx y_{ref0}/s^M$ and the plant input disturbance spectrum is defined by $D_u(s) \approx d_{u0}/s^N$. If n_{pl} represents the integral order of the asymptotic behavior of the plant at low frequencies, order n_I of the proportional integrator is defined by $n_I \geq \max(M - n_{pl}, N)$ to ensure no steady state error.
- The low-pass filter order n_F is simply $n_F \geq 0$, given that (with increasing frequency and for $\omega > \omega_F$) $n_F = 0$ ensures the constancy of the gain of the control sensitivity function $CS(j\omega)$, and $n_F \geq 1$ ensures its decrease.
- Choose the open loop gain crossover frequency ω_{cg} to reach the bandwidth specification (the closed-loop cut-off frequency will be close to ω_{cg}).
- Set $\omega_I = \omega_{cg}/\alpha_I$ with $\alpha_I \in [1, 10]$ according to settling time specifications.
- Set $\omega_F = \alpha_F\, \omega_{cg}$ with $\alpha_F \in [1, 10]$ according to noise immunity requirements.
- Determine ω_1, ω_2 and μ to obtain the required phase margin M_Φ and C_0 to obtain the desired open-loop gain crossover frequency ω_{cg}.

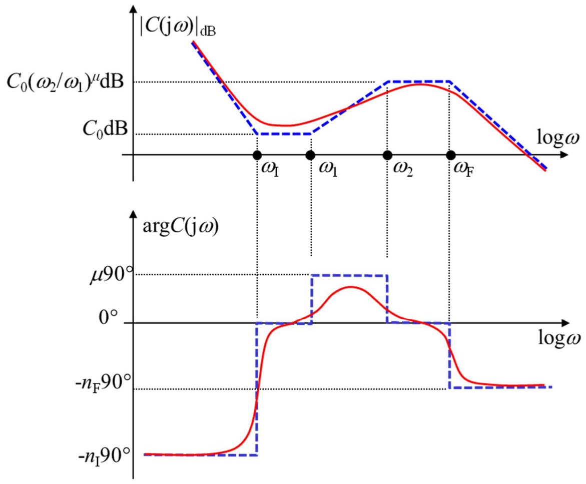

Fig. 4. Frequency response of a series fractional order PID controller for: $n_I = 2$, $n_F = 1$ and $\mu = 0.75$

As four parameters (ω_1, ω_2, μ and C_0) are used to define two requirements (ω_{cg} and M_Φ) only, up to two additional requirements can be taken into account.

3.2 Design example of a fractional order controller

Let a plant (a DC motor) be defined by:

$$G(s) = \frac{\Omega(s)}{U(s)} = \frac{K_u}{(T_e s + 1)(J_m s + f)} \qquad (10)$$

with K_u = 2.34Nm/V, T_e = 0.043s, J_m = 0.108kg.m^2 and f = 0.002Nm.s.rad^{-1}.

The PID requirements are:

- ω_{cg} = 100 rad/s
- M_Φ = 50°
- a low-pass filtering with $n_F = 1$ and $\omega_F/\omega_{cg} = 5$
- an integration with $n_I = 1$ and $\omega_{cg}/\omega_I = 5$.

To make easier the tuning of the controller, we choose $\omega_{cg} = \sqrt{\omega_1 \omega_2}$. For μ = 2, the design problem is that of the integer order PID defined by (8) and no more requirement can be taken into account. Thus, with N_D = 2, the phase margin and open-loop gain crossover frequency are ensured by $\omega_{cg}/\omega_1 = \omega_2/\omega_{cg} = 1.742$ and C_0 = 6.858. The resulting PID controller is:

$$C(s) = 6.858 \left(1 + \frac{20}{s}\right) \left(\frac{1 + s/58.02}{1 + s/172.4}\right)^2 \frac{1}{1 + s/500}\,. \qquad (11)$$

It provides a 15.2dB gain margin.
Using the fractional order PID defined by (9) with $\omega_{cg} = \sqrt{\omega_1 \omega_2}$, one more requirement can be taken into account. For instance, a 20dB gain margin can be achieved with μ = 0.815, $\omega_{cg}/\omega_1 = \omega_2/\omega_{cg} = 6.65$ and C_0 = 4.338. The controller is:

$$C(s) = 4.338 \left(1 + \frac{20}{s}\right) \left(\frac{1 + s/15.04}{1 + s/665}\right)^{0.815} \frac{1}{1 + s/500}\,, \qquad (12)$$

Fig. 5 shows how the fractional order controller can take the open-loop frequency response away from the critical point. Even if both controllers ensure the same open-loop gain crossover frequency and phase margin, Fig. 6 shows that this fractional order controller provides, on one hand, a lead phase over a greatest frequency range which could be useful in a robustness framework. But on the other hand, the fractional controller provides a lower gain at low frequencies and a greater gain at high frequencies which could respectively increase the magnitude of the *SG* input sensitivity function and of the *CS* control sensitivity function.
In order to implement the fractional order controller, the Oustaloup's approximation method (Oustaloup 2000a) is used to replace its order 0.815 part by a rational one. As there is less than two decade between ω_2 and ω_1, Fig. 6 shows that only three zero/pole pairs are needed to ensure a good approximation. Then, the controller defined by (9) can be implemented using:

$$C(s) = 4.338 \frac{1 + 20/s}{1 + s/500} \frac{1 + s/16.89}{1 + s/47.36} \frac{1 + s/54.71}{1 + s/167.5} \frac{1 + s/211.2}{1 + s/592.2} . \qquad (13)$$

Fig. 7 compares the magnitude of the four closed-loop sensitivity functions obtained with the integer order PID and with the fractional order one. They are very close even if lightly larger magnitudes of *SG* and *CS* can be detected for the fractional order controller. A lower rejection of input disturbance and a higher sensitivity of the control input is the price to pay for a wider phase lead. Fig. 8 confirms that even if the step responses to $y_{ref}(t)$ are close, the rejection of the input disturbance is slower for this fractional order controller than for this integer one.

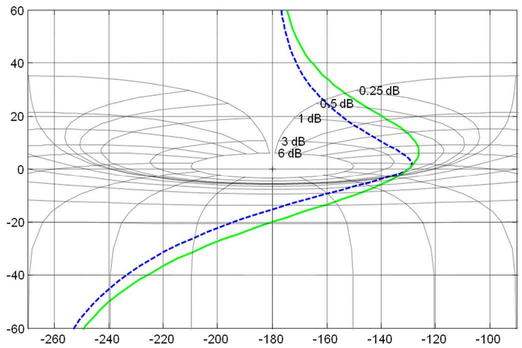

Fig. 5. Comparison of the Nichols plot of the open-loop frequency responses obtained with the integer order (dashed line) and fractional order (solid line) controllers.

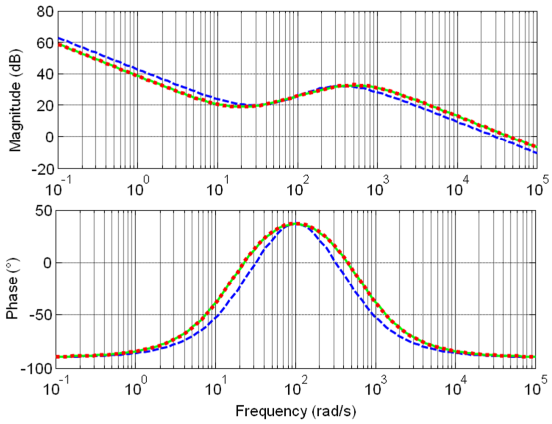

Fig. 6. Comparison of the Bode diagram of the integer order *controller (dashed line) and of the fractional order controller* before (solid line) and after (dotted line) its approximation.

4. FIRST GENERATION CRONE CONTROL-SYSTEM DESIGN

4.1 Introduction to the CRONE Control-System Design (CSD) methodology

The CRONE (a french acronym which means fractional order robust control) CSD methodology is a frequency-domain approach developed since the eighties (Oustaloup 1983, 1991a, 1995; Lanusse, 1994). It is based on the common unity-feedback configuration presented by Fig. 1. Three CRONE CSD methods have been developed, successively extending the application field. In these three methods the controller or open-loop transfer function is defined using fractional order integro-differentiation. In the frequency domain, they enable to design, simply and methodologically, LTI robust control-systems. Using frequency uncertainty domains, as in the Quantitative Feedback Theory (QFT) approach where they are called templates, the uncertainties (or perturbation) are taken into account in a fully-structured form without overestimation, thus leading to control-systems that are as less conservative and thus as high performance as possible. This is the major advantage over an H∞ approach.

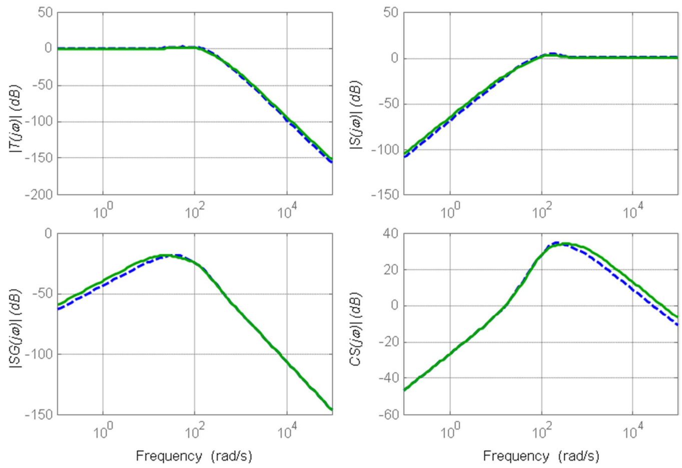

Fig. 7. Closed loop sensitivity functions for the integer order (dashed line) and fractional order (solid line) controllers.

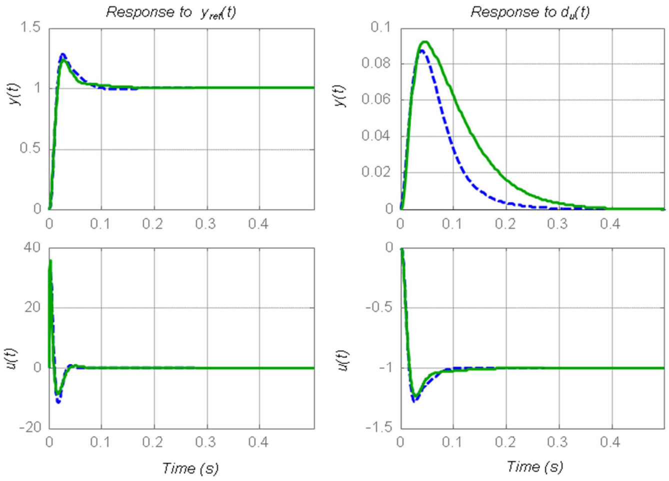

Fig. 8. Closed-loop step response for the integer order (dashed line) and fractional order (solid line) controllers.

The fractional order, which is either real or complex depending on the generation of the control design, permits parameterization of the open-loop transfer function with a small number of high-level parameters. The optimization of the control is thus reduced to only the search for the optimal values of these parameters. Unlike QFT, CRONE CSD avoids iterative nonlinear optimization of each low-level parameter of the controller's rational transfer function.

4.2 Definition of the first generation CRONE methodology

The variations of the phase margin M_Φ come both from the parametric variations of the plant (which leads to an uncertain frequency response) and from the controller phase variations over the frequency range where the frequency ω_{cg} varies. The first generation CRONE control proposes to use a controller without phase variation around frequency ω_{cg}. This strategy has to be used when frequency ω_{cg} is within a frequency range where the plant phase is constant. In this range where the plant frequency response is asymptotic (this frequency band is called a plant asymptotic-behavior band) the plant perturbation is only gain like.

The CRONE controller is defined within a frequency range $[\omega_A, \omega_B]$ around the desired frequency ω_{cg} from the fractional transfer function of an order n integro-differentiator:

$$C_F(s) = C_0 s^n \quad \text{with } n \text{ and } C_0 \in \mathrm{R}. \tag{14}$$

The constant phase $n\pi/2$ characterizes this controller around frequency ω_{cg}. When the plant gain or plant corner frequencies which are greatly different from frequency ω_{cg} vary, the constant phase controller C_F does not therefore modify the phase margin (Fig. 9). Thus, the frequency range $[\omega_A, \omega_B]$ must equal the range where frequency ω_{cg} varies.
If the plant asymptotic behavior is an order p behavior, the phase margin M_Φ is:

$$M_\Phi = (n + p + 2)\frac{\pi}{2}. \tag{15}$$

In order to make the controller biproper, the fractional order derivative of (14) has to be replaced by a band-limited derivative using corner frequencies ω_l and ω_h:

$$C_F(s) = C_0\left(\frac{\omega_l}{\omega_h}\right)^{\frac{n}{2}}\left(\frac{1+\frac{s}{\omega_l}}{1+\frac{s}{\omega_h}}\right)^n, \tag{16}$$

with $\omega_l < \omega_A$ and $\omega_h > \omega_B$.

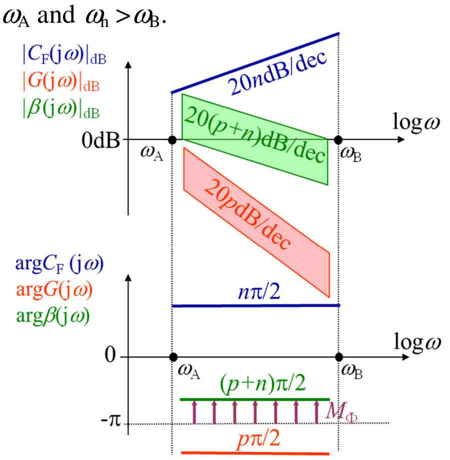

Fig. 9. Robustness of the phase margin M_Φ provided by a first generation CRONE controller

Then, to manage the steady state error and the control sensitivity level, C_F has to be complexified to include an order n_I band-limited integrator and an order n_F low-pass filter:

$$C_F(s) = C_0\left(1+\frac{\omega_I}{s}\right)^{n_I}\left(\frac{1+\frac{s}{\omega_l}}{1+\frac{s}{\omega_h}}\right)^n \frac{1}{\left(1+\frac{s}{\omega_F}\right)^{n_F}}. \tag{17}$$

This first generation CRONE controller is obviously very close to the fractional order PID controller defined by (9) and n_I and n_F are defined by the same way. The main difference is that ω_I, ω_l, ω_h and ω_F have to be such that a constant phase is achieved on a frequency range $[\omega_A, \omega_B]$ which covers the frequency range where ω_{cg} could vary. Then, the fractional order n and gain C_0 have to ensure the required nominal phase margin M_Φ and open-loop gain crossover frequency ω_{cg}

$$n = \frac{-\pi + M_\Phi - \arg G(j\omega_{cg}) + n_F \arctan\frac{\omega_{cg}}{\omega_F} + n_I\left(\frac{\pi}{2} - \arctan\frac{\omega_{cg}}{\omega_I}\right)}{\arctan\frac{\omega_{cg}}{\omega_l} - \arctan\frac{\omega_{cg}}{\omega_h}} \tag{18}$$

$$\text{and } C_0 = \left(1+\frac{\omega_{cg}^2}{\omega_F^2}\right)^{\frac{n_F}{2}} \Big/ \left|G(j\omega_{cg})\right|\left(\frac{\omega_h}{\omega_l}\right)^{\frac{n}{2}}\left(1+\frac{\omega_I^2}{\omega_{cg}^2}\right)^{\frac{n_I}{2}} \tag{19}$$

Finally, the first generation CRONE controller can be implemented using a rational transfer function obtained by using the Oustaloup's approximation:

$$C_R(s) = C_0\left(1+\frac{\omega_I}{s}\right)^{n_I}\prod_1^N \frac{1+\frac{s}{\omega'_i}}{1+\frac{s}{\omega_i}} \frac{1}{\left(1+\frac{s}{\omega_F}\right)^{n_F}}, \tag{20}$$

where

$$\omega'_1 = \sqrt{\eta}\,\omega_l,\ \omega_1 = \alpha\omega'_1,\ \omega'_{i+1} = \alpha\eta\omega'_i \text{ and } \omega_{i+1} = \alpha\eta\omega_i, \tag{21}$$

$$\text{with } \alpha = (\omega_h/\omega_l)^{n/N} \quad \text{and} \quad \eta = (\omega_h/\omega_l)^{(1-n)/N}. \tag{22}$$

4.3 Design example of a first generation CRONE controller

Let a perturbed plant G be defined by:

$$G(s) = \frac{k}{s(1+\tau s)} \quad \text{with } k = 10 \text{ and } 1000/3 \le \tau \le 3*1000. \tag{23}$$

The requirements are an open-loop gain crossover frequency ω_{cg} about 5 rad/s, a phase margin M_Φ of 50°, an integrator and a low-pass effect in the controller.
Around 5 rad/s, the plant phase is constant whereas its magnitude is uncertain. As it leads to the mean magnitude value, the time constant $\tau = \tau_{nom}$ = 1000s is chosen to defined the nominal plant G_{nom}. At ω = 5rad/s, the uncertainty of the plant frequency response is 19.1dB for the magnitude and 0.03° for the phase. Taken into account the required phase margin (and thus the desired rate of decrease of the open-loop magnitude), the open-loop gain crossover frequency could varies from about 2.3 to 11 rad/s (explained later). Within this frequency range, the nominal plant phase varies of 0.06° at least. Thus the 1st generation CRONE CSD method could be used to obtain a robust controller.
From (15) it comes that:

$$n + p = \frac{50}{90} - 2 = -1.44. \tag{24}$$

From the rate of decrease of the open-loop magnitude gain and from the plant magnitude uncertainty, in order to ensure the robustness of the phase margin, the frequency range $[\omega_A, \omega_B]$ needs to cover 0.66 decade:

$$\log_{10}\frac{\omega_B}{\omega_A} = \left|\frac{19.1}{20(n+p)}\right| = 0.66 \text{ then } \frac{\omega_B}{\omega_A} = 10^{0.66} = 4.62. \tag{25}$$

As the nominal plant leads to the mean magnitude of the open-loop frequency response around 5rad/s: $\omega_A = \omega_{cg\,nom}/\sqrt{4.62} = 2.33$rad/s, $\omega_B = \omega_{cg\,nom}\sqrt{4.62} = 10.7$rad/s. In order to achieve a constant phase, the corner frequencies ω_l and ω_h are set to $\omega_l = \omega_A/10 = 0.233$ rad/s and $\omega_h = 10\omega_B = 107$ rad/s. As n_I and n_F are set to 1, in order not to reduce the constant phase, the corner frequencies ω_I and ω_F are set to $\omega_I = \omega_{cg\,nom}/40 = 0.125$ rad/s and $\omega_F = 40\omega_{cg\,nom} = 200$rad/s. As $|G(j5)| = 4.10^{-4}$ and $\arg G(j5) = -180°$, using (18) and (19), the nominal phase margin and open-loop gain crossover frequency are ensured with $n = 0.624$ and $C_0 = 368$. Thus the fractional order CRONE controller is:

$$C(s) = 368\left(1+\frac{0.125}{s}\right)\left(\frac{1+\frac{s}{0.233}}{1+\frac{s}{107}}\right)^{0.624}\frac{1}{1+\frac{s}{200}}. \qquad (26)$$

Even if τ is perturbed, the open-loop frequency response shows that the phase and gain margins could be no less than 48° and 30dB. As predicted, the gain crossover frequency varies from 2.26 to 11.1 rad/s. Furthermore the Nichols chart magnitude contours indicate that the resonance peak of T (the complementary sensitivity function) does not vary a lot (Fig. 10.a). Its greatest value never exceeds 2.87dB. The open-loop Nyquist diagram (Fig. 10.b) shows that the CRONE controller maintains constant the phase margin.

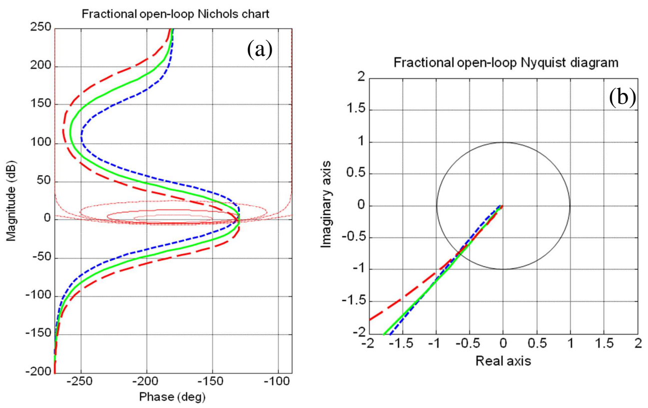

Fig. 10. Open-loop frequency responses for the 3 values of τ: $\tau = \tau_{nom}/3$ (---), $\tau = \tau_{nom}$ (—) and $\tau = \tau_{nom}*3$(– –).

The phase margin and resonance peak of T could be absolutely constant if the band-limitation achieved by ω_l and ω_h would have been lightened. Nevertheless, it could have led to greatest values of the magnitude of the CS and SG sensitivity functions. Our choice permits a well management of the performance/robustness trade-off.

As the frequency range [ω_l, ω_h] covers 3 decades, $N = 4$ is enough to well approximate the fractional order part that appears in (17). The recursive ratios are defined by $\alpha = 2.61$ and $\eta = 1.78$.

Finally, the first generation CRONE controller to be implemented is:

$$C_R(s) = 368\left(1+\frac{0.125}{s}\right)\frac{1+\frac{s}{0.310}}{1+\frac{s}{0.808}}\frac{1+\frac{s}{1.44}}{1+\frac{s}{3.75}}\frac{1+\frac{s}{6.67}}{1+\frac{s}{17.4}}\frac{1+\frac{s}{30.9}}{1+\frac{s}{80.6}}\frac{1}{1+\frac{s}{200}} \qquad (27)$$

To reduce the numerical problems, such a controller can be implemented using a parallel form obtained from a partial-fraction expansion. Fig 11 presents the step responses of the closed loop system.

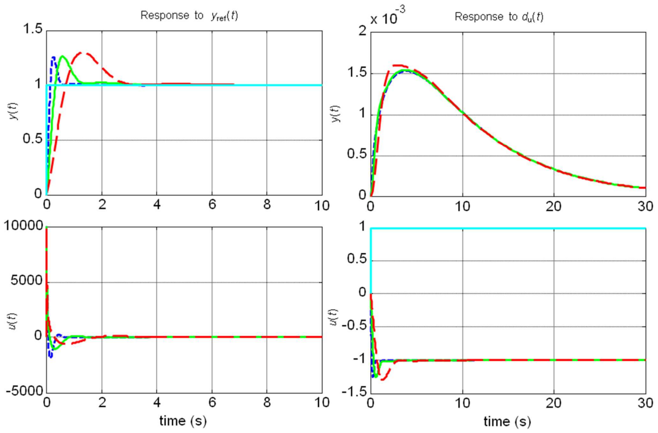

Fig. 11. Closed-loop step responses with the CRONE controller for the 3 values of τ: $\tau = \tau_{nom}/3$ (---), $\tau = \tau_{nom}$ (—) and $\tau = \tau_{nom}*3$(– –)

It shows that:

- for long time, the tracking of y_{ref} and the rejection of d_u are efficient;
- at the opposite of a PID controller (defined by (8) and which ensures the same nominal ω_{cg}), the CRONE controller provides a percentage overshoot for the response of y to y_{ref} that remains almost constant and a greatest settling time (2.5s) 4 times lower;
- the response of u to y_{ref} quickly reaches 10000, 2 times the level obtained with the PID controller;
- the response of y to d_u remains well damped when τ is not nominal even if it requires 30s to reject the disturbance (8s for the PID).

To sum up, this CRONE controller is robust even if it requires a greater control level and more time to reject an input disturbance. It illustrates perfectly the robustness/performance trade-off.

5. CONCLUSION

Using the frequency-domain approach, this paper focuses on the design of PID, fractional order PID and then on first generation CRONE controller. Fractional order controllers provide more tuning parameters that are used for the robustness purpose by the CRONE methodology. First generation CRONE controllers can be used for plants with magnitude uncertainty and constant phase. When the plant phase varies with respect to the frequency, the second CRONE generation design can be used. It proposes a required nominal open-loop transfer function defined from that of a fractional order integrator. Complex fractional orders have been used to develop the CRONE approach (Oustaloup 1991b, 1995, Lanusse 1994) and other approaches (Banos 2011). The third CRONE generation design has been developed for plants with both gain and phase uncertainties. Today, the CRONE CSD methodology permits to tackle non-minimum phase, time delay, unstable or undamped plants (Lanusse 1994, 2008, Oustaloup 1995) and linear time varying plants (Sabatier 2002). It has been extended for the control of MIMO plants (Lanusse 2000, Nelson-Gruel 2009). The CRONE toolbox for Matlab-Simulink that is being developed (Oustaloup 2000b, Lanusse 2013) is freely available for the international scientific community for research and pedagogical purposes and can be downloaded at *http://cronetoolbox.ims-bordeaux.fr*.

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