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Continuous-time linear MPC algorithms based on relaxed logarithmic barrier functions ⋆

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Abstract: In this paper, we investigate the use of relaxed logarithmic barrier functions in the context of linear model predictive control. In particular, barrier function based continuous-time algorithms are considered, in which the control input is obtained as the sampled output of a continuous-time dynamical system. We present suitable barrier function relaxations as well as results on closed-loop stability and the satisfaction of state and input constraints. The results also apply to conventional barrier function based model predictive control schemes.

1. INTRODUCTION

In model predictive control (MPC), the control action is usually computed by the repeated on-line solution of a finite horizon open-loop optimal control problem. The system behavior is predicted based on a model of the underlying plant dynamics and constraints on both the input and the system states can be considered as additional conditions in the optimization problem. Due to extensive research in the last decades, a solid theoretical foundation for MPC of constrained linear and nonlinear systems exists, providing well-understood concepts for ensuring stability properties of the closed loop, see Mayne et al. [2000]. In addition, various results on efficient algorithmic MPC implementations are available, which allow to compute the optimal control input – or at least a feasible approximation – very rapidly, e.g. Bemporad et al. [2002], Diehl et al. [2005], Zeilinger and Jones [2011].

This paper is concerned with a recently proposed class of barrier function based linear MPC algorithms, which compute a stabilizing control input based on a continuous-time dynamical system and without the need of an explicit on-line optimization [Feller and Ebenbauer, 2013]. The main idea of these algorithms is to see the MPC open-loop optimal control problem as a parameter-dependent or time-varying optimization problem and to exploit the fact that the evolution of the initial prediction state is governed by the underlying continuous-time plant dynamics. The algorithmic implementation relies on a stabilizing barrier function based, and hence smoothed, reformulation of the original open-loop optimal control problem, whose solution is then tracked asymptotically by a Newton-based continuous-time optimization algorithm. As shown in Feller and Ebenbauer [2013], this allows to formulate a continuous-time MPC algorithm which, under suitable assumptions, ensures asymptotic stability of the closed-loop system as well as strict satisfaction of all input and state constraints. Other interesting approaches towards continuous-time algorithms for real-time MPC applications, which partly also rely on barrier function based formulations, are discussed in Ohtsuka [2004] and DeHaan and Guay [2007]. Results on stabilizing barrier function based MPC schemes have been presented in Wills and Heath [2004] and Feller and Ebenbauer [2013]. In contrast to other procedures, which often consider the limiting case of the barrier function weighting parameter going to zero, these two approaches allow to guarantee asymptotic stability of the closed-loop system for any arbitrary positive weighting of the barrier functions. However, the main problem of barrier function based MPC schemes, and in particular of the outlined continuous-time MPC algorithms, is given by the fact that the underlying barrier functions are only defined in the interior of the corresponding constraint sets. This means that infeasibilities are not tolerated at all, which might be a problem in the presence of disturbances or uncertainties and particularly with regard to the intermediate feasibility of continuous-time trajectories that occur in the discussed continuous-time MPC algorithms, see Feller and Ebenbauer [2013].

Motivated by the above problem setup, we consider in this paper linear MPC formulations that are based on so-called *relaxed* barrier functions, i.e., penalty function-like extensions of the original barrier functions which are also defined outside of the respective feasible sets. We present different relaxing functions and show that, under suitable assumptions, it is still possible to guarantee asymptotic stability of the closed-loop system and, in some cases, even strict satisfaction of all input and state constraints. The results apply both to the continuous-time MPC algorithms discussed in this paper and to conventional barrier function based MPC schemes which compute the optimal control input based on an iterative optimization at each sampling instant. While the usage of relaxed barrier functions has already been studied in the context of continuous-time trajectory optimization, see Hauser and Saccon [2006], there exist to the authors knowledge no results on closed-loop stability and constraint satisfaction when considering relaxed barrier functions in the context of MPC.

Throughout the paper we will make use of the following notation: $\|x\|_{M_1} := \sqrt{x^{\mathrm{T}} M_1 x}$ for any symmetric positive semi-definite matrix M_1; M_2^i denotes the i-th row of a given Matrix M_2; for an arbitrary set S, S° will denote the open interior and ∂S the limiting boundary.

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2. BARRIER FUNCTION BASED CONTINUOUS-TIME ALGORITHMS FOR LINEAR MPC

In this section, we want to summarize briefly the main results of some previous work on barrier function based linear MPC and related continuous-time algorithms. The following summary contains mainly the underlying key ideas as all the detailed results and proofs can be found in Wills and Heath [2004] and Feller and Ebenbauer [2013].

2.1 Linear model predictive control

We consider discrete-time linear systems of the form

$$x(k+1) = A_D x(k) + B_D u(k), \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ refer to the vectors of system states and inputs, respectively. The real matrices $A_D \in \mathbb{R}^{n\times n}$, $B_D \in \mathbb{R}^{n\times m}$ are obtained by discretizing the continuous-time plant dynamics $\dot{x}(t) = A_C x(t) + B_C u(t)$ with a sampling time $T_s > 0$, where we assume (A_D, B_D) to be stabilizable. In general, the linear MPC open-loop optimal control problem for a finite prediction horizon N is given by

$$\begin{aligned} J^*(x) = \min_{\boldsymbol{u}} &\sum_{k=0}^{N-1} \ell(x_k, u_k) + F(x_N) \\ \text{s.t.}\ \ & x_{k+1} = A_D x_k + B_D u_k,\ x_0 = x(t_k) = x \\ & x_k \in \mathcal{X},\ k = 1, \ldots, N-1,\ x_N \in \mathcal{X}_f \subset \mathcal{X}, \\ & u_k \in \mathcal{U},\ k = 0, \ldots, N-1, \end{aligned} \tag{2}$$

where the stage cost $\ell(x,u)$ and the terminal cost $F(x)$ are defined as $\ell(x,u) = \|x\|_Q^2 + \|u\|_R^2$ and $F(x) = \|x\|_P^2$ for appropriately chosen weight matrices $Q = Q^{\mathrm{T}} \succeq 0$, $R = R^{\mathrm{T}} \succ 0$, $P = P^{\mathrm{T}} \succ 0$. Moreover, $\mathcal{X}_f$ refers to a closed and convex terminal constraint set and $\boldsymbol{u} := \{u_0, u_1, \ldots, u_{N-1}\}$ denotes the sequence of control inputs. The constraint sets $\mathcal{X}$ and $\mathcal{U}$ are assumed to be polytopic sets which contain the origin in their interior. By stacking the input sequence in the extended input vector $U := [u_0^{\mathrm{T}}, \cdots, u_{N-1}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nm}$ and eliminating the predicted system states x_k via $x_k(U,x) = A_D^k x + \sum_{j=0}^{k-1} A_D^j B_D u_{k-j-1},\ k = 1, \ldots, N$, problem (2) can be rewritten as a strongly convex quadratic program (QP) which is parametrized by the current system state x:

$$J^*(x) = \min_U \frac{1}{2} U^{\mathrm{T}} H U + x^{\mathrm{T}} F U + x^{\mathrm{T}} Y x \tag{3a}$$

$$\text{s.t.}\ \ GU \le w + Ex\ , \tag{3b}$$

where $0 \prec H = H^{\mathrm{T}} \in \mathbb{R}^{n_U \times n_U}$, $F \in \mathbb{R}^{n \times n_U}$, $Y \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{q \times n_U}$, $w \in \mathbb{R}^q$, and $E \in \mathbb{R}^{q\times n}$ with $n_U = Nm$.

Definition 1. Let the feasible sets $\mathcal{U}_N(x)$ and $\mathcal{X}_N$ be defined as $\mathcal{U}_N(x) := \{U \in \mathbb{R}^{n_U} : u_k \in \mathcal{U}, k = 0, \ldots, N-1,\ x_k(U,x) \in \mathcal{X}, k = 1, \ldots, N-1\ , x_N(U,x) \in \mathcal{X}_f\}$ and $\mathcal{X}_N := \{x \in \mathcal{X} : \mathcal{U}_N(x) \neq \emptyset\}$, respectively.

Definition 2. For $U \in \mathbb{R}^{n_U}$, let the projection $\mathscr{P}(U) : \mathbb{R}^{n_U} \to \mathbb{R}^m$ be defined as $\mathscr{P}(U) = [I_m\ 0\ \cdots\ 0]\, U$.

An idealized MPC scheme may then be implemented in the following way: i) measure the current system state $x = x(t_k)$, ii) compute the optimal input vector $U = U^*(x)$ by solving (3), iii) apply the first input $u(x) = u_0^*(x) = \mathscr{P}(U)$ to the plant, iv) shift the horizon and repeat the procedure at the next sampling instant. More details on idealized MPC schemes as well as results on stability and recursive feasibility can be found in Mayne et al. [2000].

2.2 Barrier function based MPC with guaranteed stability

As in the context of interior point methods, suitable barrier functions allow to eliminate the inequality constraints in (2) by including them into the cost function. As discussed later, the use of barrier functions results in an unconstrained and "smoothed" representation of problem (2) which can then be tracked by asymptotic tracking algorithms, see Section 2.3.

Let us consider the following barrier function based open-loop optimal control problem

$$\begin{aligned} \tilde{J}^*(x) = \min_{\boldsymbol{u}} &\left\{ \tilde{\ell}_0(x_0, u_0) + \sum_{k=1}^{N-1} \tilde{\ell}(x_k, u_k) + \tilde{F}(x_N) \right\} \\ \text{s.t.}\ \ & x_{k+1} = A_D x_k + B_D u_k,\ x_0 = x(t_k) = x, \end{aligned} \tag{4}$$

with $\tilde{\ell}_0(x,u) := \ell(x,u) + \varepsilon B_u(u)$, $\tilde{\ell}(x,u) := \ell(x,u) + \varepsilon B_u(u) + \varepsilon B_x(x)$, $\tilde{F}(x) := F(x) + \varepsilon B_f(x)$, where $B_u(\cdot)$, $B_x(\cdot)$ and $B_f(\cdot)$ are suitable convex barrier functions with domains $\mathcal{U}^\circ$, $\mathcal{X}^\circ$, and $\mathcal{X}_f^\circ$ with $B_u(u) \to \infty$ for $u \to \partial\mathcal{U}$, $B_x(x) \to \infty$ for $x \to \partial\mathcal{X}$, and $B_f(x) \to \infty$ for $x \to \partial\mathcal{X}_f$. The positive scalar $\varepsilon > 0$ is the barrier function weighting parameter, which determines the influence of the barrier function values on the cost objective. Two different approaches towards the stabilizing design of barrier function based MPC schemes have been presented in Wills and Heath [2004] and Feller and Ebenbauer [2013], which are both based on the concept of so-called gradient recentered barrier functions and on the standard approach of using the value function $\tilde{J}^*(x)$ as a Lyapunov function for the closed-loop system, see Mayne et al. [2000].

Definition 3. (Gradient recentered barrier function). Let $B : D \to \mathbb{R}$ be a convex barrier function on an open convex set D with $0 \in D$. Then, the function $\bar{B} : D \to \mathbb{R}$ defined as $\bar{B}(z) = B(z) - B(0) - [\nabla B(0)]^{\mathrm{T}} z$ is called the gradient recentered barrier function for B around the origin [Wills and Heath, 2004].

While the use of gradient recentered barrier functions ensures that $\tilde{J}^*(x)$ is a positive definite function with a unique minimum at the origin, suitable further conditions can be imposed on the problem parameters $B_x(\cdot)$, $B_u(\cdot)$, $\mathcal{X}_f$, $B_f(\cdot)$, and P in order to guarantee the contraction property

$$\tilde{J}^*(x^+) - \tilde{J}^*(x) \le -\tilde{\ell}_0(x, \tilde{u}_0^*(x))\ \forall x \in \mathcal{X}_N^\circ\ , \tag{5}$$

where $x^+ = A_D x + B_D \tilde{u}_0^*(x)$ denotes the next closed-loop system state. One possible approach to ensure satisfaction of (5) in the context of barrier function based MPC is summarized in Definition 4, which represents a generalization of the main ideas used in Wills and Heath [2004] and Feller and Ebenbauer [2013]. In the following, $A_K := A_D + B_D K$ describes the closed-loop dynamics for a given local controller $u = Kx$ and $B_K(x) := B_x(x) + B_u(Kx)$ refers to the corresponding combined barrier function of input and state constraints for the set $\mathcal{X}_K := \{x \in \mathcal{X} : Kx \in \mathcal{U}\}$.

Definition 4. For a given stabilizing local control gain $K \in \mathbb{R}^{n\times m}$, the parameters of the open-loop optimal control problem (4) satisfy the following conditions:

i) the barrier functions $B_u(\cdot)$ and $B_x(\cdot)$ are gradient recentered barrier functions according to Definition 3;
ii) $\exists M \in \mathbb{R}^{n\times n}, M \succeq 0$, s. t. $B_K(x) \le x^{\mathrm{T}} M x\ \forall x \in \mathcal{N}$, where $\mathcal{N} \subset \mathcal{X}_{\mathcal{K}}$ is a convex and compact set with $0 \in \mathcal{N}^\circ$ and $Kx \in \mathcal{U}\ \forall x \in \mathcal{N}$;

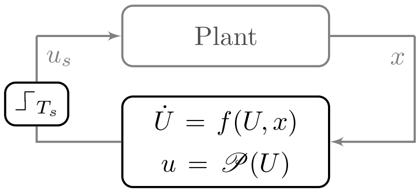

Fig. 1. The proposed continuous-time MPC algorithm.

iii) the terminal cost matrix $P \succ 0$ solves the Lyapunov equation $P = A_K^{\mathrm{T}} P A_K + Q + K^{\mathrm{T}} R K + \varepsilon M$;
iv) the terminal set is $\mathcal{X}_f = \{x \in \mathbb{R}^n : x^{\mathrm{T}} P_f x \leq 1\}$ with $P_f \succ 0$ chosen s. t. $\mathcal{X}_f \subseteq \mathcal{N}$ and $A_K^{\mathrm{T}} P_f A_K - P_f \preceq 0$;
v) the barrier function for the terminal set constraint is given by $B_f(x) = -\ln(1 - x^{\mathrm{T}} P_f x)$.

Based on these conditions, it can be shown that (5) holds true, i.e., that the value function $\tilde{J}^*(x)$ is a Lyapunov function of the closed-loop system. This proves that the origin of system (1) under the MPC feedback $u(x) = \tilde{u}_0^*(x)$ based on repeated solution of the barrier function based open-loop optimal control problem (4) is asymptotically stable for any $x_0 \in \mathcal{X}_N^\circ$. For more details and specific choices for the discussed parameters, please see Wills and Heath [2004] and Feller and Ebenbauer [2013].

2.3 A stabilizing continuous-time MPC algorithm

As in the standard case, we can rewrite the barrier function based formulation (4) in a more compact form as

$$\tilde{J}^*(x) = \min_U \; \tilde{J}(U, x) + x^{\mathrm{T}} Y x \quad \text{with} \tag{6}$$
$$\tilde{J}(U, x) = \frac{1}{2} U^{\mathrm{T}} H U + x^{\mathrm{T}} F U + \varepsilon B_c^{qp}(U, x) + \varepsilon B_f^{qp}(U, x),$$

where the barrier functions $B_c^{qp}(\cdot\,,\cdot)$ and $B_f^{qp}(\cdot\,,\cdot)$ are defined according to Feller and Ebenbauer [2013]. One way of implementing a stabilizing barrier function based MPC scheme is, of course, to solve problem (6) at each sampling instant by making use of suitable iterative optimization algorithms. In contrast to this, a system theoretic approach which computes the optimal input vector $\tilde{U}^*(x)$ based on a continuous-time dynamical system of the form

$$\dot{U} = f(U, x) \; , \quad U(t_0) = U_0 \; , \tag{7}$$

where $x = x(t)$ denotes the system state related to the continuous-time plant dynamics, has been proposed recently in Feller and Ebenbauer [2013]. The main idea is to formulate the above continuous-time dynamical system (7) in such a way that it asymptotically tracks the optimal solution of the barrier function based, and therefore smoothed, approximation of the original MPC open-loop optimal control problem. In short, the resulting solution $U(t)$ ensures satisfaction of all input and state constraints and converges asymptotically to the optimizer $\tilde{U}^*(x(t))$ as $t \to \infty$ for all initial conditions $U_0 \in \mathcal{U}_N^\circ(x_0)$ with $x_0 \in \mathcal{X}_N^\circ$. As shown in Feller and Ebenbauer [2013], this allows to formulate a continuous-time MPC algorithm which guarantees, under suitable assumptions, asymptotic convergence of the system state to the origin for all feasible initializations U_0 as well as asymptotic stability of the closed-loop system in the case of the optimal initialization $U_0 = \tilde{U}^*(x_0)$. A schematic illustration of the outlined algorithm is given in Fig. 1.

3. STABILIZING LINEAR MPC ALGORITHMS BASED ON RELAXED BARRIER FUNCTIONS

The main problem of the barrier function based MPC formulation, and in particular of the outlined continuous-time MPC algorithm, is given by the fact that the underlying barrier functions are only defined in the interior of the corresponding constraint sets and, hence, do not tolerate infeasibilities at all. When considering conventional barrier function based MPC schemes, this could be a problem in the presence of model uncertainties or disturbances which may drive the system state out of the feasible set $\mathcal{X}_N^\circ$. In the context of continuous-time MPC algorithms, however, the barrier function based formulation requires that all continuous-time state trajectories are not only feasible at the discrete sampling points but stay in the feasible set $\mathcal{X}_N^\circ$ for all times, which is a rather technical and strong assumption, see Feller and Ebenbauer [2013]. One possible approach to handle these problems is to relax the involved barrier functions by means of suitable relaxing penalty functions which are then also defined outside of the respective constraint sets. In the following, we discuss the use of relaxed logarithmic barrier functions in the context of linear MPC algorithms and present our main results on stability and feasibility properties of the resulting closed-loop system.

3.1 Relaxed logarithmic barrier functions

First, we introduce the concept of relaxed logarithmic barrier functions and discuss suitable realizations based on different relaxing functions.

Definition 5. (Relaxed logarithmic barrier function). For a constraint of the form $z \geq 0$, $z \in \mathbb{R}$, and a given scalar parameter $\delta > 0$, the function

$$\hat{B}(z) = \begin{cases} -\ln(z) & z > \delta \\ \beta(z; \delta) & z \leq \delta \end{cases} \tag{8}$$

defines a relaxed version of the logarithmic barrier function $B(z) = -\ln(z)$, where $\beta : \mathbb{R} \to \mathbb{R}$ denotes a suitable relaxing function satisfying $\mathbf{dom}\, \beta = (-\infty, \infty)$, $\beta(\delta; \delta) = -\ln(\delta)$, $\lim_{z \to -\infty} \beta(z; \delta) = \infty$, and $\beta(z; \delta)$ strictly monotone for $z \leq \delta$.

In general, it is advisable to choose $\beta(\cdot\,; \delta)$ as a strictly convex C^2 function that satisfies $\beta'(z; \delta)_{|z=\delta} = -\frac{1}{\delta}$ and $\beta''(z; \delta)_{|z=\delta} = \frac{1}{\delta^2}$. In this case, $\hat{B}(z)$ also is a strictly convex function that is twice continuously differentiable and defined on $z \in (-\infty, \infty)$.

i) Polynomial relaxation Hauser and Saccon [2006]. The first ideas on relaxed (or approximate) logarithmic barrier functions as well as a suitable choice for the function $\beta(\cdot\,; \delta)$ seem to have been presented by Hauser and Saccon [2006] in the context of continuous-time trajectory optimization. The authors make use of the polynomial relaxing function

$$\beta_k(z; \delta) = \frac{k-1}{k} \left[\left(\frac{z - k\delta}{(k-1)\delta} \right)^k - 1 \right] - \ln(\delta) \; , \tag{9}$$

where $k > 1$ is an even integer. It is easy to verify that the function $\beta_k(\cdot\,; \delta)$ has all the desired properties mentioned above. As reported in Hauser and Saccon [2006], already $k = 2$ seems to work well in practice.

ii) Exponential relaxation. In order to avoid large constraint violations, it may be beneficial if the relaxing function increases very rapidly outside the border of the feasible set. As an alternative to the polynomial relaxation above, we propose the following relaxing function

$$\beta_e(z;\delta) = \exp\left(1 - \frac{z}{\delta}\right) - 1 - \ln(\delta) \ , \qquad (10)$$

which is an upper bound for, and in some sense the limit case of, the function $\beta_k(\cdot\,;\delta)$. Clearly, the function $\beta_e(\cdot\,;\delta)$ satisfies all the conditions above and allows, therefore, a strictly convex and smooth relaxation of the original barrier function $B(z) = -\ln(z)$. Moreover, it can be easily shown that $\hat{B}(z) \leq B(z)\ \forall z > 0$ if either $\beta_k(\cdot\,;\delta)$ or $\beta_e(\cdot\,;\delta)$ are used as relaxing functions.

3.2 Problem setup: stabilizing model predictive control formulations based on relaxed logarithmic barrier functions

In the following, we consider the relaxed barrier function based MPC formulation

$$\hat{J}^*(x;\delta) = \min_{\boldsymbol{u}} \left\{ \hat{\ell}_0(x_0,u_0) + \sum_{k=1}^{N-1} \hat{\ell}(x_k,u_k) + \hat{F}(x_N) \right\}$$
$$\text{s.t.}\ \ x_{k+1} = A_D x_k + B_D u_k,\ x_0 = x(t_k) = x, \qquad (11)$$

with $\hat{\ell}_0(x,u) := \ell(x,u) + \varepsilon\hat{B}_u(u)$, $\hat{\ell}(x,u) := \ell(x,u) + \varepsilon\hat{B}_u(u) + \varepsilon\hat{B}_x(x)$, and $\hat{F}(x) := F(x) + \varepsilon\hat{B}_f(x)$, where $\hat{B}_u(\cdot)$, $\hat{B}_x(\cdot)$, and $\hat{B}_f(\cdot)$ are relaxed gradient recentered logarithmic barrier functions as defined below. We do not indicate the explicit dependence of the relaxed barrier functions on the relaxation parameter δ for the sake of notational simplicity. Moreover, we will use $\hat{B}(\cdot)$ to denote the relaxed version of a barrier function based expression $\tilde{B}(\cdot)$.

Assumption 1. The state and input constraints are given in form of compact polytopic sets that contain the origin in their interior, i.e., $\mathcal{U} = \{u \in \mathbb{R}^m : C_u u \leq d_u\}$ and $\mathcal{X} = \{x \in \mathbb{R}^n : C_x x \leq d_x\}$ with $C_x \in \mathbb{R}^{q_x \times n}$, $C_u \in \mathbb{R}^{q_u \times m}$ and $d_u > 0$, $d_x > 0$. Moreover, we assume that the feasible sets $\mathcal{X}_N$ and $\mathcal{U}_N(x)$ have nonempty interior, i.e., $\mathcal{X}_N^\circ \neq \emptyset$ and $\mathcal{U}_N^\circ(x) \neq \emptyset\ \forall x \in \mathcal{X}_N^\circ$.

Assumption 2. The barrier functions $\hat{B}_u(\cdot)$ and $\hat{B}_x(\cdot)$ for the polytopic input and state constraints are relaxed gradient recentered logarithmic barrier functions of the form $\hat{B}_u(u) = \sum_{i=1}^{q_u} \hat{B}_{u,i}(u)$ and $\hat{B}_x(x) = \sum_{i=1}^{q_x} \hat{B}_{x,i}(x)$ with, for example,

$$\hat{B}_{x,i}(x) = \begin{cases} -\ln(z_i(x)) + \ln(d_x^i) - \dfrac{C_x^i x}{d_x^i} & z_i(x) > \delta \\ \beta(z_i(x);\delta) + \ln(d_x^i) - \dfrac{C_x^i x}{d_x^i} & z_i(x) \leq \delta \end{cases}, \quad (12)$$

where the relaxing function $\beta(\cdot\,;\delta)$ is chosen according to Section 3.1 and $z_i(x) := -C_x^i x + d_x^i$. The barrier functions $\hat{B}_{u,i}(u)$ for the input constraints are defined analogously.

Assumption 3. The barrier function $\hat{B}_f(\cdot)$ for the terminal set $\mathcal{X}_f = \{x \in \mathbb{R}^n | x^{\mathrm{T}} P_f x \leq 1\}$ is a relaxed gradient recentered logarithmic barrier function of the form

$$\hat{B}_f(x) = \begin{cases} -\ln(1 - x^{\mathrm{T}} P_f x) & 1 - x^{\mathrm{T}} P_f x > \delta \\ \beta(1 - x^{\mathrm{T}} P_f x;\delta) & 1 - x^{\mathrm{T}} P_f x \leq \delta \end{cases}. \qquad (13)$$

Assumption 4. All relevant parameters of the open-loop optimal control problem (11) are chosen in such a way that the conditions in Definition 4 are satisfied, i.e., such that a barrier function based MPC scheme relying on the non-relaxed formulation (4) results in an asymptotically stable closed loop satisfying the contraction property (5).

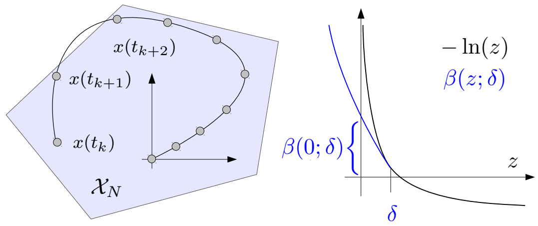

Fig. 2. Intermediate infeasibility of continuous-time trajectories (left) and an example of a relaxed logarithmic barrier function (right).

Assumption 5. In the following, the relaxation parameter $\delta \in \mathbb{R}$ satisfies $0 < \delta < \min\{d_x^1, \ldots, d_x^{q_x}, d_u^1, \ldots, d_u^{q_u}, 1\}$.

The above condition on the relaxation parameter δ ensures that the relaxed barrier functions $\hat{B}_u(u)$, $\hat{B}_x(x)$, and $\hat{B}_f(x)$ are indeed gradient recentered barrier functions according to Definition 3 with the unique minima $\hat{B}_u(0) = 0$, $\hat{B}_x(0) = 0$, $\hat{B}_f(0) = 0$. In particular, the condition $\delta < 1$ needs to be included due to the definition of the function $\hat{B}_f$ in Assumption 3. It is of course also possible to consider individual relaxation parameters δ_i for the different constraints. Here, however, we restrict ourselves to one overall δ for the sake of simplicity.
In the following, we will present two different approaches that allow to guarantee for the relaxed barrier function based formulation (11) asymptotic stability of the closed-loop system and, in one case, strict satisfaction of state and input constraints. Note that $\hat{B}(\cdot) \to \tilde{B}(\cdot)$ in the corresponding domain of definition for $\delta \to 0$, which shows that the properties of the previously discussed barrier function based MPC scheme without relaxations are recovered as the relaxation parameter $\delta > 0$ approaches zero.

3.3 Stabilization with guaranteed satisfaction of state and input constraints

In this section, we show that for any set of initial conditions $\mathcal{X}_0 \subset \mathcal{X}_N$ there exists a choice for $\delta > 0$ such that the closed-loop state and input trajectories resulting from formulation (11) stay strictly feasible and that the control law $u(x) = \hat{u}_0^*(x)$ asymptotically stabilizes the origin of system (1). Note that the following results can be extended easily to the case of barrier function based MPC of nonlinear systems.

Definition 6. Let the two positive scalars $d_{\min}$ and $\bar{\beta}(\delta)$ be defined as $d_{\min} = \min\{d_x^1, \ldots, d_x^{q_x}, d_u^1, \ldots, d_u^{q_u}\}$ and $\bar{\beta}(\delta) = \min\{\beta(0;\delta) + \ln(d_{\min}) - 1, \beta(0;\delta)\}$, respectively.

Definition 7. For δ satisfying Assumption 5, let the set $\mathcal{X}_\beta(\delta)$ be defined as $\mathcal{X}_\beta(\delta) := \{x \in \mathcal{X}_N^\circ | \hat{J}^*(x;\delta) \leq \varepsilon\bar{\beta}(\delta)\}$.

Note that $\bar{\beta}(\delta)$ is a lower bound for the values of the barrier functions $\hat{B}_{u,i}(u)$, $\hat{B}_{x,i}(x)$, and $\hat{B}_f(x)$ evaluated at the borders of the corresponding feasible sets, which leads to the following result.

Theorem 8. Let the Assumptions 1-4 hold true and let the set $\mathcal{X}_\beta(\delta)$ be defined according to Definition 7. Then, the feedback $u(x(k)) = \hat{u}_0^*(x(k))$ related to the relaxed barrier function based MPC formulation (11) asymptotically stabilizes the origin of system (1) under strict satisfaction of all input and state constraints for any $x_0 \in \mathcal{X}_\beta(\delta)$.

Proof. The proof consists of three parts. First, we show that the underlying input, state, and terminal set constraints are not violated for any $x_0 \in \mathcal{X}_\beta(\delta)$; then, we show that the value function $\hat{J}^*(x;\delta)$ will decrease; finally, we use this result to conclude that the closed-loop is asymptotically stable and that the resulting input and state trajectories are strictly feasible at all later time steps.
i) Let $x^*_{k|x} := A^k_D x + \sum_{j=0}^{k-1} A^j_D B_D \hat{u}^*_{k-j-1}(x)$, for $k = 1,\dots,N$. Since the cost function in (11) is a sum of positive definite terms, it holds that $\varepsilon\hat{B}_{x,i}(x^*_{k|x}) \leq \hat{J}^*(x;\delta)$, $i = 1,\dots,q_x$, $\varepsilon\hat{B}_{u,i}(\hat{u}^*_k(x)) \leq \hat{J}^*(x;\delta)$, $i = 1,\dots,q_u$, as well as $\varepsilon\hat{B}_f(x^*_{N|x}) \leq \hat{J}^*(x;\delta)$. For $x_0 \in \mathcal{X}_\beta(\delta) \subset \mathcal{X}^\circ_N$, we have $\hat{J}^*(x_0;\delta) \leq \varepsilon\bar{\beta}(\delta)$ and, hence, $\hat{B}_{x,i}(x^*_{k|x}) \leq \bar{\beta}(\delta)$, $\varepsilon\hat{B}_{u,i}(\hat{u}^*_k(x)) \leq \bar{\beta}(\delta)$, $i = 1,\dots,q_u$, as well as $\varepsilon\hat{B}_f(x^*_{N|x}) \leq \bar{\beta}(\delta)$, which shows due to the definition of $\bar{\beta}(\delta)$ in Definition 6 that the infeasible parts of the barrier functions, i.e. $z < 0$ in Fig. 2, are not active at x_0. Due to this, the predicted input and state trajectories are strictly feasible and the control law $u(x_0) = \hat{u}^*_0(x_0)$ results in a strictly feasible successor state $x^+_0 = A_D x_0 + B_D \hat{u}^*_0(x_0) \in \mathcal{X}^\circ_N$.
ii) Consider now the successor state $x^+_0 \in \mathcal{X}^\circ_N$. Since the local controller K and the terminal set $\mathcal{X}_f$ are chosen according to Definition 4, it remains to show that

$$\hat{F}(A_K x) - \hat{F}(x) \leq -\hat{\ell}(x, Kx)\ \forall x \in \mathcal{X}^\circ_f \tag{14}$$

in order guarantee the contraction property $\hat{J}^*(x^+_0;\delta) - \hat{J}^*(x_0;\delta) \leq -\hat{\ell}_0(x_0, \hat{u}^*_0(x_0))\ \forall x_0 \in \mathcal{X}_\beta(\delta) \subset \mathcal{X}^\circ_N$, see Mayne et al. [2000]. For $A_K := A_D + B_D K$ and $\hat{B}_K(x) := \hat{B}_x(x) + \hat{B}_u(Kx)$, condition (14) is equivalent to

$$\begin{aligned}\|A_K x\|^2_P - \|x\|^2_P + \|x\|^2_Q + \|Kx\|^2_R + \varepsilon\hat{B}_K(x)\\ + \varepsilon\hat{B}_f(A_K x) - \varepsilon\hat{B}_f(x) \leq 0\ \forall x \in \mathcal{X}^\circ_f\ .\end{aligned} \tag{15}$$

By the design of the relaxed barrier functions and the choice of $\mathcal{X}_f$ and P, see i) and ii) in Definition 4, we know that $\hat{B}_K(x) \leq B_K(x) \leq x^{\mathrm{T}} M x\ \forall x \in \mathcal{X}^\circ_f \subset \mathcal{X}_K$ and, in particular, $\|A_K x\|^2_P - \|x\|^2_P + \|x\|^2_Q + \|Kx\|^2_R + \varepsilon\hat{B}_K(x) \leq 0\ \forall x \in \mathcal{X}^\circ_f$. Furthermore, since the matrix P_f is chosen according to condition iv) of Definition 4, we have $1 - x^{\mathrm{T}} A^{\mathrm{T}}_K P_f A_K x \geq 1 - x^{\mathrm{T}} P_f x\ \forall x \in \mathcal{X}^\circ_f$, which shows, due to the monotonicity of the relaxed barrier function $\hat{B}_f(\cdot)$, that $\hat{B}_f(A_K x) - \hat{B}_f(x) \leq 0\ \forall x \in \mathcal{X}^\circ_f$. With this, it follows directly that condition (15), respectively (14), holds true. Based on standard arguments, see Mayne et al. [2000], it can be shown that this implies $\hat{J}^*(x^+_0;\delta) - \hat{J}^*(x_0;\delta) \leq -\hat{\ell}_0(x, \hat{u}^*_0(x_0))\ \forall x_0 \in \mathcal{X}_\beta(\delta)$.
iii) The fact that the value function decreases shows that $\hat{J}^*(x^+_0;\delta) \leq \varepsilon\bar{\beta}(\delta)$ and hence $x^+_0 \in \mathcal{X}_\beta(\delta) \subset \mathcal{X}^\circ_N$ for any $x_0 \in \mathcal{X}_\beta(\delta)$. By repeating this argument, we see that the resulting closed-loop system state satisfies $x(k) \in \mathcal{X}_\beta(\delta)\ \forall k \geq 0$ for any $x(0) = x_0 \in \mathcal{X}_\beta(\delta)$, which shows that all future state and input trajectories stay strictly feasible. Moreover, by the construction of the relaxed gradient recentered barrier functions, one can show that $\hat{J}^*(x;\delta)$ is a well-defined, positive definite and radially unbounded function. Hence, it can be used as a Lyapunov function that allows to prove asymptotic stability of the origin with a guaranteed region of attraction of at least $\mathcal{X}_\beta(\delta)$. □

Remark 9. Note that it is also possible to consider $\check{\mathcal{X}}_\beta(\delta) := \{x \in \mathcal{X}^\circ_N | \hat{J}^*(x;\delta) \leq \varepsilon\check{\beta}(\delta)\}$ with $\check{\beta}(\delta) = \min\{\beta(\delta;\delta) + \ln(d_{\min}) - 1, \beta(\delta;\delta)\}$. It can be shown that in this case the *relaxing parts* of the underlying barrier functions are never activated for any $x_0 \in \check{\mathcal{X}}_\beta(\delta)$. Thus, such an approach could be seen as an exact relaxation of the original barrier function based MPC formulation (4). Of course, $\check{\mathcal{X}}_\beta(\delta) \subset \mathcal{X}_\beta(\delta)$ for any $\delta > 0$.

The following results state some useful properties of the set $\mathcal{X}_\beta(\delta)$ and show that the feasible set $\mathcal{X}^\circ_N$ of the non-relaxed barrier function based MPC scheme can be approximated arbitrarily close by decreasing δ.

Lemma 10. Let the set $\mathcal{X}_\beta(\delta)$ be defined according to Definition 7. Then, for any δ satisfying Assumption 5, the set $\mathcal{X}_\beta(\delta)$ is a nonempty compact and convex set. Furthermore, it holds that $\mathcal{X}_\beta(\delta) \to \mathcal{X}^\circ_N$ as $\delta \to 0$.

Proof. Since $J^*(x;\delta)$ is a strongly convex and positive definite function and $\bar{\beta}(\delta) > 0$ by definition, the first part follows trivially. As $\bar{\beta}(\delta) \to \infty$ for $\delta \to 0$, the second part is a direct result of the definition of the set $\mathcal{X}_\beta(\delta)$. □

Corollary 11. For any set $\mathcal{X}_0 \subset \mathcal{X}^\circ_N$ there exists a $\bar{\delta} > 0$ such that the closed-loop system is asymptotically stable and satisfies all input and state constraints for any $x_0 \in \mathcal{X}_0$ if $\delta \leq \bar{\delta}$. In particular, the feasible set $\mathcal{X}^\circ_N$ of the non-relaxed case is recovered for $\delta \to 0$.

Remark 12. Note that when considering conventional barrier function based MPC schemes with iterative optimization at every sampling instant, the underlying barrier functions are now defined globally, which makes the MPC scheme more robust against disturbances or uncertainties. Moreover, in the context of the continuous-time MPC algorithm outlined in Section 2.3, the approach allows for continuous-time trajectories which are strictly feasible at the discrete sampling points but may activate the infeasible parts of the relaxed barrier functions in between, see Section 3.5 as well as Fig. 2 on the previous page.

Remark 13. Depending on the set of initial conditions $\mathcal{X}_0$, the parameter $\bar{\delta}$ that satisfies the condition $\mathcal{X}_0 \subseteq \mathcal{X}_\beta(\delta)$ $\forall\delta \leq \bar{\delta}$ may be very small. However, the discussed choice for $\bar{\delta}$ merely represents a sufficient condition for stability and may be conservative. In general, a possibly much larger value for δ could be chosen in practice without loosing the desired closed-loop behavior.

3.4 Global stabilization with upper bounds for the maximal violation of state and input constraints

In the previous Section, we showed how asymptotic stability of the closed-loop system as well as strict satisfaction of all input and state constraints can be guaranteed for any initial condition $x_0 \in \mathcal{X}^\circ_N$ by choosing the relaxation parameter $\delta > 0$ small enough. In the following, we present a different approach which allows to prove global asymptotic stabilization of the closed loop system while giving an upper bound for the maximally occurring violation of input and state constraints for a given δ. In contrast to the previous discussion, we assume controllability of the discrete-time system (1) and do not relax the barrier function of the terminal set constraint.

Assumption 6. The pair (A_D, B_D) is controllable and the prediction horizon satisfies the condition $N \geq n$, i.e., the matrix $\Omega = \left[A^{N-1}_D B_D \ \cdots \ A_D B_D \ B_D\right] \in \mathbb{R}^{n \times n_U}$ has rank n. Moreover, the Assumptions 1-2 and 4 hold true and the barrier function $\hat{B}_f(\cdot)$ for the terminal set constraint is not relaxed, i.e., $\hat{B}_f(x) = B_f(x)$.

Theorem 14. Let Assumption 6 hold true. Then, independently of the relaxation parameter δ, the feedback $u(x(k)) = \hat{u}_0^*(x(k))$ based on the relaxed barrier function based MPC formulation (11) asymptotically stabilizes the origin of system (1) for any initial condition $x_0 \in \mathbb{R}^n$. Moreover, an upper bound for the maximal violation of input and state constraints is given by $\hat{z}(x_0;\delta) = |\min\{\hat{z}_1, \dots, \hat{z}_{q_x+q_u}, 0\}|$, where $\hat{z}_i < 0$ is a solution to

$$\beta(\hat{z}_i;\delta) + \ln(d^i) + \frac{\hat{z}_i}{d^i} - 1 = \frac{1}{\varepsilon}\hat{J}^*(x_0;\delta) \tag{16}$$

with $d =: [d_x^{\mathrm{T}}, d_u^{\mathrm{T}}]^{\mathrm{T}}$ and $i = 1, \dots, q_x + q_u$.

Proof. Due to the above controllability assumption, there exists for any $x_0 \in \mathbb{R}^n$ an input vector $U(x_0)$ such that $x_N(U(x_0), x_0) \in \mathcal{X}_f^\circ$. Hence, $\hat{U}^*(x_0)$ and $\hat{J}^*(x_0;\delta)$ are defined for any $x_0 \in \mathbb{R}^n$. Since the terminal state satisfies $x^*_{N|x_0} \in \mathcal{X}_f^\circ$, the local controller $u = Kx$ can be used to construct a feasible control sequence for the successor state $x_0^+ = A_D x_0 + B_D \hat{u}_0^*(x_0)$. Since all parameters are chosen according to Definition 4, basically the same arguments as in part ii) of the proof of Theorem 8 can be used in order to show that $\hat{J}^*(x_0^+;\delta) - \hat{J}^*(x_0;\delta) \le -\hat{\ell}_0(x, \hat{u}_0^*(x_0))\ \forall x_0 \in \mathbb{R}^n$. Hence, the value function $\hat{J}^*(x;\delta)$ can in this case be employed as a Lyapunov function for proving global asymptotic stability of the origin, see part iii) of the proof of Theorem 8. Furthermore, due to the decrease of the value function and following the same arguments as in part i) of the proof of Theorem 8, the values of all barrier functions are bounded by $\frac{1}{\varepsilon}\hat{J}^*(x_0;\delta)$, which shows that the element-wise worst case violations for the state and input constraints $z_i(x) = -C_x^i x + d_x^i \ge 0$, $i = 1, \dots, q_x$, and $z_j(u) = -C_u^j u + d_u^j \ge 0$, $j = 1, \dots, q_u$, are given by the solutions to (16). If no solution $\hat{z}_i < 0$ exists, no constraint violations will occur in the closed-loop system. □

Thus, by relaxing the state and input constraint barrier functions, it is possible to get rid of the usually inherent infeasibility problems and to design MPC schemes that guarantee, under rather mild assumptions, global asymptotic stability of the closed-loop system. Furthermore, along the lines of Lemma 10 and Corollary 11, we can state the following result, which allows to satisfy a given constraint violation tolerance for a set of initial conditions.

Corollary 15. For any given tolerance $\hat{z}_{\max} > 0$ and any set of initial conditions $\mathcal{X}_0 \subset \mathcal{X}_N^\circ$, there exists a relaxation parameter $\tilde{\delta} > 0$ such that the maximal constraint violation satisfies $\hat{z}(x_0;\delta) \le \hat{z}_{\max}$ for any $x_0 \in \mathcal{X}_0$ and any $\delta \le \tilde{\delta}$.

Remark 16. In general, the relaxation parameter $\tilde{\delta}$ needed for satisfying a, possibly small, given constraint violation tolerance for a set of initial conditions may be much larger than the one needed for a strictly feasible stabilization according to Section 3.3, i.e., $\tilde{\delta} \gg \bar{\delta}$. In comparison, this makes the global stabilization approach more suitable for a practical implementation.

If the relaxing function $\beta(\cdot\,;\delta)$ is chosen as $\beta_e(\cdot\,;\delta)$ or $\beta_k(\cdot\,;\delta)$ with $k > 2$, a nonlinear equation solver can be used to find solutions of Eq. (16). In the case of $\beta(\cdot\,;\delta) = \beta_k(\cdot\,;\delta)$ with $k = 2$, however, Eq. (16) reduces to a quadratic equation in $\hat{z}_i$ and a closed form expression for the maximal constraint violation $\hat{z}(x_0;\delta)$ can be given. This result is summarized in the following Corollary, whose proof is omitted here for the sake of brevity.

Corollary 17. For a given initial condition $x_0 \in \mathbb{R}^n$ and $\beta(\cdot\,;\delta) = \beta_k(\cdot\,;\delta)$ with $k = 2$, the maximal possible violation of input and state constraints in the closed-loop system can be computed explicitly and is given by $\hat{z}(x_0;\delta) = |\min\left\{\delta\left(\gamma_1 - \sqrt{\gamma_1^2 - \gamma_2}\right), 0\right\}|$ with $\gamma_1 := 2 - \frac{\delta}{d_{\min}}$ and $\gamma_2 := 1 + 2\ln\left(\frac{d_{\min}}{\delta}\right) - \frac{2}{\varepsilon}\hat{J}^*(x_0;\delta)$.

Remark 18. Note that the above constraint violation satisfies $\hat{z}(x_0;\delta) < 0$ exactly in the case $\gamma_2 < 0$, which leads to $\delta > \bar{\delta} := d_{\min}\exp(\frac{1}{2} - \frac{1}{\varepsilon}\hat{J}^*(x_0;\delta))$. In fact, it can be shown easily that $x_0 \in \mathcal{X}_\beta(\delta)$ if $\delta \le \bar{\delta}$, see Theorem 8. This illustrates again that the parameter δ that is sufficient for guaranteeing strict constraint satisfaction based on the arguments in Section 3.3 may be very small.

3.5 Continuous-time linear MPC algorithms based on relaxed logarithmic barrier functions

In this section, we combine the above results on stabilizing relaxed barrier function based MPC with the continuous-time asymptotic tracking algorithm proposed in Feller and Ebenbauer [2013]. As a result, we present a continuous-time MPC algorithm that obtains a stabilizing control input as the output of a dynamical system and does, in contrast to previous work, not rely on any form of intermediate feasibility assumption.

By eliminating the equality constraints related to the system dynamics, problem (11) can again be rewritten in the following, more compact, form

$$\hat{J}^*(x;\delta) = \min_U\ \hat{J}(U,x) + x^{\mathrm{T}}Yx \quad \text{with} \tag{17}$$

$$\hat{J}(U,x) = \frac{1}{2}U^{\mathrm{T}}HU + x^{\mathrm{T}}FU + \varepsilon\hat{B}_c^{qp}(U,x) + \varepsilon\hat{B}_f^{qp}(U,x),$$

where $\hat{B}_c^{qp}(\cdot\,,\cdot)$ and $\hat{B}_f^{qp}(\cdot\,,\cdot)$ denote the relaxed versions of the barrier functions $B_c^{qp}(\cdot\,,\cdot)$ and $B_f^{qp}(\cdot\,,\cdot)$ in problem (6). By the design of the relaxed barrier functions, the function $\hat{J}(U,x)$ is strongly convex in U with $\nabla_U^2\hat{J}(U,x) \succeq \lambda_{\min}(H)I_{n_U}$ for any given x. Note that, with respect to the optimal control input, the problems (17) and (11) are completely equivalent. Therefore, if problem (11) is formulated in such a way that the conditions discussed in either Section 3.3 or Section 3.4 are satisfied, the optimizer $\hat{U}^*(x)$ related to (17) yields a control law $u(x) = \hat{u}_0^*(x) = \mathscr{P}(\hat{U}^*(x))$ which is asymptotically stabilizing in the corresponding region of attraction. By considering the constantly changing system state x as a parameter, problem (17) can be seen as minimization of a time-varying strongly convex C^2 function. Suppose $x(t_0) = x_0 \in \mathbb{R}^n$ and $x(t)$ evolves in time according to $\dot{x} = A_C x + B_C u$, where $u = u(t)$ may be any measurable function of time. Consider now the continuous-time dynamical system

$$\dot{U} = -\left(\frac{\partial^2\hat{J}(U,x)}{\partial U^2}\right)^{-1}\left(\frac{\partial\hat{J}(U,x)}{\partial U}^{\mathrm{T}} + \frac{\partial^2\hat{J}(U,x)}{\partial x\partial U}\,\dot{x}\right), \tag{18}$$

with an initial condition $U(t_0) = U_0$ satisfying the condition $\hat{J}(U_0,x_0) < \infty$, i.e., the vector U_0 is feasible with respect to the considered barrier function specifications. Clearly, the right-hand side of Eq. (18) is well-defined for any $(U(t),x(t))$ in the domain of $\hat{J}(U,x)$. Based on the above results and the ideas presented in Feller and Ebenbauer [2013], we now propose the following continuous-time relaxed barrier MPC algorithm, see also Fig. 1.

Algorithm 1 *(Continuous-time rbMPC algorithm)*

Initialization for $t = t_0$:
(i) choose a suitable initialization $U(t_0) = U_0$
(ii) set $u_s = \mathscr{P}(U_0)$
Integration for $t > t_0$:
∟ apply $u = u_s$ to the plant;
∟ measure the current state $x(t)$ and obtain $U(t)$ by integrating the dynamical system (18) with $\dot{x} = A_C x + B_C u_s$;
∟ *Sampling* whenever $t = t_k = kT_s$, $k \in \mathbb{N}_{>0}$:
  ∟ set $u_s = \mathscr{P}(U(t_k))$.

We first combine Algorithm 1 with the relaxed barrier function approach presented in Section 3.3, which allows us to state the following results.

Theorem 19. Let the cost $\hat{J}(U, x)$ and the barrier functions in (17) satisfy the conditions in Theorem 8 and let the set $\mathcal{X}_\beta(\delta)$ be defined according to Definition 7. Then, for the optimal initial condition $U_0 = \hat{U}^*(x_0)$, the feedback $u(k) = u_s(t_k)$ obtained from Algorithm 1 asymptotically stabilizes the origin of the discrete-time system (1) for any $x_0 \in \mathcal{X}_\beta(\delta)$ while guaranteeing strict satisfaction of all input and state constraints.

Proof. Following Feller and Ebenbauer [2013], consider the positive semi-definite Lyapunov function candidate $W(U, x) = \frac{1}{2}\frac{\partial \hat{J}(U,x)}{\partial U}^{\mathrm{T}}\frac{\partial \hat{J}(U,x)}{\partial U}$, which satisfies $W(U, x) \geq 0$, $W(U, x) = 0 \Leftrightarrow \nabla_U \hat{J}(U, x) = 0$. If all elements of $\hat{J}(U, x)$ in (17) are chosen according to Section 3.3, the function $W(U, x) : \mathbb{R}^{n_U} \times \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and well-defined due to the relaxation of all involved logarithmic barrier functions. Using (18), the directional time derivative of W along U and x is given by $\dot{W} = -2W(U, x) \leq 0$. In the case $U_0 = \hat{U}^*(x_0)$, it holds that $W(U_0, x_0) = 0$ and $\dot{W}(U_0, x_0) = 0$ and, hence, $W(U(t), x(t)) \equiv 0$, which is equivalent to $\nabla_U \hat{J}(U(t), x(t)) \equiv 0 \ \forall t \geq t_0$. Since $\hat{J}(U, x)$ in (17) is strongly convex in U, this obviously implies $U(t) \equiv \hat{U}^*(x(t)) \ \forall t \geq t_0$. This shows that in the case $U_0 = \hat{U}^*(x_0)$, the solution $U(t)$ of (18) tracks the optimal solution of problem (17), and therefore of problem (11), exactly. Thus, Algorithm 1 is equivalent to the conventional MPC implementation relying on the relaxed barrier function based formulation (11), and both strict satisfaction of constraints and asymptotic stability of the closed-loop system follow directly from Theorem 8. □

Remark 20. In the case $U_0 \neq \hat{U}^*(x_0)$, conditions for the guaranteed existence of $U(t)$ and $x(t)$ for all $t \geq t_0$ as well as for the convergence of the system state to the origin are still an open problem. In the practical implementation, however, Algorithm 1 seems to achieve very good results also for suboptimal or even infeasible initializations U_0.

Remark 21. Note that Theorem 19 can be seen as a direct extension of the results presented in Feller and Ebenbauer [2013], that allows to eliminate the there needed intermediate feasibility assumption.

In the following, we combine Algorithm 1 with the globally stabilizing relaxed barrier function based MPC scheme discussed in Section 3.4.

Definition 22. Let $\mathcal{U}_{N,f}(x) := \{U \in \mathbb{R}^{n_U} : x_N(U, x) \in \mathcal{X}_f\}$ denote the feasible set for U when considering only the non-relaxed terminal set constraint.

Theorem 23. Let the cost function $\hat{J}(U, x)$ and the barrier functions in (17) satisfy the condition in Theorem 14. Then, for the optimal initial condition $U_0 = \hat{U}^*(x_0)$, Algorithm 1 asymptotically stabilizes the origin of the discrete-time closed-loop system (1) for any $x_0 \in \mathbb{R}^n$. Furthermore, the maximal violation of state and input constraints is upper bounded by the value $\hat{z}(x_0; \delta)$ specified in Theorem 14. In the case $U_0 \neq \hat{U}^*(x_0)$, $U_0 \in \mathcal{U}^\circ_{N,f}(x_0)$, Algorithm 1 still achieves asymptotic convergence of the system state to the origin for any $x_0 \in \mathbb{R}^n$.

Proof. Consider again the Lyapunov function candidate $W(U, x)$ from above. The proof consists of three parts. First, we deal with the case of an optimal initialization. Then, we show that for a feasible initialization both $x(t)$ and the solution $U(t)$ of (18) exist for all times and that $U(t)$ asymptotically tracks the optimal solution $\hat{U}^*(x(t))$. Based on this, we then prove asymptotic convergence of the system state to the origin for $U_0 \in \mathcal{U}^\circ_{N,f}(x_0)$.

i) In case of an optimal initialization $U_0 = \hat{U}^*(x_0)$, we have $W(U(x(t)), x(t)) \equiv 0$, which shows that $U(t) \equiv \hat{U}^*(x(t))$ for $t \geq t_0$, see the proof of Theorem 19 above. Hence, the claimed stability properties follow from Theorem 14 and the fact that the optimal solution is tracked exactly.

ii) Consider now the case $U_0 \neq \hat{U}^*(x_0)$, $U_0 \in \mathcal{U}^\circ_{N,f}(x_0)$. For any given $x \in \mathbb{R}^n$, the function $W(U, x) : \mathcal{U}^\circ_{N,f}(x) \to \mathbb{R}$ is well-defined and $W(U, x) \to \infty$ whenever $U \to \partial\mathcal{U}_{N,f}(x)$ due to the non-relaxed barrier function $\hat{B}_f(\cdot)$. Moreover, because of the above controllability assumption, the set $\mathcal{U}^\circ_{N,f}(x)$ is nonempty for any $x \in \mathbb{R}^n$. However, the fact that $W(U, x)$ is only positive semidefinite requires to show that both $U(t)$ and $x(t)$ exist for all times, i.e., that $U(t)$ stays bounded for all $t \geq t_0$. To see this, we use that any strongly convex C^2 function $f : \mathcal{D} \subseteq \mathbb{R}^p \to \mathbb{R}$ with $\nabla^2 f(y) \succeq mI_p$ satisfies $\|\nabla f(y)\| \geq m\|y - y^*\| \ \forall y \in \mathcal{D}$, where y^* is the unique minimizer of f [Polyak, 1987, p. 11]. Hence, $W(U, x) = \|\nabla_U \hat{J}(U, x)\|^2 \geq (\lambda_{\min}(H))^2\|U - \hat{U}^*(x)\|^2 \ \forall U \in \mathcal{U}^\circ_{N,f}(x)$, $x \in \mathbb{R}^n$, which shows that $W(U, x)$ is radially unbounded in the deviation $R(t) := U(t) - \hat{U}^*(x(t))$, where $\hat{U}^*(x(t))$ is always bounded and unique. Since $W(U_0, x_0) := W_0 < \infty$ for $U_0 \in \mathcal{U}^\circ_{N,f}(x_0)$ and $\dot{W}(U, x) = -2W(U, x) \leq 0$, the function $W(U(t), x(t))$ is monotonically decreasing. Thus, we have $W(U(t), x(t)) \leq W_0 < \infty$ as well as $\|R(t)\| \leq 1/\lambda_{\min}(H)\sqrt{W_0}$ for $t \geq t_0$. This shows that $U(t)$ stays both feasible and bounded, i.e., $U(t) \in \mathcal{U}^\circ_{N,f}(x(t))$ and $\|U(t)\| < \infty$ for $t \geq t_0$. Then, being the solution of a linear system with a bounded input, $x(t)$ is also defined for all $t \geq t_0$. Hence, the right-hand side of (18) is defined for all times and its solution $U(t)$ satisfies $U(t) = \hat{U}^*(x(t)) + R(t)$ with $R(t)$ bounded as well as $U(t) \to \hat{U}^*(x(t))$ for $t \to \infty$ for any $U_0 \in \mathcal{U}^\circ_{N,f}(x_0)$.

iii) Based on the above arguments, we know that Algorithm 1 generates an input sequence of the form $\bar{u}_0(k) = \hat{u}_0^*(x(k)) + r(k)$ with $r(k)$ bounded and $r(k) \to 0$ for $k \to \infty$. Using the fact that $u(k) = \hat{u}_0^*(x(k))$ is globally asymptotically stabilizing and following the proof of Theorem 3 in Feller and Ebenbauer [2013], this allows to show that $\forall\ \bar{c} > 0\ \exists\ \bar{k}\ :\ \|x(k)\| \leq \bar{c}\ \forall\ k \geq \bar{k}$. Hence, we have that $\lim_{k\to\infty}\|x(k)\| = 0$, which proves asymptotic convergence of the system state to the origin for $U_0 \neq \hat{U}^*(x_0)$, $U_0 \in \mathcal{U}^\circ_{N,f}(x_0)$ for any $x_0 \in \mathbb{R}^n$. □

4. NUMERICAL EXAMPLES

In the following, we briefly illustrate the closed-loop behavior of the proposed MPC algorithm by means of an academic numerical example. We consider a double integrator system with the discrete-time system model

$$x(k+1) = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} T_s^2 \\ T_s \end{bmatrix} u(k) \ , \tag{19}$$

where the discretization time is chosen to be $T_s = 0.1$ s. The linear MPC open-loop optimal control problem is formulated over a prediction horizon of $N = 10$, using a quadratic cost function with the weight matrices $Q = \mathrm{diag}(1,\ 0.1)$, $R = 1$ and the input and state constraints $|u| \le 1$, $|x_1| \le 2.8$, and $|x_2| \le 0.8$. The parameters of the barrier function based MPC formulation are chosen according to the design procedure presented in Feller and Ebenbauer [2013] with $\varepsilon = 10^{-2}$ and $\bar{\gamma} = 30$. We implemented Algorithm 1 in MATLAB and tested the closed-loop behavior for different initial conditions and varying values of the relaxation parameter δ for both of the two approaches presented in Section 3.3 and Section 3.4, respectively. Exemplary results are illustrated in Fig. 3 and Fig. 4 together with some comments. In general, we can say that the novel implementation based on relaxed logarithmic barrier functions produces very reliable and numerically robust results.

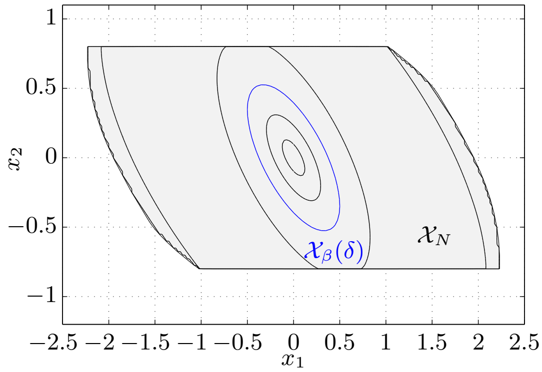

Fig. 3. Regions in the state space for which a given maximal constraint violation of $\hat{z}_{\max} = 10^{-3}$ is guaranteed for $\beta(\cdot;\delta) = \beta_k(\cdot;\delta)$ with $k = 2$ and $\delta \in \{5\times10^{-4}, 10^{-4}, 3\times10^{-5}, 10^{-5}, 10^{-6}\}$. The feasible set $\mathcal{X}_N$ is approximately recovered for small δ. Also depicted (blue) is the set $\mathcal{X}_\beta(\delta)$ in which no constraint violation will occur for $\delta = 10^{-100}$.

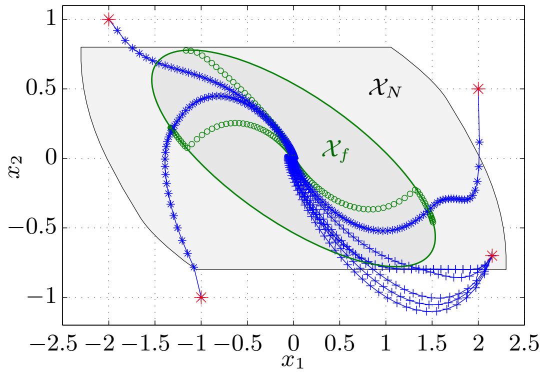

Fig. 4. Closed loop behavior for $x_0 = [2.15, -0.7] \in \mathcal{X}_N$ and $\delta \in \{0.5, 0.1, 5\times10^{-2}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-6}\}$ [+]: if δ is small enough, the state trajectories stay feasible. Also shown is the behavior for different infeasible initial conditions with $\delta = 10^{-2}$ [$*$] together with the corresponding predicted terminal states $x^*_{N|x(t_k)}$ [∘].

5. CONCLUSION

In this paper, we presented two approaches for the design of stabilizing MPC algorithms that are based on relaxed logarithmic barrier functions. While the first approach ensures asymptotic stability and strict constraint satisfaction for a bounded set of feasible initial conditions, the second approach allows for global stability guarantees in combination with upper bounds for the maximal constraint violation. Based on a Newton-based asymptotic tracking algorithm, we then proposed a continuous-time MPC algorithm which allows to implement linear MPC without the need of an iterative on-line optimization and does not, as in previous results, rely on any assumption on the intermediate feasibility of continuous-time trajectories. It is our hope that the presented results do not only represent a natural and useful extension of the continuous-time MPC algorithm proposed in previous work but that they may help to understand and justify the use of relaxed barrier functions in the context of MPC in general.

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