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The regulator equations for regular linear systems

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Abstract: We investigate the state feedback regulator problem for exponentially stable plants belonging to the class of regular linear systems. The plant is driven by a linear completely unstable exosystem via a disturbance signal. The exosystem also generates the reference signal for the plant output. The state feedback regulator problem is to design a controller that has access to the states of the exosystem and the plant and which, while guaranteeing the stability of the closed-loop system without the exosystem, drives the tracking error to zero. We show that, under suitable assumptions, this problem is solvable if and only if a pair of algebraic equations, called the regulator equations, is solvable. We present an example that illustrates our theory.

Keywords: Exosystem, regular linear system, regulator equations, state feedback regulator problem.

1. INTRODUCTION

We investigate the tracking and disturbance rejection problem, also called the *regulator problem*, for a linear infinite-dimensional plant from the special class of regular linear systems, when the reference and disturbance signals are produced by a linear unstable signal generator called the *exosystem*. *Regular systems* model many physical systems involving waves, beams, plates, shells, elastic media, heat propagation, etc, see Byrnes et al. (2002); Chai and Guo (2009, 2010a,b); Guo and Shao (2005, 2006); Guo and Zhang (2005); Staffans (2004), and they usually have unbounded control and observation operators. However, in the literature on the regulator problem, in order to avoid technical difficulties, it is assumed that these operators are bounded. In this paper we overcome this limitation.

There are two standard versions of the regulator problem: In the first, called the *state feedback regulator problem*, the controller is provided with full information of the state of the plant and the exosystem, while in the second version, called the *error feedback regulator problem*, only the tracking error is available to the controller. In this work we will focus on the state feedback version alone, and under the assumption that the plant is exponentially stable. Indeed, we think that stabilizing the plant and solving the regulator problem are two distinct issues, and it would only obfuscate the theory to present them mixed together. The exponential stability implies that for the state feedback we actually only need the state of the exosystem. We plan to address the error feedback regulator problem in a follow-up paper. We mention that it is easy, in principle, to design an error feedback controller if the plant together with the exosystem are detectable via the tracking error. Indeed, the straightforward approach is to use a full state observer, which of course is infinite-dimensional. This is the approach taken in several references, for example, in Byrnes et al. (2000) or in Immonen and Pohjolainen (2006). For plants that are already stable, the real challenge (that we shall address in our follow-up paper) is to design a finite-dimensional error feedback controller.

Regulator theory for infinite-dimensional linear systems with bounded control and observation operators has been significantly advanced by a group of researchers at Tampere University of Technology who have developed a sophisticated theory of infinite-dimensional exosystems, see for instance Hämäläinen and Pohjolainen (2010); Immonen and Pohjolainen (2005, 2006); Paunonen and Pohjolainen (2010, 2012). The state feedback regulator problem for exponentially stabilizable linear plants driven by infinite-dimensional exosystems generating periodic signals was addressed in Immonen and Pohjolainen (2005).

In this work we restrict the state operator of the linear, unstable and possibly infinite-dimensional exosystem to be bounded. Our reason for imposing this restriction is to avoid the following robustness problem: according to the internal model principle due to Davison (1976) or Francis and Wonham (1975), all the unstable eigenvalues of the exosystem will be poles of the controller. Thus, in the case of an exosystem with an unbounded set of unstable eigenvalues, the closed-loop system cannot be robustly stable with respect to small delays, see (Logemann et al., 1996, Theorem 1.2). Closely related negative results are in Georgiou and Smith (1989), and this issue is explained in more detail in (Weiss and Haefele, 1999, Section 4).

We extend the key results in Byrnes et al. (2000) on the state feedback regulator problem to plants with unbounded control and observation operators. There is con-

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siderable interest in plants with boundary control and/or boundary observation, for which the control and/or observation operators are unbounded, see for instance Staffans (2004), Tucsnak and Weiss (2009). Probably the most general class of distributed parameter systems for which there is a well established and relatively simple representation and feedback theory, are the regular linear systems (see Staffans (2004); Weiss (1994b,a)). In this work we have chosen to formulate the state feedback regulator problem for regular plants. We also assume that the plant is exponentially stable and not just stabilizable, the latter assumption being customary. This is not limiting, since in regulator theory the problems of stabilization and regulation can be decoupled and addressed sequentially. Hence we shall assume that the plant has been stabilized via a suitable feedback and we shall solve the regulator problem for the stable plant.

2. THE PLANT, THE EXOSYSTEM AND THE ERROR

In this section we describe the basic assumptions about the plant to be controlled and the exosystem, and we derive some simple consequences of these assumptions. The *plant* is described by the equations

$$\begin{cases} \dot z(t) = Az(t) + Bu(t) + B^1 d(t), \\ y(t) = C_\Lambda z(t) + Du(t) + D^1 d(t). \end{cases} \tag{2.1}$$

The state of this system is $z(t)$, its input signal is $\left[\begin{smallmatrix} u \\ d \end{smallmatrix}\right]$ and its output signal is y. We regard u as the control input (to be generated by a controller) while d is a disturbance. We have $z(t) \in Z$, where the state space Z is assumed to be a Hilbert space. We have $u(t) \in U$, $d(t) \in U^1$ and $y(t) \in Y$, where U, U^1 and Y are Hilbert spaces. The operator A is the generator of an *exponentially stable* operator semigroup $\mathbb{T}$ on Z. The control operator $B \in \mathcal{L}(U, Z_{-1})$ is admissible for $\mathbb{T}$, while $B^1 \in \mathcal{L}(U^1, Z_{-1})$ (not necessarily admissible). Here, as usual in the theory of well-posed systems, Z_{-1} denotes the completion of Z with respect to the norm $\|z\|_{-1} = \|A^{-1}z\|$. The observation operator $C \in \mathcal{L}(Z, Y)$ is admissible for $\mathbb{T}$, $D \in \mathcal{L}(U, Y)$ and $D^1 \in \mathcal{L}(U^1, Y)$. As is customary in the theory of well-posed linear systems, we have denoted

$$C_\Lambda z = \lim_{\lambda \to +\infty} C\lambda(\lambda I - A)^{-1} z,$$

which is an operator with domain consisting of those $z \in Z$ for which the above limit exists. We assume that the triple (A, B, C) is regular (as defined, for instance, in Weiss and Curtain (1997)) and for some (hence, for every) $s \in \rho(A)$, the product $C_\Lambda(sI - A)^{-1}B^1$ exists (which is weaker than requiring that the triple (A, B^1, C) be regular).

We assume that there exists a linear system with no input, referred to as the *exosystem* (sometimes called the exogenous system), that produces the reference output $r(t)$ and the disturbance signal $d(t)$:

$$\begin{cases} \dot w(t) = Sw(t), \\ r(t) = Q^1 w(t), \quad d(t) = C^1 w(t). \end{cases} \tag{2.2}$$

Here $S \in \mathcal{L}(W)$, where W is a Hilbert space, and its spectrum $\sigma(S)$ is a subset of $\overline{\mathbb{C}_0^+}$, i.e., the exosystem is completely unstable. In the applications that we have in mind, $\sigma(S)$ is on the imaginary axis. We have $Q^1 \in \mathcal{L}(W, Y)$ and $C^1 \in \mathcal{L}(W, U^1)$. We refer to the difference between the measured and reference outputs as the *error*:

$$\begin{aligned} e(t) = y(t) - r(t) &= C_\Lambda z(t) + Du(t) + D^1 d(t) - Q^1 w(t) \\ &= C_\Lambda z(t) + Du(t) + Qw(t), \end{aligned}$$

where $Q \in \mathcal{L}(W, Y)$ is defined by $Q = D^1C^1 - Q^1$.

We will also need to consider the *combined plant* Σ_p representing the plant and the exosystem together, on the combined state space $X = Z \times W$, with the state

$$x = \begin{bmatrix} z \\ w \end{bmatrix} \in X = Z \times W,$$

with input space U and output space Y, described by the equations

$$\dot x(t) = A_p x(t) + B_p u(t), \tag{2.3}$$
$$e(t) = C_{p\Lambda} x(t) + D_p u(t), \tag{2.4}$$

where $A_p = \begin{bmatrix} A & P \\ 0 & S \end{bmatrix}$, $B_p = \begin{bmatrix} B \\ 0 \end{bmatrix}$, $C_p = [C_\Lambda \ Q]$, $D_p = D$, $P = B^1C^1$ and

$$\mathcal{D}(A_p) = \mathcal{D}(C_p) = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in X \;\middle|\; Az + Pw \in Z \right\}. \tag{2.5}$$

Lemma 2.1. The operator A_p with domain as defined in (2.5) generates an operator semigroup $\mathbb{T}^p$ on X.

We omit the simple proof. For the combined plant, we have $X_1 = \mathcal{D}(A_p)$, of course, and $X_{-1} = Z_{-1} \times W$, which is easy to verify. (Here X_{-1} is the analogue of Z_{-1} for the generator A_p.) The domain of C_p is (by definition) $\mathcal{D}(A_p)$ and $C_{p\Lambda}$ in (2.4) is the Λ-extension of C_p. This combined plant is partially stable (since A is stable) but not stabilizable, because there is no way to influence the component w of the state. The problem we want to solve in this paper is to make the output signal e of Σ_p small, meaning that it belongs to a certain weighted L^2 space.

Proposition 2.2. The combined plant Σ_p from (2.3)–(2.5) is regular. In particular, B_p and C_p are admissible for $\mathbb{T}^p$ and the transfer function of Σ_p is

$$\mathbf{G}_p(s) = C_{p\Lambda}(sI - A_p)^{-1}B_p + D_p = C_\Lambda(sI - A)^{-1}B + D.$$

The operator $C_{p\Lambda}$ can be described as follows:

$$\mathcal{D}(C_{p\Lambda}) = \mathcal{D}(C_\Lambda) \times W \quad \text{and} \quad C_{p\Lambda} \begin{bmatrix} z \\ w \end{bmatrix} = C_\Lambda z + Qw.$$

The *Sylvester* equation

$$\Pi S = A\Pi + P + BL, \tag{2.6}$$

which must be solved for Π, when $L \in \mathcal{L}(W, U)$ is given, will play an important role in the sequel. The intuitive meaning of this equation is as follows: Consider the combined plant Σ_p from (2.3)–(2.5) with the linear state feedback $u = Lw$, as shown in Figure 1. Then it can be shown (using the exponential stability of A) that in steady state we have $z(t) = \Pi w(t)$, i.e., $\lim_{t\to\infty} \|z(t) - \Pi w(t)\| = 0$. Next we define a subspace of Z as follows:

$$\underset{\sim}{Z} = \mathcal{D}(A) + (\lambda I - A)^{-1}PW + (\lambda I - A)^{-1}BU, \tag{2.7}$$

It is easy to check that $\underset{\sim}{Z} \subset \mathcal{D}(C_\Lambda)$.

Lemma 2.3. The Sylvester equation (2.6) has a unique solution $\Pi \in \mathcal{L}(W, Z)$, moreover $\operatorname{Ran}\Pi \subset \underset{\sim}{Z}$, so that the product $C_\Lambda\Pi$ exists and is in $\mathcal{L}(W, Y)$.

Proof. Suppose that (2.6) has a solution Π. Then for each $t \geq 0$ and $w \in W$,

$$\mathbb{T}_t \Pi S e^{-St} w - \mathbb{T}_t A \Pi e^{-St} w = \mathbb{T}_t (P + BL) e^{-St} w,$$

which is equivalent to

$$-\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{T}_t \Pi e^{-St} w) = \mathbb{T}_t (P + BL) e^{-St} w.$$

Integrating the above equation in Z_{-1} on the interval $[0, \infty)$, we get

$$\Pi w = \int_0^\infty \mathbb{T}_t (BL + P) e^{-St} w \mathrm{d}t \qquad (2.8)$$

which shows that Π is unique. Via integration by parts,

$$\Pi w = -A^{-1}(BL + P)w + \int_0^\infty \mathbb{T}_t A^{-1}(BL + P) e^{-St} S w \,\mathrm{d}t. \qquad (2.9)$$

Since $A^{-1}(P + BL) \in \mathcal{L}(W, Z)$, we can conclude from (2.9) that $\Pi \in \mathcal{L}(W, Z)$. Multiplying both sides of (2.9) by $A \in \mathcal{L}(Z, Z_{-1})$, we can verify that Π as defined in (2.8) solves (2.6). From (2.6) we obtain that

$$\Pi = A^{-1} \Pi S - A^{-1} P - A^{-1} BL,$$

which implies that $\operatorname{Ran} \Pi \subset \underset{\sim}{Z}$ and therefore $C_\Lambda \Pi$ exists. This operator is the strong limit of the operators $C\lambda(\lambda I - A)^{-1}\Pi \in \mathcal{L}(W, Y)$ as $\lambda \to +\infty$. Applying the uniform boundedness principle we obtain that $C_\Lambda \Pi \in \mathcal{L}(W, Y)$. $\square$

3. THE LINEAR STATE FEEDBACK REGULATOR PROBLEM

In this section we formulate and solve the linear state feedback regulator problem. For any $\alpha \in \mathbb{R}$ define

$$L^2_\alpha([0, \infty); Y) = e_\alpha L^2([0, \infty); Y),$$

where $(e_\alpha v)(t) = e^{\alpha t} v(t)$, with the norm $\|e_\alpha v\|_{L^2_\alpha} = \|v\|_{L^2}$.

Problem 3.1. **The linear state feedback regulator problem:** For the combined plant Σ_p from (2.3)–(2.5), find a feedback control law in the form $u = Lw$, with $L \in \mathcal{L}(W, U)$, such that for the resulting closed-loop system with no input, described by

$$\begin{bmatrix} \dot{z} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A & P + BL \\ 0 & S \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} = A_p^L \begin{bmatrix} z \\ w \end{bmatrix}, \qquad (3.1)$$

$$e = [C_\Lambda \;\; Q + DL] \begin{bmatrix} z \\ w \end{bmatrix}, \qquad (3.2)$$

we have $e \in L^2_\alpha([0, \infty); Y)$ for some $\alpha < 0$ and for all initial conditions $z(0) = z_0 \in Z$ and $w(0) = w_0 \in W$ (i.e., for any initial state in X).

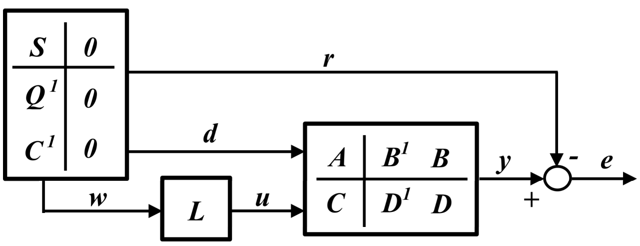

Figure 1. The closed-loop system corresponding to the state feedback regulator problem. This system is not asymptotically stable, but the error is in $L^2_\alpha([0, \infty); Y)$ with $\alpha < 0$, as if it were exponentially stable.

Lemma 3.2. If $R \in \mathcal{L}(W, Y)$ is such that the function $m(t) = R e^{St} w_0$ belongs to $L^2_\alpha([0, \infty); Y)$ for some $\alpha < 0$ and all $w_0 \in W$, then $R = 0$.

Proof. Fix $0 < \beta < |\alpha|$ and $w_0 \in W$ and define the function $v(t) = e^{\beta t} m(t)$. We can factor $v(t) = e^{(\alpha + \beta)t}[e^{-\alpha t} m(t)]$. Since $\alpha + \beta < 0$ and $m \in L^2_\alpha([0, \infty); Y)$, both factors are in L^2, so that $v \in L^1([0, \infty); Y)$ and hence its Laplace transform satisfies

$$\hat{v}(s) = R((s - \beta)I - S)^{-1} w_0, \qquad \hat{v} \in H^\infty(\mathbb{C}_0^+; Y). \quad (3.3)$$

This needs some explanation. The above formula for $\hat{v}$ only holds for $s - \beta$ in some right half-plane contained in $\rho(S)$. Nevertheless, $\hat{v}$ is defined on all of $\mathbb{C}_0^+$ and it is an analytic continuation of the function defined by the above formula.

Let $\omega = \beta/2$. Since $\sigma(S) \subset \mathbb{C}_0^+$, we have the following bound for the semigroup e^{-St}: $\|e^{-St}\| \leq M_\omega e^{\omega t}$ for some $M_\omega \geq 1$ and all $t \geq 0$. This implies that

$$\|(sI + S)^{-1}\| \leq \frac{M_\omega}{\operatorname{Re} s - \omega} \qquad \forall\, s \in \mathbb{C}_\omega^+,$$

or equivalently that

$$\|((s - \beta)I - S)^{-1}\| \leq \frac{M_\omega}{\omega - \operatorname{Re} s} \quad \text{for } \operatorname{Re} s < \omega.$$

This, along with (3.3), shows that $\hat{v}$ has a bounded analytic continuation to the left half-plane where $\operatorname{Re} s < \omega$. Since we already know that $\hat{v}$ is bounded and analytic on $\mathbb{C}_0^+$, it follows that it is a bounded entire function. By Liouville's theorem, $\hat{v}$ is constant. It is easy to see that the limit of $\hat{v}$ at infinity is 0, and hence $\hat{v} = 0$. Therefore, for every $t \geq 0$ we have $v(t) = 0$ and hence $m(t) = 0$. Since this is true for every $w_0 \in W$, we get that $R = 0$. $\square$

The next theorem is our main result. It gives necessary and sufficient conditions for the solvability of the state feedback regulator problem.

Theorem 3.3. The linear state feedback regulator problem is solvable iff there exist mappings $\Pi \in \mathcal{L}(W, Z)$ and $\Gamma \in \mathcal{L}(W, U)$ satisfying the *regulator equations*

$$\Pi S = A\Pi + B\Gamma + P, \qquad (3.4)$$

$$0 = C_\Lambda \Pi + D\Gamma + Q. \qquad (3.5)$$

The first regulator equation holds in $\mathcal{L}(W, Z)$ and the second holds in $\mathcal{L}(W, Y)$. In this case a feedback law solving the state feedback regulator problem is

$$u(t) = \Gamma w(t). \qquad (3.6)$$

Proof. Let us first suppose that $u(t) = Lw(t)$ solves the linear state feedback regulator problem. According to Lemma 2.3 the Sylvester equation (2.6) has a unique solution $\Pi \in \mathcal{L}(W, Z)$ for this L. Hence $\Gamma = L$ and Π satisfy the first regulator equation (3.4). We want to show that this Γ and Π solve the second regulator equation (3.5) also. For any $w_0 \in W$, we claim that $\left[\begin{smallmatrix} \Pi w_0 \\ w_0 \end{smallmatrix}\right] \in \mathcal{D}(A_p^L)$ and

$$\mathbb{T}_t^{p,L} \begin{bmatrix} \Pi w_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} \Pi e^{St} w_0 \\ e^{St} w_0 \end{bmatrix} \qquad \forall\, t \geq 0. \qquad (3.7)$$

Indeed, if we differentiate the right-hand side, using (2.6) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \Pi e^{St} w_0 \\ e^{St} w_0 \end{bmatrix} = \begin{bmatrix} (A\Pi + BL + P) e^{St} w_0 \\ S e^{St} w_0 \end{bmatrix} = A_p^L \begin{bmatrix} \Pi e^{St} w_0 \\ e^{St} w_0 \end{bmatrix}.$$

Recall that $\operatorname{Ran} \Pi \subset \underset{\sim}{Z} \subset \mathcal{D}(C_\Lambda)$. Thus we can apply $[C_\Lambda \;\; Q + DL]$ to (3.7) and use (3.2) to obtain that

$$e(t) = (C_\Lambda \Pi + DL + Q) e^{St} w_0 \qquad \forall\, t \geq 0, \qquad (3.8)$$

when the initial condition for (3.1) is $\left[\begin{smallmatrix} \Pi w_0 \\ w_0 \end{smallmatrix}\right]$. By assumption, for some $\alpha < 0$, $e \in L^2_\alpha([0,\infty);Y)$ for all $w_0 \in W$ and $\sigma(S) \subset \overline{\mathbb{C}^+_0}$. According to Lemma 3.2 (with $R = C_\Lambda \Pi + DL + Q$) we get that $C_\Lambda \Pi + DL + Q = 0$. Thus, (3.5) holds with $\Gamma = L$.

Conversely, suppose that the regulator equations (3.4) and (3.5) are satisfied by $\Pi \in \mathcal{L}(W,Z)$ and $\Gamma \in \mathcal{L}(W,U)$. Define $L = \Gamma$, then the first regulator equation becomes (2.6). We define a closed subspace X^+ of $X = Z \times W$ as the graph of the mapping Π, i.e.,

$$X^+ = \left\{ \begin{bmatrix} \Pi w \\ w \end{bmatrix} \;\middle|\; w \in W \right\}.$$

As already mentioned at (3.7), $X^+ \subset \mathcal{D}(A^L_p)$ and X^+ is $\mathbb{T}^{p,L}_t$-invariant. For any initial condition $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in X$ we have

$$\mathbb{T}^{p,L}_t \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} = \mathbb{T}^{p,L}_t \begin{bmatrix} \Pi w_0 \\ w_0 \end{bmatrix} + \mathbb{T}^{p,L}_t \begin{bmatrix} z_0 - \Pi w_0 \\ 0 \end{bmatrix}.$$

Formula (3.7) and the upper triangular form of $\mathbb{T}^{p,L}_t$ imply

$$\mathbb{T}^{p,L}_t \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} \Pi e^{St} w_0 \\ e^{St} w_0 \end{bmatrix} + \begin{bmatrix} \mathbb{T}_t (z_0 - \Pi w_0) \\ 0 \end{bmatrix}.$$

Applying $[C_\Lambda \;\; Q + DL]$ to the above equation, using (3.2) and $C_\Lambda \Pi + Q + DL = 0$, we get that

$$e(t) = C_\Lambda \mathbb{T}_t (z_0 - \Pi w_0).$$

Since $\mathbb{T}$ is exponentially stable and C_Λ is an admissible observation operator for $\mathbb{T}$, it follows that $e \in L^2_\alpha([0,\infty);Y)$ for some $\alpha < 0$. Therefore, the linear state feedback regulator problem is solved by $u = \Gamma w$. □

4. EXAMPLE

We present an example of state feedback regulation which illustrates our theory. In this example the plant is an one-dimensional heat equation on the domain $[0,1]$ with the control acting via the boundary conditions. The output of this plant must track a sinusoid while rejecting constant (or linearly growing) disturbances. We will solve the regulator equations to obtain the desired feedback control law and demonstrate its performance via numerical simulation.

Consider a one-dimensional heat equation on the interval $[0,1]$ with a Robin boundary control $u(t)$ at the right end point ($x = 1$) and a Neumann boundary disturbance $d(t)$ acting at the left end:

$$\frac{\partial z}{\partial t}(x,t) = \frac{\partial^2 z}{\partial x^2}(x,t), \quad x \in (0,1), \qquad (4.1)$$

$$-\frac{\partial z}{\partial x}(0,t) = d(t), \quad \frac{\partial z}{\partial x}(1,t) + kz(1,t) = u(t), \qquad (4.2)$$

where $k > 0$ is a constant. The initial condition $z(x,0) = \varphi(x) \in L^2(0,1)$. Assume that the output $y(t)$ is obtained via point evaluation of the state $z(x,t)$ at a prescribed point $x_1 \in [0,1]$:

$$y(t) = Cz(t) = z(x_1,t). \qquad (4.3)$$

Our objective is to design a state feedback law which guarantees that the output (4.3) tracks a sinusoidal reference signal $r(t) = M \sin(\alpha t + \psi)$ (with $\alpha > 0$ known but $M > 0$ and $\psi \in \mathbb{R}$ unknown) while rejecting a disturbance of the form $d(t) = c_1 + c_2 t$, with $c_1, c_2 \in \mathbb{R}$ unknown. For this, we consider the exosystem as in (2.2), with

$$S = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \begin{aligned} Q^1 &= [1\ 0\ 0\ 0], \\ C^1 &= [0\ 0\ 1\ 0]. \end{aligned} \qquad (4.4)$$

We intend to apply Theorem 3.3 to find a feedback law of the form $u(t) = \Gamma w(t)$, where $\Gamma \in \mathbb{R}^4$ ensures that for some $\delta < 0$, the error $e = y - r$ is in $L^2_\delta([0,\infty);\mathbb{C})$ for every initial condition of the plant and the exosystem.

The system (4.1)–(4.3) can be reformulated in the abstract form (2.1) (see also Byrnes et al. (2002) where a similar heat equation is considered in higher dimensions). Here we merely list the relevant spaces and operators, working with real-valued functions only:

(1) $Z = L^2[0,1]$, $U = Y = \mathbb{R}$, $A = \mathrm{d}^2/\mathrm{d}x^2$ with $\mathcal{D}(A) = \{\varphi \in \mathcal{H}^2(0,1) \,|\, \varphi'(0) = \varphi'(1) + k\varphi(1) = 0\}$.
(2) We have $A^* = A$, so that we may regard Z_{-1} as the dual of Z_1 with respect to the pivot space Z. In particular, the distributions δ_ξ (Dirac pulse at the point ξ) are in Z_{-1} for any $\xi \in [0,1]$.
(3) Also $B = \delta_1$ and $B^1 = \delta_0$. Hence $B, B^1 \in \mathcal{L}(U, Z_{-1})$.
(4) The operator $C \in \mathcal{L}(Z_1, Y)$ is defined by $C\phi = \phi(x_1)$. It can be verified that $\mathcal{D}(C_\Lambda) \supset \mathcal{H}^1(0,1)$.

Thus we can replace the original plant (4.1)–(4.3) with the following system:

$$z_t = Az + B^1 d + Bu, \quad y = C_\Lambda z. \qquad (4.5)$$

The well-posedness and regularity of the system (4.5) can be established via trivial modifications to the results in Byrnes et al. (2002). It is easy to verify that A is strictly negative and generates an exponentially stable semigroup.

A straightforward calculation shows that for all $s \in \mathbb{C}$ with $\mathrm{Re}\, s \geq 0$ and $x_1 \in [0,1]$, the transfer function for the system (4.5) is (see Byrnes et al. (1994))

$$\mathbf{G}(s) = C_\Lambda (sI - A)^{-1} B = \frac{\cosh(x_1\sqrt{s})}{\sqrt{s}\sinh(\sqrt{s}) + k\cosh(\sqrt{s})}.$$

Furthermore it is easy to see that for each $k > 0$, there exists a constant C_k so that

$$\sup_{s \in \mathbb{C}^+_0} |\mathbf{G}(s)| \leq C_k \qquad \text{and} \qquad \lim_{s \to \infty} \mathbf{G}(s) = 0.$$

For the plant (4.5) driven by the exosystem determined in (4.4), we seek a control law in the form $u = \Gamma w$ that solves the regulator problem. Thus we seek mappings $\Pi \in \mathcal{L}(\mathbb{R}^4, L^2(0,1))$ and $\Gamma \in \mathcal{L}(\mathbb{R}^4, \mathbb{R})$ which satisfy the regulator equations

$$\Pi S w = A \Pi w + B \Gamma w + B^1 C^1 w, \qquad (4.6)$$

$$0 = C_\Lambda \Pi w - Q^1 w, \qquad (4.7)$$

for all $w \in \mathbb{R}^4$. Note that these equations consist of a coupled system of one dimensional elliptic boundary value problems (4.6) subject to the algebraic constraint (4.7) and can be easily solved using elementary techniques. With the notation $\Gamma = [\gamma_1\ \gamma_2\ \gamma_3\ \gamma_4]$ where $\gamma_j \in \mathbb{R}$, we obtain

$$\gamma_1 = \frac{\mathrm{Re}\,(\mathbf{G}(i\alpha))}{|\mathbf{G}(i\alpha)|^2}, \qquad \gamma_2 = \frac{\mathrm{Im}\,(\mathbf{G}(i\alpha))}{|\mathbf{G}(i\alpha)|^2},$$

$$\gamma_3 = kx_1 - k - 1, \qquad \gamma_4 = -\frac{k}{3}x_1^3 + \frac{k}{2}x_1 + x_1 - \frac{k}{6} - \frac{1}{2}.$$

For the numerical simulation we have taken the initial condition $\varphi(x) = 3\cos(\pi x)$, $k = 0.5$, the disturbance signal $d(t) = 0.1(t+1)$, the reference signal $r(t) = \sin(2t)$ and the observation point $x_1 = 0.25$. The results are presented in Figures 2, 3 and 4.

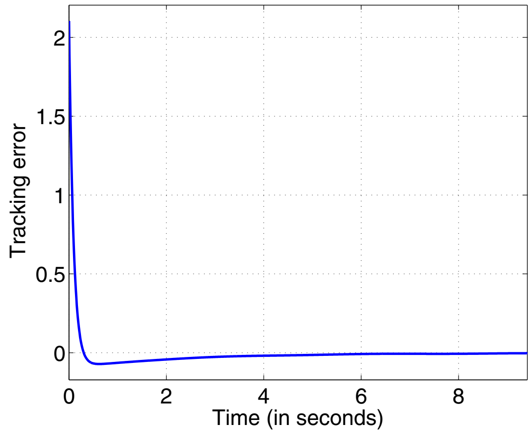

Figure 2. Plot of the tracking error in the example. The error tends to zero.

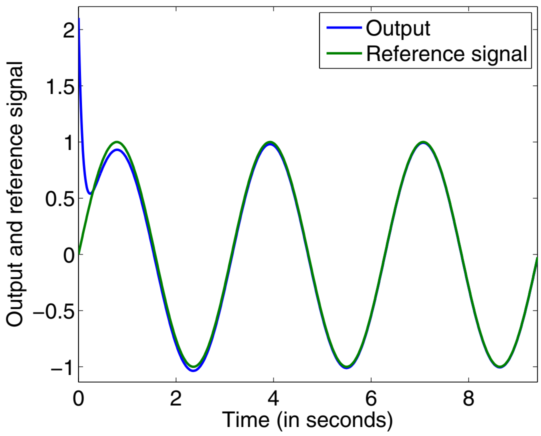

Figure 3. Plot of the reference signal $\sin(2t)$ and of the plant output.

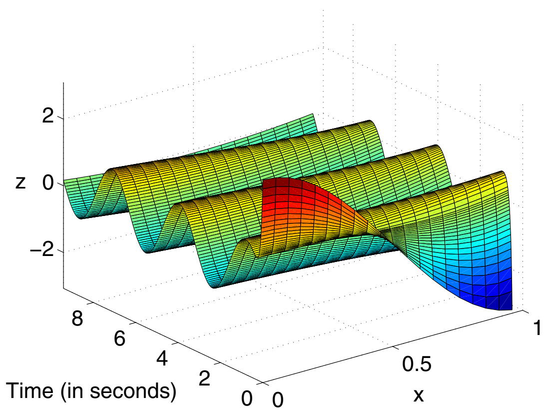

Figure 4. Temperature profile on the unit interval as a function of time.

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