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Distributed Estimation over Analog Fading Channels Using Constant-Gain Estimators

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Abstract: This paper considers the distributed estimation of an unstable target via constant-gain estimators under local communications and channel fading. The communication graph is assumed to be fixed and undirected, and the channel fading is assumed to be identical. Necessary and sufficient conditions on communication network over which the state of the unstable target can be estimated in the mean square sense are given for both continuous-time and discrete-time cases, which reveal the fundamental limitation on distributed estimation induced by local communications, channel fading, and target dynamics. In addition, our results for the case without channel fading and the case with separate communications are consistent with the results in the literature.

Keywords: Analog fading channel; constant-gain estimator; distributed estimation; mean square detectability.

1. INTRODUCTION

Distributed estimation and control for multi-agent systems has attracted much attention from the control community recently, due to its advantages such as low cost, high scalability, simple maintenance, etc. A survey on multi-agent systems can be found in Olfati-Saber et al. [2007], and the consensusability condition for general linear multi-agent systems is provided in Ma and Zhang [2010] for the continuous-time case and in You and Xie [2011] for the discrete-time case. Distributed estimation generally requires to construct an algorithm for each node to estimate the state of one or several targets by using only local information. The target(s) may only transmit information to a subset of nodes, and for those nodes who cannot get information from the target(s) directly, we should provide algorithms using only information from their neighboring nodes to estimate the state of the target(s). The consensus-based Kalman filter is studied in Olfati-Saber [2005, 2007, 2009]. Other distributed estimation algorithms using the theory of Kalman filtering include gossip-based Kalman filter as in Kar and Moura [2011] and diffusion-based Kalman filter as in Hu et al. [2012]. In order to achieve simpler design and lower complexity than time-varying Kalman filter, a distributed estimation algorithm with a common static gain for both the innovation and the state-consensus information is proposed in Zhou et al. [2013]. The estimation gain design proposed in Zhou et al. [2013] follows the idea in Hong et al. [2006], Hong and Wang [2009].

Channel fading represents the fluctuation experienced by transmitted signals due to the effects of multipath and shadowing in wireless communication systems. It has attracted recurring research interests from the communication community; see, for example, Tse and Viswanath [2005], Goldsmith [2005], Proakis and Salehi [2005]. Generally, there are two options to transmit a signal: analog or digital. Elia [2005] considers the mean square stabilization over analog fading channels and shows that the minimum mean square capacity for stabilization can be given in terms of the unstable poles of the single-input plant under investigation. Xiao et al. [2012] further presents the network requirement for both state feedback and output feedback stabilization of multi-input-multi-output plants over multiple fading channels. Xiao et al. [2013] deals with the state feedback stabilization over digital fading channels. Kalman filtering over analog fading channels is studied in Dey et al. [2009], while Kalman filtering over digital fading channels is addressed in Quevedo et al. [2012, 2013].

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In this paper, we consider the distributed estimation of an unstable target under constraints on both network connectivity and channel fading. Next, we summarize the contribution of this paper. First of all, a framework of distributed estimation is presented by using constant-gain estimators introduced in Zhou et al. [2013], and it accommodates both network connectivity and fading phenomenon. Secondly, necessary and sufficient conditions on communication network over which the unstable target can be estimated in the mean square sense are provided for both continuous-time and discrete-time cases. Those conditions show the fundamental limitation on distributed estimation induced by local communications, channel fading, and target dynamics. The results for two special cases: the case without channel fading and the case with separate communications, also demonstrate the consistency between our results and the existing results in the literature.

The remainder of the paper is organized as follows. The distributed estimation problem is formulated in Section 2, where the communication graph, the system setup, as well as the assumptions adopted in this paper are given. Sections 3 and 4 provide conditions on communication network for mean square detectablity in the continuous-time setting and the discrete-time setting, respectively. Finally, Section 5 concludes the paper and discusses future work.

The notation used in this paper is mostly standard. The symbol := means "defined as". The set of real numbers, the set of nonnegative real numbers, and the set of nonnegative integers are denoted by $\mathbb{R}$, $\mathbb{R}_0$, and $\mathbb{N}_0$, respectively. Use $\mathrm{Re}(\lambda)$ and $|\lambda|$ to denote the real part and the magnitude of the complex variable λ. The kronecker product is denoted by $\otimes$. Furthermore, the mathematical expectation operator is denoted by $\mathrm{E}\{\cdot\}$.

2. PROBLEM FORMULATION

Consider the scenario where a set of N nodes/estimators is deployed to estimate the state (e.g., position and velocity) of a target. The target can transmit information to a subset of nodes, and each node can only communicate with its neighboring nodes. The communication graph and the system setup are described as follows.

2.1 Communication Graph

We set $\beta_i(t) > 0$ if node i can obtain information from the target at time t, otherwise $\beta_i(t) = 0$. Similarly, $\alpha_{ij}(t) > 0$ if node i can get information from node j at time t, otherwise $\alpha_{ij}(t) = 0$. Self loop is excluded by setting $\alpha_{ii}(t) = 0$ for all $t \geq 0$ and $i = 1, 2, \ldots, N$.

The interaction topology can be conveniently characterized by an algebraic graph $\mathcal{G} := \{\mathcal{V}, \mathcal{E}(t)\}$ with a vertex set $\mathcal{V} := \{v_1, v_2, \ldots, v_N\}$ and an edge set $\mathcal{E}(t) := \{(i,j) : i, j \in \mathcal{V}\}$. The vertex v_i represents the i-th node, and $(i,j) \in \mathcal{E}(t)$ if node i can get information from node j at time t, i.e., $\alpha_{ij}(t) > 0$. Define the neighbor set of node i at time t by $\mathcal{N}_i(t) := \{j : (i,j) \in \mathcal{E}(t)\}$. The weighted adjacency matrix of $\mathcal{G}$ is defined as $G(t) := [\alpha_{ij}(t)]_{i,j=1,2,\ldots,N}$. The corresponding degree matrix is given by $D(t) := \mathrm{diag}\{d_1(t), d_2(t), \ldots, d_N(t)\}$ with $d_i(t) := \sum_{j=1}^{N} \alpha_{ij}(t)$. Then, the Laplacian of $\mathcal{G}$ is $L_p(t) := D(t) - G(t)$. For the convenience of later analysis, further let $B_p(t) := \mathrm{diag}\{\beta_1(t), \beta_2(t), \ldots, \beta_N(t)\}$ and

$$H_p(t) := B_p(t) + L_p(t). \tag{1}$$

2.2 System Setup

The dynamics of the target is described by

$$\delta x_0(t) = Ax_0(t) + w_0(t), \tag{2}$$

where $x_0(t) \in \mathbb{R}^m$ is the state of the target, and $w_0(t) \in \mathbb{R}^m$ represents the white process noise with zero mean and covariance matrix Q. In the continuous-time setting [1], $\delta x_0(t) = \dot{x}_0(t)$, $t \in \mathbb{R}_0$, while in the discrete-time setting, $\delta x_0(t) = x_0(t+1)$, $t \in \mathbb{N}_0$.

On the basis of the communication graph described in Section 2.1, the i-th node gets information from the target and its neighboring nodes via analog fading channels according to

$$\begin{aligned} y_i(t) &= \beta_i(t)\left[\eta_i(t)Cx_0(t) + v_i(t)\right], \\ z_{ij}(t) &= \alpha_{ij}(t)\left[\xi_{ij}(t)Cx_j(t) + n_{ij}(t)\right], \end{aligned} \tag{3}$$

where $y_i(t) \in \mathbb{R}^q$ and $z_{ij}(t) \in \mathbb{R}^q$ are channel outputs, $x_j(t) \in \mathbb{R}^m$ denotes the estimation of the target's state at node j, $v_i(t)$ and $n_{ij}(t)$ are the zero-mean white communication noises with covariance matrices R_{vi} and R_{nij}, and $\eta_i(t) \geq 0$, $\xi_{ij}(t) \geq 0$ represent the random channel fading gains which are assumed to be known at node i. Note that the network connectivity is characterized by α_{ij} and β_i, while the channel fading is described by ξ_{ij} and η_i.

In this situation, the sum of innovation and state-consensus information at node i can be constructed as

$$\begin{aligned} \phi_i(t) :=& y_i(t) - \eta_i(t)\beta_i(t)Cx_i(t) \\ &+ \sum_{j=1}^{N} \left[z_{ij}(t) - \xi_{ij}(t)\alpha_{ij}(t)Cx_i(t)\right] \\ =& y_i(t) - \eta_i(t)\beta_i(t)Cx_i(t) \\ &+ \sum_{j \in \mathcal{N}_i(t)} \left[z_{ij}(t) - \xi_{ij}(t)\alpha_{ij}(t)Cx_i(t)\right]. \end{aligned} \tag{4}$$

The following constant-gain distributed estimators are employed in this paper:

$$\delta x_i(t) = Ax_i(t) + L\phi_i(t), \quad \forall i = 1, 2, \ldots, N, \tag{5}$$

where L is the constant estimation gain to be designed. Note that only local information is required at each estimator.

Denote the estimation error at node i by $e_i(t) := x_i(t) - x_0(t)$. According to (2)-(5), we have

$$\begin{aligned} \delta e_i(t) =& \delta x_i(t) - \delta x_0(t) \\ =& Ax_i(t) + L\phi_i(t) - Ax_0(t) - w_0(t) \\ =& Ae_i(t) - L\eta_i(t)\beta_i(t)Ce_i(t) + L\beta_i(t)v_i(t) \\ &+ L \sum_{j \in \mathcal{N}_i(t)} \xi_{ij}(t)\alpha_{ij}(t)C[e_j(t) - e_i(t)] \\ &+ L \sum_{j \in \mathcal{N}_i(t)} \alpha_{ij}(t)n_{ij}(t) - w_0(t). \end{aligned} \tag{6}$$

[1] A mathematically precise expression for the continuous-time stochastic system (2) is $dx_0(t) = Ax_0(t)dt + dw_0(t)$ and the theory of stochastic differential equations applies here. We use $\dot{x}_0(t) = Ax_0(t) + w_0(t)$ for (2) with a slight abuse of notation, which would not affect the results obtained in this paper.

Let $e(t) := [e_1(t)'\ e_2(t)' ... e_N(t)']'$ and $P(t) := \mathrm{E}\{e(t)e(t)'\}$, where the expectation is taken over $\{w_0\}_0^t$, $\{\eta_i\}_0^t$, $\{v_i\}_0^t$, $\{\xi_{ij}\}_0^t$, and $\{n_{ij}\}_0^t$. The mean square detectability is defined as follows.

Definition 1. The target (2) is mean square detectable via constant-gain distributed estimators (5), if $P(t)$ is well defined for all $t \geq 0$ and

$$\lim_{t\to\infty} P(t) \leq M, \tag{7}$$

where $M > 0$ is a constant matrix. □

2.3 Assumptions

Next, we list assumptions adopted throughout this paper on communication graph, channel fading, and target dynamics, respectively.

Assumption 1. The graph $\mathcal{G}$ is assumed to be

(a) undirected: $\alpha_{ij}(t) = \alpha_{ji}(t)$ for all $t \geq 0$ and $i, j = 1, 2, \ldots, N$;
(b) connected: at least one node in each maximal connected branch [2] of $\mathcal{G}$ receives information from the target for all $t \geq 0$;
(c) fixed: $\beta_i(t) = \beta_i$, $\alpha_{ij}(t) = \alpha_{ij}$, for all $t \geq 0$ and $i, j = 1, 2, \ldots, N$, where β_i and α_{ij} are constant scalars. □

Assumption 2. The channel fading is assumed to be identical, i.e., all communication channels experience the same analog fading as

$$\eta_i(t) = \xi_{ij}(t) = \xi(t), \quad \forall i, j = 1, 2, \ldots, N, \tag{8}$$

where $\xi(t) \in \mathbb{R}$ is white with mean $\mu_\xi \neq 0$ and covariance $\sigma_\xi^2 \geq 0$. □

Assumption 3. The pair (A, C) is assumed to be detectable, A is unstable, and C has full-row rank. □

Under (a) and (b) of Assumption 1, it has been proved in Hong et al. [2006] that $H_p(t)$ defined in (1) is positive definite for all $t \geq 0$. Further under Assumption 1(c), $H_p(t)$ is reduced to a constant matrix $H_p > 0$, and thus there exists a unitary matrix U_p such that $U_p H_p U_p' = \Lambda_p = \mathrm{diag}\{\lambda_{p1}, \lambda_{p2}, \ldots, \lambda_{pN}\}$, where $\lambda_{p1}, \lambda_{p2}, \ldots, \lambda_{pN}$ are eigenvalues of H_p with ordering $0 < \lambda_{p1} \leq \lambda_{p2} \leq \cdots \leq \lambda_{pN}$. In addition, under Assumption 1(c), the condition (7) for mean square detectability can be replaced by $\lim_{t\to\infty} P(t) = M$ for some $M > 0$. In view of the linearity of estimation error dynamics (6), when only the mean square detectability is concerned, we can ignore all additive noises in the system by assuming that $w_0(t) = 0, v_i(t) = 0, n_{ij}(t) = 0$ for all $t \geq 0$ and $i, j = 1, 2, \ldots, N$, without loss of generality. In this case, the estimation error dynamics (6) becomes

$$\begin{aligned}\delta e_i(t) =& A e_i(t) - L\eta_i(t)\beta_i C e_i(t) \\ &+ L \sum_{j\in\mathcal{N}_i(t)} \xi_{ij}(t)\alpha_{ij} C[e_j(t) - e_i(t)],\end{aligned} \tag{9}$$

and the condition (7) for mean square detectability is further reduced to $\lim_{t\to\infty} P(t) = 0$. If the network is also identical as in Assumption 2, then we have

$$\begin{aligned}\delta e_i(t) =& A e_i(t) - L\xi(t)\beta_i C e_i(t) \\ &+ L \sum_{j\in\mathcal{N}_i(t)} \xi(t)\alpha_{ij} C[e_j(t) - e_i(t)],\end{aligned} \tag{10}$$

and

$$\delta e(t) = [(I \otimes A) - \xi(t) H_p \otimes LC] e(t). \tag{11}$$

Remark 2. To focus on the case with fixed undirected graph and identical channel fading simplifies later analysis and can still characterize the fundamental limitation on distributed estimation induced by local communications and channel fading. The case with directed switching topology and nonidentical channel fading is one of our future research directions. □

Under Assumption 3, we can take (A, C) to be of the form:

$$A = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix}, \quad C = [C_s \ \ C_u], \tag{12}$$

where A_s is stable, all poles of A_u are unstable, and (A_u, C_u) is observable.

In the following two sections, we will establish conditions for mean square detectability in both continuous-time and discrete-time settings.

3. CONTINUOUS-TIME CASE

The proposition below summarizes a series of necessary and sufficient conditions for mean square detectability in the continuous-time setting.

Proposition 3. Suppose Assumptions 1-3 hold. The following statements are equivalent in the continuous-time setting.

(a) The target (2) is mean square detectable via constant-gain distributed estimators (5).
(b) There exists an L such that the sequence $\{P(t)\}_{t\geq 0}$ computed by

$$\begin{aligned}&\dot{P}(t)\\ =&(I \otimes A)P(t) + P(t)(I \otimes A') - \mu_\xi (H_p \otimes LC)P(t)\\ &+ \sigma_\xi^2 (H_p \otimes LC)P(t)(H_p \otimes C'L')\\ &- \mu_\xi P(t)(H_p \otimes C'L')\end{aligned}$$

with any $P(0) \geq 0$ is convergent to 0 as t approaches ∞.
(c) There exist $P > 0$ and L such that

$$\begin{aligned}0 >&(I \otimes A)P + P(I \otimes A') - \mu_\xi (H_p \otimes LC)P\\ &+ \sigma_\xi^2 (H_p \otimes LC)P(H_p \otimes C'L')\\ &- \mu_\xi P(H_p \otimes C'L').\end{aligned}$$

(d) There exist $W > 0$ and L such that

$$\begin{aligned}0 >&(I \otimes A')W + W(I \otimes A) - \mu_\xi (H_p \otimes C'L')W\\ &+ \sigma_\xi^2 (H_p \otimes C'L')W(H_p \otimes LC)\\ &- \mu_\xi W(H_p \otimes LC).\end{aligned}$$

(e) There exist $P_i > 0$, $i = 1, 2, \ldots, N$, and L such that

$$\begin{aligned}0 >&AP_i + P_i A' + \sigma_\xi^2 \lambda_{pi}^2 LCP_iC'L'\\ &- \mu_\xi \lambda_{pi} LCP_i - \mu_\xi \lambda_{pi} P_i C'L'\end{aligned} \tag{13}$$

for all $i = 1, 2, \ldots, N$.
(f) The target (2) is mean square detectable via constant-gain distributed estimators (5) with (A, C) replaced by (A_u, C_u) given in (12). □

[2] The graph $\tilde{\mathcal{G}} = \{\tilde{\mathcal{V}}, \tilde{\mathcal{E}}(t)\}$ is a maximal connected branch of $\mathcal{G}$, if $\tilde{\mathcal{V}} \in \mathcal{V}$, $\tilde{\mathcal{E}}(t) \in \mathcal{E}(t)$, no other vertices in $\mathcal{V} - \tilde{\mathcal{V}}$ connected to $\tilde{\mathcal{G}}$, and $\tilde{\mathcal{G}}$ is connected.

Consider the linear matrix inequality

$$\begin{bmatrix} SA + A'S - YC - C'Y' & \sqrt{\frac{1}{\bar{g}}}YC \\ \sqrt{\frac{1}{\bar{g}}}C'Y' & -S \end{bmatrix} < 0, \qquad (14)$$

and define

$$g_c := \begin{cases} \sum_{k=1}^{m} \max\{2\mathrm{Re}(\lambda_k(A)), 0\}, & \text{if } q = 1; \\ \max_{k=1,2,\ldots,m} \{2\mathrm{Re}(\lambda_k(A))\}, & \text{if } q = m; \\ \inf_{S>0,Y} \bar{g}, \text{ s.t. (14)}, & \text{otherwise}, \end{cases} \qquad (15)$$

where $\lambda_1(A), \lambda_2(A), \ldots, \lambda_m(A)$, are the eigenvalues of A counting algebraic multiplicity. The next lemma is vital in obtaining explicit conditions on network connectivity and channel fading for mean square detectability.

Lemma 4. Under Assumption 3, there exists a solution $P_0 > 0$ to the following modified Riccati inequality in the continuous-time setting

$$0 > AP_0 + P_0A' - \tau P_0C'(CP_0C')^{-1}CP_0, \qquad (16)$$

if and only if $\tau > g_c$. □

Proof. In view of the equivalence between Proposition 3(a) and 3(f), we assume that all the eigenvalues of A are in the closed-right half of the complex plane without loss of generality.

Note that the existence of $P_0 > 0$ to (16) is equivalent to the existence of $P_0 > 0$ and L_0 to

$$0 > (A - L_0C)P_0 + P_0(A' - C'L_0') + \frac{1}{\tau}L_0CP_0C'L_0', \quad (17)$$

which is further equivalent to (14) with $S = P_0^{-1}$, $Y = SL_0$, and $\bar{g} = \tau$. Observe that (14) holds for all $\bar{g} \geq \bar{g}_a$ if (14) is true for some $\bar{g} = \bar{g}_a$. Therefore, the optimization in the last line of (15) provides the critical value of τ for any $q = 1, 2, \ldots, m$.

Next, consider the special case: $q = 1$. We will first show the necessity. Note that (16) is equivalent to

$$0 > P_0^{-\frac{1}{2}}AP_0^{\frac{1}{2}} + P_0^{\frac{1}{2}}A'P_0^{-\frac{1}{2}} - \tau P_0^{\frac{1}{2}}C'(CP_0C')^{-1}CP_0^{\frac{1}{2}}.$$

By taking the trace for the both sides of the above inequality, we obtain

$$\begin{aligned} 0 >& \mathrm{tr}(P_0^{-\frac{1}{2}}AP_0^{\frac{1}{2}}) + \mathrm{tr}(P_0^{\frac{1}{2}}A'P_0^{-\frac{1}{2}}) \\ &- \mathrm{tr}(\tau P_0^{\frac{1}{2}}C'(CP_0C')^{-1}CP_0^{\frac{1}{2}}) \\ =& \mathrm{tr}(A) + \mathrm{tr}(A') - \mathrm{tr}(\tau) \\ =& \sum_{k=1}^{m} \max\{2\mathrm{Re}(\lambda_k(A)), 0\} - \tau. \end{aligned} \qquad (18)$$

Thus, the existence of $P_0 > 0$ to (16) implies $\tau > g_c = \sum_{k=1}^{m} \max\{2\mathrm{Re}(\lambda_k(A)), 0\}$. To establish the sufficiency, we note that, as shown in Xiao and Xie [2010], the existence of $P_0 > 0$ to (16) is also equivalent to

$$\begin{aligned} 1 >& \inf_{L_0, \text{ s.t. } (A-L_0C) \text{ stable}} \left\| \sqrt{\frac{1}{\tau}} C(sI - A - L_0C)^{-1}L_0 \right\|_2^2 \\ =& \frac{1}{\tau} \inf_{L_0, \text{ s.t. } (A-L_0C) \text{ stable}} \left\| C(sI - A - L_0C)^{-1}L_0 \right\|_2^2 \\ =& \frac{1}{\tau} \sum_{k=1}^{m} \max\{2\mathrm{Re}(\lambda_k(A)), 0\}, \end{aligned}$$

where the last equation follows from Theorem II.1 of Braslavsky et al. [2007]. Therefore, $\tau > g_c$ also implies the existence of $P_0 > 0$ to (16).

For the case $q = m$, (16) becomes

$$0 > AP_0 + P_0A' - \tau P_0. \qquad (19)$$

It follows from the property of generalized eigenvalue as studied in Boyd et al. [1994] that $\tau > g_c = \max_{k=1,2,\ldots,m}\{2\mathrm{Re}(\lambda_k(A))\}$ is a necessary and sufficient condition for the existence of P_0 to (16). □

The next theorem gives conditions on communication network over which the target with unstable dynamics can be estimated in the continuous-time setting.

Theorem 5. Under Assumptions 1-3, the target (2) is mean square detectable via constant-gain distributed estimators (5) in the continuous-time setting if

$$\tau_c := \frac{\mu_\xi^2}{\sigma_\xi^2} \times \frac{4\lambda_{pN}\lambda_{p1}}{(\lambda_{pN} + \lambda_{p1})^2} > g_c, \qquad (20)$$

and only if

$$\frac{\mu_\xi^2}{\sigma_\xi^2} > g_c. \qquad (21)$$

Moreover, if (20) holds, then there exists a solution $P_0 > 0$ to the modified Riccati inequality (16) with $\tau = \tau_c$, and an estimation gain ensuring the mean square detectability is given by

$$L_c = \frac{2\mu_\xi}{(\lambda_{pN} + \lambda_{p1})\sigma_\xi^2} P_0C'(CP_0C')^{-1}, \qquad (22)$$

where $P_0 > 0$ is any solution to (16) with $\tau = \tau_c$. □

Proof. First, the sufficiency of (20) will be shown. Based on Lemma 4, (20) is equivalent to the existence of $P_0 > 0$ to (16) with $\tau = \tau_c$. Using the estimation gain given in (22) and $P_i = P_0$, $i = 1, 2, \ldots, N$, we can derive that the right hand side of (13) in Proposition 3 becomes

$$\begin{aligned} &AP_0 + P_0A' + \sigma_\xi^2\lambda_{pi}^2L_cCP_0C'L_c' - \mu_\xi\lambda_{pi}L_cCP_0 \\ &- \mu_\xi\lambda_{pi}P_0C'L_c' \\ =& AP_0 + P_0A' + \left(\frac{\lambda_{pi}^2}{\lambda_{pN}\lambda_{p1}} - \frac{\lambda_{pi}}{\lambda_{p1}} - \frac{\lambda_{pi}}{\lambda_{pN}} \right) \\ &\times \tau_cP_0C'(CP_0C')^{-1}CP_0 \\ =& AP_0 + P_0A' - \left[1 - \left(\frac{\lambda_{pi}}{\lambda_{p1}} - 1 \right) \left(\frac{\lambda_{pi}}{\lambda_{pN}} - 1 \right) \right] \\ &\times \tau_cP_0C'(CP_0C')^{-1}CP_0. \end{aligned}$$

It follows from $0 < \lambda_{p1} \leq \lambda_{p2} \leq \cdots \leq \lambda_{pN}$ that

$$\left(\frac{\lambda_{pi}}{\lambda_{p1}} - 1 \right) \left(\frac{\lambda_{pi}}{\lambda_{pN}} - 1 \right) \leq 0, \qquad (23)$$

for all $i = 1, 2, \ldots, N$. Therefore, the solvability of (16) with $\tau = \tau_c$ implies (13) in Proposition 3 with $P_i = P_0$, $i = 1, 2, \ldots, N$, and L_c given in (22), which completes the proof of sufficiency.

Next, we will prove the necessity of (21). Suppose the target is mean square detectable via constant-gain distributed estimators. The condition (13) in Proposition 3 implies

$$0 > AP_i + P_iA' - \frac{\mu_\xi^2}{\sigma_\xi^2}P_iC'(CP_iC')^{-1}CP_i. \qquad (24)$$

It follows from Lemma 4 again that (21) is true, which completes the proof of necessity. □

Remark 6. Note that the sufficient condition (20) in Theorem 5 shows an explicit relationship among channel fading, local communications, and target dynamics. Reducing the gap between (20) and (21) deserves further investigation.

We are also interested in the following two special cases.

- Case without channel fading: $\sigma_\xi^2 = 0$.
- Case with separate communications: $\beta_i = 1$ and $\alpha_{ij} = 0$ for all $i, j = 1, 2, \ldots, N$.

We can obtain the next corollary.

Corollary 7. Suppose Assumptions 1-3 hold.

- For the case without channel fading, the target (2) is always mean square detectable via constant-gain distributed estimators (5) in the continuous-time setting.
- For the case with separate communications, the target (2) is mean square detectable via constant-gain distributed estimators (5) in the continuous-time setting if and only if (21) is true. □

Remark 8. The above corollary is consistent with the results in Zhou et al. [2013] and Xiao and Xie [2010] [3]. □

4. DISCRETE-TIME CASE

We can derive the next proposition on mean square detectability in the discrete-time setting.

Proposition 9. Suppose Assumptions 1-3 hold. The following statements are equivalent in the discrete-time setting.

(a) The target (2) is mean square detectable via constant-gain distributed estimators (5).

(b) There exists an L such that the sequence $\{P(t)\}_{t\geq 0}$ computed by

$$\begin{aligned} &P(t+1)\\ =&(I\otimes A)P(t)(I\otimes A') - \mu_\xi(H_p\otimes LC)P(t)(I\otimes A')\\ &+(\mu_\xi^2+\sigma_\xi^2)(H_p\otimes LC)P(t)(H_p\otimes C'L')\\ &-\mu_\xi(I\otimes A)P(t)(H_p\otimes C'L') \end{aligned}$$

with any $P(0)\geq 0$ is convergent to 0 as t approaches ∞.

(c) There exist $P > 0$ and L such that

$$\begin{aligned} P >&(I\otimes A)P(I\otimes A') - \mu_\xi(H_p\otimes LC)P(I\otimes A')\\ &+(\mu_\xi^2+\sigma_\xi^2)(H_p\otimes LC)P(H_p\otimes C'L')\\ &-\mu_\xi(I\otimes A)P(H_p\otimes C'L'). \end{aligned}$$

(d) There exist $W > 0$ and L such that

$$\begin{aligned} W >&(I\otimes A')W(I\otimes A) - \mu_\xi(H_p\otimes C'L')W(I\otimes A)\\ &+(\mu_\xi^2+\sigma_\xi^2)(H_p\otimes C'L')W(H_p\otimes LC)\\ &-\mu_\xi(I\otimes A')W(H_p\otimes LC). \end{aligned}$$

(e) There exist $P_i > 0$, $i = 1, 2, \ldots, N$, and L such that

$$\begin{aligned} P_i >&AP_iA' + (\mu_\xi^2+\sigma_\xi^2)\lambda_{pi}^2 LCP_iC'L'\\ &-\mu_\xi\lambda_{pi}LCP_iA' - \mu_\xi\lambda_{pi}AP_iC'L' \end{aligned} \tag{25}$$

for all $i = 1, 2, \ldots, N$.

(f) The target (2) is mean square detectable via constant-gain distributed estimators (5) with (A, C) replaced by (A_u, C_u) given in (12). □

Consider the following linear matrix inequality

$$\begin{bmatrix} -S & SA+YC & \sqrt{\frac{1}{\bar g-1}}YC\\ A'S+C'Y' & -S & 0\\ \sqrt{\frac{1}{\bar g-1}}C'Y' & 0 & -S \end{bmatrix} < 0, \tag{26}$$

and define

$$g_d := \begin{cases} \prod_{k=1}^{m}\max\{|\lambda_k(A)|^2, 1\}, & \text{if } q=1;\\ \max_{k=1,2,\ldots,m}|\lambda_k(A)|^2, & \text{if } q=m;\\ \inf_{S>0,Y}\bar g, \text{ s.t. (26)}, & \text{otherwise.} \end{cases} \tag{27}$$

The next lemma can be considered as a discrete-time counterpart of Lemma 4, whose proof can be found in Schenato et al. [2007].

Lemma 10. Under Assumption 3, there exists a solution $P_0 > 0$ to the following modified Riccati inequality in the discrete-time setting

$$P_0 > AP_0A' - \tau AP_0C'(CP_0C')^{-1}CP_0A', \tag{28}$$

if and only if $\tau > 1 - 1/g_d$. □

The theorem below provides necessary and sufficient conditions on communication network for mean square detectability in the discrete-time setting.

Theorem 11. Under Assumptions 1-3, the target (2) is mean square detectable via constant-gain distributed estimators (5) in the discrete-time setting if

$$\tau_d := \frac{\mu_\xi^2}{\mu_\xi^2+\sigma_\xi^2}\times\frac{4\lambda_{pN}\lambda_{p1}}{(\lambda_{pN}+\lambda_{p1})^2} > 1-\frac{1}{g_d}, \tag{29}$$

and only if

$$\frac{\mu_\xi^2}{\mu_\xi^2+\sigma_\xi^2} > 1-\frac{1}{g_d}. \tag{30}$$

Moreover, if (29) holds, then there exists a solution $P_0 > 0$ to the modified Riccati inequality (28) with $\tau = \tau_d$, and a estimation gain ensuring the mean square detectability is given by

$$L_d = \frac{2\mu_\xi}{(\lambda_{pN}+\lambda_{p1})(\mu_\xi^2+\sigma_\xi^2)}AP_0C'(CP_0C')^{-1}, \tag{31}$$

where $P_0 > 0$ is any solution to (28) with $\tau = \tau_d$. □

Remark 12. Similarly to (20) for the continuous-time case, the sufficient condition (29) in Theorem 11 also clearly connects channel fading and local communications with system dynamics for the discrete-time case. □

The corollary below can be derived directly from Theorem 11.

Corollary 13. Suppose Assumptions 1-3 hold.

- For the case without channel fading, the target (2) is mean square detectable via constant-gain distributed estimators (5) in the discrete-time setting if $\frac{4\lambda_{pN}\lambda_{p1}}{(\lambda_{pN}+\lambda_{p1})^2} > 1-\frac{1}{g_d}$.
- For the case with separate communications, the target (2) is mean square detectable via constant-gain

[3] Note that the stabilization problem, as a counterpart of the estimation problem, is considered in Xiao and Xie [2010].

distributed estimators (5) in the discrete-time setting if and only if (30) holds. □

Remark 14. The second part of Corollary 13 is consistent with the results in Xiao et al. [2011] and Xiao et al. [2012] [4], where (30) is a necessary and sufficient condition for each individual node to estimate the target's state if every node can obtain information from the target and there is no communication between any two nodes. By comparing the first part of Corollary 13 and that of Corollary 7, we can see that, without channel fading, local communications may still have a negative effect on distributed estimation in the discrete-time case, which is negligible in the continuous-time case. □

5. CONCLUSIONS

In this paper, the distributed estimation of an unstable target has been addressed via constant-gain estimators under constraints on both network connectivity and channel fading. Conditions on communication network for mean square detectablity have been given in terms of network connectivity, channel fading and target dynamics, where both the continuous-time case and the discrete-time case have been considered. Our results also covers some existing results in the literature as special cases.

Possible future research directions include distributed estimation with directed switching topology and nonidentical channel fading, and distributed control over fading channels.

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