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Realization of a nonlinear system in the feedforward form: a polynomial approach

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Abstract: The paper studies the problem of finding a realization of a nonlinear system described by a higher-order differential equation in the so-called feedforward form. A necessary and sufficient condition for the problem to be solvable is given, and it is represented by the requirement that the corresponding polynomial description of such an equation can be factorized. Then certain one-forms associated to the factorization are always integrable which allows to write down the realization. The results are also applied to derive a necessary and sufficient condition for the transformation of a nonlinear state equations into the feedforward form.

Keywords: nonlinear systems, feedforward form, polynomial approach, realization problem

1. INTRODUCTION

The feedforward form plays important role in the control theory of nonlinear systems. Mainly for the reason that once a system is in the feedforward form one can find a solution to the system equations simply by integrating recursively the respective state equations. Therefore, the feedforward form is useful for instance for designing stabilizers for the systems (see e.g. Mazenc and Praly (1996); Zigang et al. (2001); Respondek and Tall (2004)). In the continuous-time case, the geometric characterization of the feedforward form has been studied in Astolfi and Kaliora (2005) in terms of invariant distributions. However, no algorithm for computing such distributions was given. In that respect, the corresponding result for discrete-time systems, which was studied in Aranda-Bricaire and Moog (2004), is stronger, as it is accomplished by an algorithm. Note also that in Respondek and Tall (2004) the problem has been studied in terms of vector fields. However, the adaptation of the formalism of Aranda-Bricaire and Moog (2004) to the continuous-time case is not trivial, as it requires to find a solution to a set of higher-order partial differential equations.

In this paper we study the problem of a realization and transformation of a nonlinear system into the feedforward form. The systems under consideration are autonomous (i.e. without input). A necessary and sufficient condition is given for the both problems to be solvable. In the problem formulation and solution the so-called polynomial / transfer function approach is employed (Zheng et al., 2001; Halás, 2008). In comparison to the analogous approach for linear systems there are two main differences. First, the respective polynomials relate the differentials of the system variables, and second, the polynomials belong to the (non-commutative) skew polynomial ring. Such an approach has already been employed to study several control problems for nonlinear systems (see e.g. Zheng et al. (2001); Halás et al. (2008); Halás and Kotta (2012)). Employing this formalism, a necessary and sufficient condition for the existence of the realization of a nonlinear system in the feedforward form is derived. The only requirement for the solution to exist is that the corresponding polynomial description of the higher order differential equation can be factorized, as it is shown that certain one-forms associated to the factorization are always integrable. Then the results are extended further to derive a necessary and sufficient condition for the transformation of nonlinear state equations into the feedforward form. In this case the solution consists of finding, if possible, a candidate for the system output being fully observable, and then of applying the results of the realization problem. An interesting result here is that one can possibly transform even a linear system into the feedforward form (by a nonlinear change of coordinates) while this might not be possible by any linear change of coordinates.

2. PRELIMINARIES

In this paper we will use the mathematical setting of Conte et al. (2007); Zheng et al. (2001); Halás (2008); Halás and Kotta (2012) adapted to the case of autonomous systems (i.e. systems without input).

Consider a nonlinear system defined either by state-space equations of the form

$$\dot{x} = f(x) \tag{1}$$

where $x \in \mathbf{R}^n$, or by a higher-order differential equation of the form

$$y^{(n)} = F(y, \dot{y}, \dots, y^{(n-1)}) \tag{2}$$

In (1) and (2) the functions f and F respectively are assumed to be elements of the differential field of meromorphic functions of variables $\{x_1, \dots, x_n\}$ or $\{y, \dot{y}, \dots, y^{(n-1)}\}$ respectively, denoted by $\mathcal{K}$. The time-derivative operator $\mathrm{d}/\mathrm{d}t$ acts on $\mathcal{K}$ in the usual way. In particular, $\mathrm{d}x/\mathrm{d}t = f(x)$ and, respectively, $\mathrm{d}y^{(n-1)}/\mathrm{d}t = F(y, \dots, y^{(n-1)})$.

Define the formal vector space of differential one-forms

$$\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi; \xi \in \mathcal{K}\}$$

Elements of $\mathcal{E}$ are called (differential) one-forms.

A one-form $\omega \in \mathcal{E}$ is called exact, if there exists $F \in \mathcal{K}$ such that $\mathrm{d}F = \omega$, and it is called integrable, if there exists $\alpha \in \mathcal{K}$ such that $\alpha\omega$ is exact.
A subspace $\mathcal{V} \subset \mathcal{E}$ is called exact or integrable if it has a basis that consists of exact one-forms only.
Exactness and integrability of a one-form (or a subspace of one-forms) can be inspected by applying the Poincaré and, respectively, the Frobenius theorem.

Frobenius theorem. Let $\mathcal{V} = \mathrm{span}_{\mathcal{K}}\{\omega_1, \dots, \omega_r\}$ be a subspace of $\mathcal{E}$. Then $\mathcal{V}$ is integrable if and only if

$$\mathrm{d}\omega_i \wedge \omega_1 \wedge \dots \wedge \omega_r = 0$$

for all $i = 1, \dots, r$.

The operator $\mathrm{d}/\mathrm{d}t$ defined on $\mathcal{K}$ induces the time-derivative operator, which is by abuse of notation denoted by the same symbol $\mathrm{d}/\mathrm{d}t$, that acts on $\mathcal{E}$ as follows.
Let $v = \sum_i \alpha_i \mathrm{d}\xi_i$ be in $\mathcal{E}$, then

$$\dot{v} = \sum_i (\dot{\alpha}_i \mathrm{d}\xi_i + \alpha_i \mathrm{d}\dot{\xi}_i)$$

The operator $\mathrm{d}/\mathrm{d}t$ defined on $\mathcal{E}$ induces the left skew polynomial ring $\mathcal{K}[s]$ of polynomials in s over $\mathcal{K}$ with the usual addition, and the (non-commutative) multiplication defined by the commutation rule

$$sa = as + \dot{a} \tag{3}$$

where $a \in \mathcal{K}$.
Thus, the ring $\mathcal{K}[s]$ represents the ring of linear ordinary differential operators that act on any $v \in \mathcal{E}$ as follows

$$\left(\sum_{i=0}^{k} \alpha_i s^i\right) v = \sum_{i=0}^{k} \alpha_i v^{(i)}$$

Note that, using the proposed definitions, the time-derivative operator $\mathrm{d}/\mathrm{d}t$ and (exterior) differential operator d commute

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} & \mathcal{E} \\ \mathrm{d}\uparrow & & \uparrow\mathrm{d} \\ \mathcal{K} & \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}t}} & \mathcal{K} \end{array}$$

Lemma 1. (Ore, 1931, 1933) For all non-zero $a, b \in \mathcal{K}[s]$, there exist non-zero $a_1, b_1 \in \mathcal{K}[s]$ such that $a_1 b = b_1 a$. Moreover, $\deg a_1$ and $\deg b_1$ are not greater than $\deg b + \deg a$.

To the (higher order) differential equation (2) one can now associate a polynomial from the ring $\mathcal{K}[s]$.
After differentiating (2) we get $\mathrm{d}y^{(n)} - \sum_{i=0}^{n-1} \frac{\partial F}{\partial y^{(i)}} \mathrm{d}y^{(i)} = 0$ or alternatively

$$a(s)\mathrm{d}y = 0 \tag{4}$$

where $a(s) = s^n - \sum_{i=0}^{n-1} \frac{\partial F}{\partial y^{(i)}} s^i$ is in $\mathcal{K}[s]$.

Example 2. Consider the system described by the differential equation

$$\ddot{y} = \dot{y} + \dot{y}^2/y$$

The polynomial description of the system can be obtained as

$$\begin{aligned} \ddot{y} - \dot{y} - \dot{y}^2/y &= 0 \\ \mathrm{d}\ddot{y} - (1 + 2\dot{y}/y)\mathrm{d}\dot{y} + \dot{y}^2/y^2\,\mathrm{d}y &= 0 \\ \left(s^2 - (1 + 2\dot{y}/y)s + \dot{y}^2/y^2\right)\mathrm{d}y &= 0 \end{aligned}$$

Remark 3. Note that in the mathematical setting employed in this paper, we are interested neither in local nor global, but in the so-called generic properties, i.e. in the properties that hold almost everywhere. Therefore, important point to notice is that though, for instance, dimension of a vector space can decrease on a set of singular points, it is of the same dimension everywhere else. Hence, we say it is of (generically) constant dimension.

2.1 Problem statement

We will focus on finding conditions under which for the higher-order differential equation (2) there exists a state-space representation (1) in the so-called feedforward form.

Definition 4. A system of the form (1) is said to be in the feedforward form, if

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \dots, x_n) \\ \dot{x}_2 &= f_2(x_2, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_n) \end{aligned} \tag{5}$$

where $\partial f_i/\partial x_{i+1} \neq 0$ for $i = 1, \dots, n-1$.

As will be shown in Section 3.1 any system of the form (5) can equivalently be transformed to the following simplified feedforward form

$$\begin{aligned} \dot{z}_1 &= \varphi_1(z_1, z_2) \\ \dot{z}_2 &= \varphi_2(z_2, z_3) \\ &\vdots \\ \dot{z}_n &= \varphi_n(z_n) \end{aligned} \tag{6}$$

where $\partial \varphi_i/\partial z_{i+1} \neq 0$ for $i = 1, \dots, n-1$.

Therefore, for the problem to be solvable it is not restrictive to look for the conditions under which the system (2) can be realized in the form (6).

Problem statement. For the nonlinear system (2) find, if possible, a state-space representation of the form (6).

3. REALIZATION IN THE FEEDFORWARD FORM

A necessary and sufficient condition for the problem to be solvable is based on the fact that whenever the polynomial $a(s)$ in (4) can be factorized in n factors (of the order 1), that is

$$a(s) = a_n(s) \dots a_2(s) a_1(s)$$

the respective one-forms $\omega_i = a_i(s) \dots a_1(s)\mathrm{d}y$, $i = 1, \dots, n$ are integrable.

The complete proof that the one-forms ω_i are always integrable is not trivial. It consists of proving several lemmas first. Its sketch is presented in Appendix.

Theorem 5. For the (higher-order) differential equation (2) there exists a state-space realization in the feedforward form (6) if and only if the polynomial $a(s)$ in (4) can be factorized as

$$a(s) = a_n(s) \dots a_2(s) a_1(s)$$

where $a_i(s) \in \mathcal{K}[s]$ and $\deg a_i(s) = 1$; $i = 1, \dots, n$.

Proof. *Sufficiency.* By the results presented in Appendix, the one-forms

$$\omega_i = a_i(s) \dots a_1(s) \mathrm{d}y$$

$i = 1, \dots, n$ are integrable ($\omega_n = a(s)\mathrm{d}y$ being exact). Hence, there exist n integrating factors $k_i \in \mathcal{K}$ such that

$$\mathrm{d}\varphi_i = k_i \omega_i$$

for some $\varphi_i \in \mathcal{K}$, $i = 1, \dots, n$. Note that $k_n = 1$ and $\varphi_n = y^{(n)} - F(y, \dots, y^{(n-1)})$.
Then the polynomial $a(s)$ can be written as

$$\begin{aligned} a(s) &= a_n(s) \dots \frac{1}{k_2} \cdot k_2 a_2(s) \frac{1}{k_1} \cdot k_1 a_1(s) \\ a(s) &= \tilde{a}_n(s) \dots \tilde{a}_2(s) \tilde{a}_1(s) \end{aligned}$$

where $\tilde{a}_1(s) = k_1 a_1(s)$, $\tilde{a}_i(s) = k_i a_i(s) \frac{1}{k_{i-1}}$, $i = 2, \dots, n$. That is

$$\mathrm{d}\varphi_i = \tilde{a}_i(s) \dots \tilde{a}_1(s) \mathrm{d}y \tag{7}$$

$i = 1, \dots, n$.

In other words, if the polynomial $a(s)$ can be factorized in n factors, $a(s) = a_n(s) \dots a_2(s) a_1(s)$, then there always exists a factorization $a(s) = \tilde{a}_n(s) \dots \tilde{a}_2(s) \tilde{a}_1(s)$ such that $\tilde{a}_i(s) \dots \tilde{a}_2(s) \tilde{a}_1(s) \mathrm{d}y$, $i = 1, \dots, n$, are exact one-forms.

From (7) it also implies that

$$\mathrm{d}\varphi_i = \tilde{a}_i(s) \mathrm{d}\varphi_{i-1}$$

$i = 1, \dots, n$ where we set $\mathrm{d}\varphi_0 := \mathrm{d}y$. Hence $\varphi_i = \varphi_i(\dot{\varphi}_{i-1}, \varphi_{i-1})$, where $\varphi_0 = y$.

Finally, the choice $x_i = \varphi_{i-1}$, $i = 1, \dots, n$, yields a realization in the feedforward form (6).

Necessity. It suffices to show that for the system (6) there exists a candidate for the system output which gives us a (higher-order) differential equation of the system that can be factorized in n factors (of the order 1).

Such a candidate can simply be chosen as (any nonzero) function of z_1, $y = h(z_1)$. After differentiating (6) and $y = h(z_1)$ we get

$$\begin{aligned} \mathrm{d}\dot{z}_1 &= a_{11} \mathrm{d}z_1 + a_{12} \mathrm{d}z_2 \\ \mathrm{d}\dot{z}_2 &= a_{22} \mathrm{d}z_2 + a_{23} \mathrm{d}z_3 \\ &\vdots \\ \mathrm{d}\dot{z}_n &= a_{nn} \mathrm{d}z_n \\ \mathrm{d}y &= c_1 \mathrm{d}z_1 \end{aligned}$$

or alternatively

$$\begin{aligned} (s - a_{11}) \mathrm{d}z_1 &= a_{12} \mathrm{d}z_2 \\ (s - a_{22}) \mathrm{d}z_2 &= a_{23} \mathrm{d}z_3 \\ &\vdots \\ (s - a_{nn}) \mathrm{d}z_n &= 0 \\ \mathrm{d}y &= c_1 \mathrm{d}z_1 \end{aligned} \tag{8}$$

where $a_{ij} = \partial\varphi_i / \partial z_j$, $i, j = 1, \dots, n$, and $c_1 = \partial h / \partial z_1$.

To obtain a polynomial description of the higher-order differential equation, we eliminate all $\mathrm{d}z_i$'s.

Step 1
We start with

$$\mathrm{d}y = c_1 \mathrm{d}z_1$$

and eliminate $\mathrm{d}z_1$.
By Ore condition (Lemma 1) there exist $(s - \alpha_1)$ and γ_1 in $\mathcal{K}[s]$ such that [1] $(s - \alpha_1)c_1 = \gamma_1(s - a_{11})$. Hence

$$(s - \alpha_1)\mathrm{d}y = \gamma_1 (s - a_{11}) \mathrm{d}z_1$$

and by (8)

$$(s - \alpha_1)\mathrm{d}y = c_2 \mathrm{d}z_2$$

where $c_2 = \gamma_1 a_{12}$.

Step k+1
Suppose that after Step k we have

$$(s - \alpha_k) \dots (s - \alpha_2)(s - \alpha_1)\mathrm{d}y = c_{k+1} \mathrm{d}z_{k+1}$$

We eliminate $\mathrm{d}z_{k+1}$.
By Ore condition there exist $(s - \alpha_{k+1})$ and γ_{k+1} in $\mathcal{K}[s]$ such that $(s - \alpha_{k+1})c_{k+1} = \gamma_{k+1}(s - a_{k+1,k+1})$. Hence

$$(s - \alpha_{k+1}) \dots (s - \alpha_1)\mathrm{d}y = \gamma_{k+1}(s - a_{k+1,k+1}) \mathrm{d}z_{k+1}$$

and by (8)

$$(s - \alpha_{k+1}) \dots (s - \alpha_2)(s - \alpha_1)\mathrm{d}y = c_{k+2} \mathrm{d}z_{k+2}$$

where $c_{k+2} = \gamma_{k+1} a_{k+1,k+2}$.

After *Step n*, we have

$$(s - \alpha_n) \dots (s - \alpha_2)(s - \alpha_1)\mathrm{d}y = 0 \tag{9}$$

Hence, the polynomial description (4) of the higher-order differential equation of the system (6), for the chosen output $y = h(z_1)$, can be factorized in n factors. □

Example 6. Consider the system from Example 2

$$\ddot{y} = \dot{y}^2/y + \dot{y}$$

with the polynomial description

$$\left(s^2 - (1 + 2\dot{y}/y)s + \dot{y}^2/y^2\right) \mathrm{d}y = 0$$

The polynomial $a(s) = s^2 - (1 + 2\dot{y}/y)s + \dot{y}^2/y^2$ can be factorized

$$(s - \dot{y}/y - 1)(s - \dot{y}/y)\mathrm{d}y = 0$$

The one-forms

$$\begin{aligned} \omega_1 &= (s - \dot{y}/y)\mathrm{d}y \\ \omega_2 &= (s - \dot{y}/y - 1)(s - \dot{y}/y)\mathrm{d}y \end{aligned}$$

are integrable (ω_2 being exact). The integrating factor for ω_1 is $1/y$

$$\frac{1}{y}\omega_1 = \frac{1}{y}\mathrm{d}\dot{y} - \frac{\dot{y}}{y^2}\mathrm{d}y = \mathrm{d}\left(\frac{\dot{y}}{y}\right)$$

Hence, $\varphi_0 = y$ and $\varphi_1 = \dot{y}/y$. Thus, the choice

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y}/y \end{aligned}$$

yields a realization in the feedforward form (6)

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= x_2 \end{aligned}$$

3.1 *Equivalence of the feedforward forms*

Now, one can show that it is not restrictive to look for a realization in the form (6), rather than in the form (5).

[1] From $(s - \alpha_1)c_1 = \gamma_1(s - a_{11})$ one has $c_1 = \gamma_1$, $\dot{c}_1 - \alpha_1 = -\gamma_1 a_{11}$. This set of equations can always be solved for γ_1 and α_1.

Theorem 7. For any system in the feedforward form (5) there exists a transformation $z = \phi(x)$ that transforms the system into the feedforward form (6).

Proof. The proof is an adaption of the proof of the necessity part of Theorem 5. It suffices to show that there exists a candidate for the system output which gives a higher-order differential equation of the system that can be factorized in n factors (of the order 1). Again, such a candidate can simply be chosen as (any nonzero) function of x_1, $y = h(x_1)$. Then after differentiating (5) and $y = h(x_1)$, we have

$$\begin{aligned}
\mathrm{d}\dot{x}_1 &= a_{11}\mathrm{d}x_1 + a_{12}\mathrm{d}x_2 + \ldots + a_{1n}\mathrm{d}x_n \\
\mathrm{d}\dot{x}_2 &= a_{22}\mathrm{d}x_2 + \ldots + a_{2n}\mathrm{d}x_n \\
&\vdots \\
\mathrm{d}\dot{x}_n &= a_{nn}\mathrm{d}x_n \\
\mathrm{d}y &= c_1\mathrm{d}x_1
\end{aligned}$$

or alternatively

$$\begin{aligned}
(s - a_{11})\mathrm{d}x_1 &= a_{12}\mathrm{d}x_2 + a_{13}\mathrm{d}x_3 + \ldots + a_{1n}\mathrm{d}x_n \\
(s - a_{22})\mathrm{d}x_2 &= a_{23}\mathrm{d}x_3 + \ldots + a_{2n}\mathrm{d}x_n \\
&\vdots \\
(s - a_{nn})\mathrm{d}x_n &= 0 \\
\mathrm{d}y &= c_1\mathrm{d}x_1
\end{aligned} \tag{10}$$

where $a_{ij} = \partial f_i/\partial x_j$, $i, j = 1, \ldots, n$, and $c_1 = \partial h/\partial x_1$.

To obtain a polynomial description of the higher-order differential equation, all $\mathrm{d}x_i$'s are eliminated.

Step 1
We start with

$$\mathrm{d}y = c_1\mathrm{d}x_1$$

and eliminate $\mathrm{d}x_1$.
By Ore condition there exist $(s - \alpha_1)$ and γ_1 in $\mathcal{K}[s]$ such that $(s - \alpha_1)c_1 = \gamma_1(s - a_{11})$. Hence

$$(s - \alpha_1)\mathrm{d}y = \gamma_1(s - a_{11})\mathrm{d}x_1$$

and by (10)

$$(s - \alpha_1)\mathrm{d}y = c_2\mathrm{d}x_2 + c_3\mathrm{d}x_3 + \ldots + c_n\mathrm{d}x_n$$

where $c_i = \gamma_1 a_{1i}$, $i = 2, \ldots, n$.

Step k+1
Suppose that after Step k we have

$$\begin{aligned}
(s - \alpha_k) &\ldots (s - \alpha_2)(s - \alpha_1)\mathrm{d}y = \\
&= c_{k+1}\mathrm{d}x_{k+1} + c_{k+2}\mathrm{d}x_{k+2} + \cdots + c_n\mathrm{d}x_n
\end{aligned}$$

We eliminate $\mathrm{d}x_{k+1}$.
By Ore condition there exist $(s - \alpha_{k+1})$ and γ_{k+1} in $\mathcal{K}[s]$ such that $(s - \alpha_{k+1})c_{k+1} = \gamma_{k+1}(s - a_{k+1,k+1})$. Hence

$$\begin{aligned}
(s - \alpha_{k+1}) \; &(s - \alpha_k)\ldots(s - \alpha_2)(s - \alpha_1)\mathrm{d}y = \\
&= \gamma_{k+1}(s - a_{k+1,k+1})\mathrm{d}x_{k+1} + \\
&\quad + (s - \alpha_{k+1})[c_{k+2}\mathrm{d}x_{k+2} + \cdots + c_n\mathrm{d}x_n]
\end{aligned}$$

and by (10)

$$\begin{aligned}
(s - \alpha_{k+1}) \; &(s - \alpha_k)\ldots(s - \alpha_2)(s - \alpha_1)\mathrm{d}y = \\
&= \gamma_{k+1}(a_{k+1,k+2}\mathrm{d}x_{k+2} + \cdots + a_{k+1,n}\mathrm{d}x_n) \\
&\quad + (s - \alpha_{k+1})[c_{k+2}\mathrm{d}x_{k+2} + \cdots + c_n\mathrm{d}x_n]
\end{aligned}$$

Since $s\,\mathrm{d}x_i = a_{ii}\mathrm{d}x_i + \cdots + a_{in}\mathrm{d}x_n$ for all $i = 1, \ldots, n$, one, after substituting and rearrangement, gets

$$\begin{aligned}
(s - \alpha_{k+1})(s - \alpha_k)\ldots \; &(s - \alpha_2)(s - \alpha_1)\mathrm{d}y = \\
&= c_{k+2}\mathrm{d}x_{k+2} + \cdots + c_n\mathrm{d}x_n
\end{aligned}$$

for some $c_{k+2}, \ldots, c_n$ in $\mathcal{K}$.

After **Step n**, we have

$$(s - \alpha_n)\ldots(s - \alpha_2)(s - \alpha_1)\mathrm{d}y = 0 \tag{11}$$

Hence, the polynomial description (4) of the higher-order differential equation of the system (5), for the chosen output $y = h(x_1)$, can be factorized in n factors.

Finally, applying Theorem 5 this implies that there exists a transformation $z = \phi(x)$ which transforms the system (5) with $y = h(x_1)$ to the system (6) with $y = z_1$. □

Example 8. Consider the system

$$\begin{aligned}
\dot{x}_1 &= x_2x_3 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_3
\end{aligned}$$

which is in the form (5). One can choose $y = x_1$, for instance. Then the higher-order differential equation can be computed as

$$\begin{aligned}
y &= x_1 \\
\dot{y} &= x_2x_3 \\
\ddot{y} &= x_3^2 + x_2x_3 = x_3^2 + \dot{y} \\
y^{(3)} &= 2x_3^2 + \ddot{y} = 3\ddot{y} - 2\dot{y}
\end{aligned}$$

The polynomial description of the equation can be factorized

$$\begin{gathered}
y^{(3)} - 3\ddot{y} + 2\dot{y} = 0 \\
(s^3 - 3s^2 + 2s)\,\mathrm{d}y = 0 \\
(s - 2)(s - 1)s\,\mathrm{d}y = 0
\end{gathered}$$

The choice $\mathrm{d}z_1 = \mathrm{d}y$, $\mathrm{d}z_2 = s\mathrm{d}y$, $\mathrm{d}z_2 = (s - 1)s\mathrm{d}y$, i.e. $(z_1, z_2, z_3) = (y, \dot{y}, \ddot{y} - \dot{y})$ gives a realization in the form (6)

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_2 + z_3 \\
\dot{z}_3 &= 2z_3
\end{aligned}$$

The corresponding transformation can be found from $y = x_1 = z_1$, $\dot{y} = x_2x_3 = z_2$, and $\ddot{y} = x_3^2 + x_2x_3 = z_2 + z_3$. That is, $(z_1, z_2, z_3) = (x_1, x_2x_3, x_3^2)$.

4. TRANSFORMATION TO THE FEEDFORWARD FORM

An interesting problem is to find a (necessary and sufficient) condition under which a system of the form (1) can be transformed to the feedforward form (6) (it has been shown that it is not restrictive to look for a realization a nonlinear system in such a simplified feedforward form).

To find a solution one can apply the ideas of the realization problem presented in Section 3.

Theorem 9. For a system of the form (1) there exists a state transformation $z = \phi(x)$ which transforms the system into the feedforward form (6) if and only if there exists a function $h(x) \in \mathcal{K}$, a candidate for the output $y = h(x)$, such that it is fully observable, i.e.

$$\mathrm{rank}_{\mathcal{K}}\left(\frac{\partial(y, \dot{y}, \ldots, y^{(n-1)})}{\partial x}\right) = n \tag{12}$$

and the conditions of Theorem 5 are fulfilled.

Proof. First, note that for a state-space system (1), with some output function $y = h(x)$, a state transformation $z = \phi(x)$ preserves the observability of the output y. That is, $\mathrm{rank}_{\mathcal{K}}\left(\partial(y, \dot{y}, \ldots, y^{(n-1)})/\partial x\right)$ is invariant under $z = \phi(x)$.
For the system (6) any nonzero function of z_1, $y = h(z_1)$,

is fully observable, i.e. $\text{rank}_{\mathcal{K}}\left(\partial(y,\dot{y},\ldots,y^{(n-1)})/\partial z\right)=n$. Therefore, a necessary condition for the system (1) to be transformable into the feedforward form (6) is the existence of a function $h(x)$ (a candidate for the output $y=h(x)$) such that (12) holds. Then, a necessary and sufficient condition is given by applying the results of Theorem 5. □

In general, the problem is how to find, if possible, for the system (1) a function $h(x)$ (a candidate for the output) such that condition (12) holds. In practice, it is usually not difficult to find such a function and it seems it "almost always" exists. However, at the moment the authors are not aware of any procedure or algorithm that will give such a function for the system (1), if it exists. This forms an open problem left for future research. In what follows, we give some discussion about the topic, though.

A fully observable output does not exist for the system

$$\dot{x}=0$$

where $x\in\mathbb{R}^n$ and $n>1$. Clearly, for any (nonzero) function $h(x)$ one has $\dot{h}(x)=0$, and rank of the matrix in (12) always equals 1.

However, if at least one of the zeros is replaced by a nonzero constant we can find a fully observable output function. Consider, for example, the system

$$\begin{aligned}\dot{x}_1&=0\\ \dot{x}_2&=1\end{aligned}$$

Then, for instance, for $y=x_1x_2$ we have $\dot{y}=x_1$, and rank of the matrix in (12) equals 2.

Since one, in general, looks for $h(x)$ in a class of nonlinear functions, it is possible to find a fully observable output even if there is none when the attention is restricted to the linear theory. This gives a chance to transform a linear system to the feedforward form by a nonlinear change of coordinates, though this might not be possible by any linear change of coordinates. This is demonstrated by the following example.

Example 10. Consider the system

$$\begin{aligned}\dot{x}_1&=x_1\\ \dot{x}_2&=x_2\end{aligned}$$

For any linear function of x_1 and x_2, i.e. $y=c_1x_1+c_2x_2$ with nonzero $c_1,c_2\in\mathbb{R}$, one has $\dot{y}=c_1x_1+c_2x_2$, and thus rank of the matrix in (12) equals 1. This also implies there is no linear transformation that could transform the system in to the feedforward form (any linear transformation $z=kx$ gives $\dot{z}_1=z_1$ and $\dot{z}_2=z_2$).

Nevertheless, if for instance $y=x_1+x_2^2$ then $\dot{y}=x_1+2x_2^2$, and rank of the matrix in (12) is 2.
In this case the higher-order differential equation can be found as

$$\begin{aligned}y&=x_1+x_2^2\\ \dot{y}&=x_1+2x_2^2=y+x_2^2\\ \ddot{y}&=\dot{y}+2x_2^2=\dot{y}+2(\dot{y}-y)=3\dot{y}-2y\end{aligned}$$

The corresponding polynomial description can be factorized

$$\begin{gathered}\ddot{y}-3\dot{y}+2y=0\\ (s^2-3s+2)\,\mathrm{d}y=0\\ (s-1)(s-2)\,\mathrm{d}y=0\end{gathered}$$

Therefore, the choice $\mathrm{d}z_1=\mathrm{d}y$, $\mathrm{d}z_2=(s-2)\mathrm{d}y$, i.e. $(z_1,z_2)=(y,\dot{y}-2y)$ gives a realization in the feedforward form

$$\begin{aligned}\dot{z}_1&=2z_1+z_2\\ \dot{z}_2&=z_2\end{aligned}$$

The corresponding transformation can be found from $y=x_1+x_2^2=z_1$ and $\dot{y}=x_1+2x_2^2=2z_1+z_2$. That is, $(z_1,z_2)=(x_1+x_2^2,-x_1)$.

5. CONCLUSIONS

This paper studied the problem of a realization of a nonlinear higher-order differential equation in the feedforward form. First, it was shown that it is not restrictive to look for a realization in the simplified feedforward form. Then the existence of the solution only requires that the corresponding polynomial description of the higher-order differential equation can be factorized, for the respective one-forms associated to the factorization are always integrable. The results were also applied to derive a necessary and sufficient condition for the transformation of a nonlinear state-space system into the feedforward form. In this case an interesting result is that one can possibly transform a linear system into the feedforward form by a nonlinear change of coordinates while this might not be possible by any linear change of coordinates. An open problem here consists of finding, if possible, a candidate for the system output such that it is fully observable.

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Appendix A. INTEGRABILITY

Let K be the field of meromorphic functions of variables $\{y^{(k)}; k \geq 0\}$, and define $\frac{\mathrm{d}}{\mathrm{d}t}y^{(k)} = y^{(k+1)}$. That is, all $y^{(k)}$, $k \geq 0$, are independent. Define $E = \operatorname{span}_K\{\mathrm{d}\xi; \xi \in K\}$, and let $K[s]$ be the corresponding ring of skew polynomials that act as differential operators over E.

Lemma 11. Let ν be a one-form in E. Then ν is exact if and only if $\dot{\nu}$ is exact.

Proof. *Sufficiency.* We show that ν non-exact implies $\dot{\nu}$ is non-exact.
Assume that $\nu = \sum_i c_i \mathrm{d}y^{(i)} \neq 0$ is not exact, that is

$$\mathrm{d}\nu = \sum_i \mathrm{d}(c_i) \wedge \mathrm{d}y^{(i)} = \sum_{i,j} \frac{\partial c_i}{\partial y^{(j)}} \mathrm{d}y^{(j)} \wedge \mathrm{d}y^{(i)} \neq 0$$

Since $\mathrm{d}y^{(j)} \wedge \mathrm{d}y^{(i)} = -\mathrm{d}y^{(i)} \wedge \mathrm{d}y^{(j)}$ and $\mathrm{d}y^{(i)} \wedge \mathrm{d}y^{(i)} = 0$ for all i, j, one can write $\mathrm{d}\nu$ in the canonical representation

$$\mathrm{d}\nu = \sum_{i>j} a_{ij} \mathrm{d}y^{(j)} \wedge \mathrm{d}y^{(i)} \neq 0$$

where i is strictly greater than j, and at least one of $a_{ij} = \partial c_i/\partial y^{(j)} - \partial c_j/\partial y^{(i)}$ is non-zero. Denote by k the maximum number of i and by l the maximum number of j such that $a_{kl} \neq 0$.
Then

$$\begin{aligned}\mathrm{d}\dot{\nu} = \sum_{i>j} (\dot{a}_{ij} \mathrm{d}y^{(j)} \wedge \mathrm{d}y^{(i)} \; + \; a_{ij} \mathrm{d}y^{(j+1)} \wedge \mathrm{d}y^{(i)} \\ + \; a_{ij} \mathrm{d}y^{(j)} \wedge \mathrm{d}y^{(i+1)})\end{aligned}$$

and since $a_{kl} \neq 0$ one has $a_{kl}\mathrm{d}y^{(l)} \wedge \mathrm{d}y^{(k+1)} \neq 0$ which implies $\mathrm{d}\dot{\nu} \neq 0$.

Necessity. Note that ν exact implies there exists φ in K such that $\mathrm{d}\varphi = \nu$. Since $\dot{\varphi}$ is also in K and $\mathrm{d}\dot{\varphi} = \dot{\nu}$, one has $\dot{\nu}$ is exact. □

In a similar manner one can show the corresponding results for the integrability.

Lemma 12. Let ν be a one-form in E. Then the following statements are equivalent

- $\operatorname{span}_K\{\nu\}$ is integrable,
- $\operatorname{span}_K\{\dot{\nu}\}$ is integrable,
- $\operatorname{span}_K\{\nu, \dot{\nu}\}$ is integrable.

Proof. The proof is omitted due to the lack of space. □

Lemma 13. Let ω be an exact one-form in E and assume $\omega = (s + \alpha)\nu$ for some $s + \alpha$ in $K[s]$ and ν in E. Then ν is an integrable one-form.

Proof. For $\alpha = 0$ it implies from Lemma 11 that ν must be an exact one-form, therefore also integrable.

For $\alpha \neq 0$. By assumption

$$\omega = (s + \alpha)\nu = \dot{\nu} + \alpha\nu$$

is an exact one-form. That is

$$\mathrm{d}(\dot{\nu} + \alpha\nu) = \mathrm{d}\dot{\nu} + \mathrm{d}\alpha \wedge \nu + \alpha \mathrm{d}\nu = 0$$

Hence

$$\mathrm{d}\dot{\nu} \wedge \nu \wedge \dot{\nu} + \mathrm{d}\alpha \wedge \nu \wedge \nu \wedge \dot{\nu} + \alpha \mathrm{d}\nu \wedge \nu \wedge \dot{\nu} = 0$$

$$\mathrm{d}\dot{\nu} \wedge \nu \wedge \dot{\nu} + \alpha \mathrm{d}\nu \wedge \nu \wedge \dot{\nu} = 0 \quad \text{(A.1)}$$

From (A.1) either both summands are zero or both are nonzero. If

$$\begin{aligned} \mathrm{d}\dot{\nu} \wedge \nu \wedge \dot{\nu} &\neq 0 \\ \mathrm{d}\nu \wedge \nu \wedge \dot{\nu} &\neq 0 \end{aligned}$$

then by Frobenius theorem $\operatorname{span}_K\{\nu, \dot{\nu}\}$ is completely non-integrable. That is, there does not exist any linear combination of ν and $\dot{\nu}$ being an exact one-form, which is however a contradiction, for $\dot{\nu} + \alpha\nu$ is an exact one-form by assumption.
Therefore, from (A.1) one necessarily has

$$\begin{aligned} \mathrm{d}\dot{\nu} \wedge \nu \wedge \dot{\nu} &= 0 \\ \mathrm{d}\nu \wedge \nu \wedge \dot{\nu} &= 0 \end{aligned}$$

Hence, by Frobenius theorem $\operatorname{span}_K\{\nu, \dot{\nu}\}$ is integrable and by Lemma 12 so is ν. □

Lemma 14. Let $\omega = a_0 \mathrm{d}y + \cdots + a_{r+1}\mathrm{d}y^{(r+1)}$ be an exact one-form, where every a_i is a meromorphic function of $y, \ldots, y^{(r+1)}$, and $a_{r+1} \neq 0$. Assume $\omega = (s + \alpha)\nu$ for some meromorphic function α of $y, \ldots, y^{(r+1)}$ and ν in E. Then ν is integrable and can be expressed as $\nu = b_0 \mathrm{d}y + \cdots + b_r \mathrm{d}y^{(r)}$, where every b_i is a meromorphic function of $y, \ldots, y^{(r)}$, and $b_r \neq 0$.

Proof. From Lemma 13, ν is integrable. Because of $\omega = (s + a)\nu$, ν can be represented as $\nu = b_0 dy + \cdots + b_r dy^{(r)}$, where $b_i \in K$ $(i = 0, 1, \ldots, r)$. Since ν is integrable, there exist an integrating element $c \in K$ and a meromorphic function φ of $y, \ldots, y^{(r)}$ such that $\mathrm{d}\varphi = c\nu$. It suffices to show that c is a meromorphic function of $y, \ldots, y^{(r)}$. For $\mathrm{d}\varphi$, there exist meromorphic functions $\bar{a}$ and $\bar{b}$ of $y, \ldots, y^{(r+1)}$ such that $(\bar{a}s + \bar{b})\mathrm{d}\varphi = \omega$. Thus, we obtain

$$\begin{aligned}(\bar{a}s + \bar{b})\mathrm{d}\varphi = \omega = (s + a)\nu = (s + a)c^{-1}\mathrm{d}\varphi \\ = (c^{-1}s + ac^{-1} + \dot{(c^{-1})})\mathrm{d}\varphi.\end{aligned}$$

Thus, $\bar{a} = c^{-1}$ and $\bar{b} = ac^{-1} + \dot{(c^{-1})} = a\bar{a} + \dot{\bar{a}}$ hold. Since a and $\bar{b}$ are meromorphic functions of $y, \ldots, y^{(r+1)}$, $\bar{a} = c^{-1}$ is a meromorphic function of $y, \ldots, y^{(r)}$.

Finally, by recursively applying Lemma 14, one can prove the integrability of the one-forms ω_i in the proof of Theorem 5.
First, note that ω_{n-1}, satisfying

$$\mathrm{d}(y^{(n)} - F(y, \dot{y}, \ldots, y^{(n-1)})) = (s + a_n)\omega_{n-1}$$

for some $s + a_n$ in $\mathcal{K}[s]$, does not depend on $\{y^{(k)}, k \geq n\}$.
And second, $\omega_i = a_i(s)\omega_{i-1}$ with $\deg a_i(s) = 1$ for $i = 1, \ldots, n$.