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Computation of the Largest Constraint Admissible Set for Linear Continuous-Time Systems with State and Input Constraints ⋆

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Abstract: We present a new method for the approximation of the largest constraint admissible set (CAS) for linear continuous-time systems with state and input constraints. The CAS is the set of initial states for which the controlled system does not violate the input or state constraints. The presented approach is based on a suitable discretization of the continuous-time system. In fact, we will show that CAS in the continuous-time case can be computed analogously to the discrete-time case, given an appropriate sampling time was chosen. We stress that the computation of CAS for continuous-time systems is considerably more difficult than for discrete-time systems, since one has to guarantee that the system does not violate the constraints in between the sampling instances.

Keywords: linear continuous-time systems, constraints, terminal control, predictive control

1. INTRODUCTION

Constraint admissible sets (CAS) are important for the analysis of controlled systems with constraints. The CAS is defined to be the set of initial conditions x_0 for which the closed-loop trajectory emanating from x_0 does not violate the input and state constraints (see, e.g, Gilbert and Tan (1991) or Wang et al. (2009)). CAS are of special interest in the framework of constrained linear quadratic regulation (Scokaert and Rawlings, 1998) or model predictive control (MPC). In fact, CAS are instrumental for reformulating the linear-quadratic infinite-horizon optimal control problem as a finite-dimensional optimization problem (see, e.g., Sznaier and Damborg (1987), Chmielewski and Manousiouthakis (1996) or Scokaert and Rawlings (1998)). With regard to MPC, CAS serve as terminal sets guaranteeing (asymptotic) stability of the controlled system (De Doná et al., 2002).

The paper deals with the computation of the largest CAS for constrained linear systems. In the discrete-time case, an iterative procedure for the exact computation of the largest CAS, which we denote by $\widehat{\mathcal{C}}_\infty$ (where the accent "$\widehat{\ }$" indicates the discrete-time case), is stated in Gilbert and Tan (1991). We extend the findings in Gilbert and Tan (1991) by providing an upper bound for the required number of iterations to obtain $\widehat{\mathcal{C}}_\infty$. Based on the results for discrete-time systems, we tackle the more complex continuous-time case. Of course, for a continuous-time system, an outer approximation of the largest CAS (denoted by $\mathcal{C}_\infty$) can be found by discretizing the system and computing $\widehat{\mathcal{C}}_\infty$ for the resulting discrete-time system (Gilbert and Tan, 1991). Unfortunately, outer approximations $\overline{\mathcal{C}}_\infty \supseteq \mathcal{C}_\infty$ are of little use, since they may contain initial states $x_0 \in \overline{\mathcal{C}}_\infty$ for which the closed-loop trajectory emanating from x_0 violates the input or state constraints. Here, we intend to derive a method for the computation of tight inner approximations $\underline{\mathcal{C}}_\infty$ of $\mathcal{C}_\infty$. Precisely, for any given $\epsilon \in (0,1)$, we show how to compute a set $\underline{\mathcal{C}}_\infty$ such that

$$\underline{\mathcal{C}}_\infty \subseteq \mathcal{C}_\infty \quad \text{and} \quad (1-\epsilon)\,\mathcal{C}_\infty \subseteq \underline{\mathcal{C}}_\infty. \tag{1}$$

Essentially, we provide a criterion for an appropriate choice of the sampling time (see Eq. (29)).

The paper is organized as follows. We introduce some notation and preliminaries in the remainder of this section. Afterwards, we recall how to calculate the largest CAS for constrained discrete-time systems in the first part of Sect. 2. We extend the established results by providing an upper bound for the number of iterations necessary to identify $\widehat{\mathcal{C}}_\infty$. The main result of the paper, i.e., the computation of a tight inner approximation of $\mathcal{C}_\infty$, is presented in Sect. 3. Finally, we analyze two numerical examples and state conclusions in Sects. 4 and 5, respectively.

1.1 Preliminaries and Notation

We denote matrices by capital letters, vectors and scalars by lowercase letters and sets by calligraphic letters. Let $A, P \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ with $P = P^T$ and $n, m \in \mathbb{N}$. By $\|A\|_2$ and $\mu_2(A)$ denote the spectral norm and the associated logarithmic norm (see Söderlind (2006)) of A. By $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the smallest and the largest eigenvalue of the (symmetric) matrix P, respectively. By $\mathcal{B}(r) := \{x \in \mathbb{R}^n \,|\, \|x\|_2 \le r\}$, denote a ball in $\mathbb{R}^n$ with radius $r \in \mathbb{R}_+$, where $\mathbb{R}_+ := \{r \in \mathbb{R} \,|\, r > 0\}$. Let $\mathcal{E}(P,\rho) := \{x \in \mathbb{R}^n \,|\, \|x\|_P^2 \le \rho^2\}$ be shorthand notation for an ellipsoid, where $\|x\|_P^2 := x^T P x$ and where $\rho \in \mathbb{R}_+$. Define $\mathbb{R}_0 := \{r \in \mathbb{R} \,|\, r \ge 0\}$, $\mathbb{N}_j := \{i \in \mathbb{N} \,|\, j \le i\}$ and $\mathbb{N}_{j,k} := \{i \in \mathbb{N} \,|\, j \le i \le k\}$. Finally, let $\mathcal{T} \subseteq \mathbb{R}^n$ and $\lambda \in (0,1)$ and define $\lambda\mathcal{T} := \{\lambda x \,|\, x \in \mathcal{T}\}$. The set $\mathcal{T}$ is called A-invariant, if $Ax \in \mathcal{T}$ for every $x \in \mathcal{T}$. The following lemmas summarize some properties which we use in later sections.

Lemma 1. Let $\mathcal{T} \subseteq \mathbb{R}^n$ be A-invariant and let $\epsilon \in (0,1)$. Then, (i) $(1-\epsilon)\,\mathcal{T}$ is A-invariant and (ii) $A^k x \in \mathcal{T}$ for every $x \in \mathcal{T}$ and every $k \in \mathbb{N}$.

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Lemma 2. Let $\mathcal{T} \subset \mathbb{R}^n$ be a convex and compact set with $0 \in \mathrm{int}(\mathcal{T})$, let $\epsilon \in (0,1)$ and let $\underline{r} \in \mathbb{R}_+$ be such that $\mathcal{B}(\underline{r}) \subseteq \mathcal{T}$. Let $x \in (1-\epsilon)\mathcal{T}$ and let $\Delta x \in \mathbb{R}^n$. If $\|\Delta x\|_2 \leq \epsilon\, \underline{r}$, then $x + \Delta x \in \mathcal{T}$.

2. THE DISCRETE-TIME CASE

In this section, we consider discrete-time linear systems

$$\hat{x}(k+1) = \widehat{A}\,\hat{x}(k) + \widehat{B}\,\hat{u}(k), \quad \hat{x}(0) = \hat{x}_0 \tag{2}$$

with input and state constraints of the form

$$\hat{u}(k) \in \widehat{\mathcal{U}} \quad \text{and} \quad \hat{x}(k) \in \widehat{\mathcal{X}} \quad \text{for every } k \in \mathbb{N}, \tag{3}$$

where $\widehat{A} \in \mathbb{R}^{n\times n}$ and $\widehat{B} \in \mathbb{R}^{n\times m}$ and where $\widehat{\mathcal{U}} \subset \mathbb{R}^m$ and $\widehat{\mathcal{X}} \subset \mathbb{R}^n$ are assumed to be convex and compact sets with the origin as an interior point. We further assume that a linear control law

$$\hat{u}(k) = -\widehat{K}\,\hat{x}(k) \tag{4}$$

with $\widehat{K} \in \mathbb{R}^{n\times m}$ is given, such that the closed-loop system

$$\hat{x}(k+1) = (\widehat{A} - \widehat{B}\,\widehat{K})\,\hat{x}(k), \quad \hat{x}(0) = \hat{x}_0 \tag{5}$$

is Schur stable. In the following, we recall the procedure for the computation of the largest CAS in the discrete-time case. In order to clearly divide between established results and new findings, we introduce the subsections *State of the art* and *Improvement* (here and in Sect. 3).

2.1 State of the art: Exact computation of $\widehat{\mathcal{C}}_\infty$

Let $\hat{\varphi}(k, \hat{x}_0)$ denote the solution of the closed-loop system (5) at step $k \in \mathbb{N}$ for initial condition x_0 and define the set

$$\widehat{\mathcal{C}}_{\max} := \{\hat{x} \in \widehat{\mathcal{X}} \,|\, -\widehat{K}\,\hat{x} \in \widehat{\mathcal{U}}\}, \tag{6}$$

which collects all states $\hat{x} \in \mathcal{X}$ for which the application of the feedback law (4) does not violate the inputs constraints $\mathcal{U}$. Then, the largest CAS $\widehat{\mathcal{C}}_\infty$ can be expressed as (Gilbert and Tan, 1991)

$$\widehat{\mathcal{C}}_\infty := \{\hat{x}_0 \in \widehat{\mathcal{C}}_{\max} \,|\, \hat{\varphi}(k, \hat{x}_0) \in \widehat{\mathcal{C}}_{\max}, \forall k \in \mathbb{N}\}. \tag{7}$$

We collect some statements which are important for the characterization of $\widehat{\mathcal{C}}_\infty$.

Lemma 3. Let $\mathcal{T} \subseteq \widehat{\mathcal{C}}_{\max}$ be $\widetilde{A}$-invariant with $\widetilde{A} = \widehat{A} - \widehat{B}\,\widehat{K}$. Then, $\mathcal{T} \subseteq \widehat{\mathcal{C}}_\infty$.

The proof of Lem. 3 can be found in Athans and Falb (1966) as cited in Chmielewski and Manousiouthakis (1996). Now, let $\widehat{\mathcal{C}}_i$ denote the set of all initial states $\hat{x}_0 \in \widehat{\mathcal{C}}_{\max}$ for which the closed-loop trajectory $\hat{\varphi}(k, \hat{x}_0)$ respects the constraints for at least the first $i \in \mathbb{N}$ steps. Formally,

$$\widehat{\mathcal{C}}_i := \{\hat{x}_0 \in \widehat{\mathcal{C}}_{\max} \,|\, \hat{\varphi}(k, \hat{x}_0) \in \widehat{\mathcal{C}}_{\max}, \forall k \in \mathbb{N}_{0,i}\}. \tag{8}$$

Then

$$\widehat{\mathcal{C}}_\infty \subseteq \widehat{\mathcal{C}}_{i+1} \subseteq \widehat{\mathcal{C}}_i \subseteq \widehat{\mathcal{C}}_{\max} \tag{9}$$

for every $i \in \mathbb{N}$ (cf. relation (2.2) in Gilbert and Tan (1991)). Lemma 4 states an interesting implication.

Lemma 4. Let $i \in \mathbb{N}$. If $\widehat{\mathcal{C}}_i = \widehat{\mathcal{C}}_{i+1}$, then $\widehat{\mathcal{C}}_i$ is $\widetilde{A}$-invariant with $\widetilde{A} = \widehat{A} - \widehat{B}\widehat{K}$.

The statement in Lem. 4 results from the proof of Thm. 2.2 in Gilbert and Tan (1991). Combining Lems. 3, 4 and relation (9) leads to Thm. 5, which is equivalent to Thm. 2.2 in Gilbert and Tan (1991).

Theorem 5. Let $i \in \mathbb{N}$. If $\widehat{\mathcal{C}}_i = \widehat{\mathcal{C}}_{i+1}$, then $\widehat{\mathcal{C}}_\infty = \widehat{\mathcal{C}}_i$.

Theorem 5 suggests to implement the following algorithm for the iterative computation of $\widehat{\mathcal{C}}_\infty$ (cf. Alg. 3.1 in Gilbert and Tan (1991)).

Algorithm 1. Exact computation of $\widehat{\mathcal{C}}_\infty$.

(1) Set $i = 0$ and $\widehat{\mathcal{C}}_0 = \widehat{\mathcal{C}}_{\max}$.
(2) Compute $\widehat{\mathcal{C}}_{i+1}$.
(3) If $\widehat{\mathcal{C}}_i = \widehat{\mathcal{C}}_{i+1}$, return $\widehat{\mathcal{C}}_\infty = \widehat{\mathcal{C}}_i$ and terminate, otherwise increase i by one an go to step 2.

2.2 Improvement: Maximal number of iterations

In order for Alg. 1 to be useful in practice, we have to guarantee that the termination criterion $\widehat{\mathcal{C}}_i = \widehat{\mathcal{C}}_{i+1}$ is met after a finite number of iterations. In principle, Thm. 4.1 in Gilbert and Tan (1991) provides this guaranty. We extend this theorem by providing an upper bound for the number of iterations required by Alg. 1. The basic idea is to study the convergence rate of the unconstrained system to the origin. A similar approach was used in Fiacchini et al. (2007) to compute λ-contractive polytopic sets for linear and nonlinear discrete-time systems. Basically, the proof of the following lemma represents a specialized variant of the proof of the well-known Lyapunov criterion for asymptotic stability (see, e.g., (Vidyasagar, 2002, pp. 165 ff.)). As a preparation, note that, for every positive definite matrix $\widehat{Q} = \widehat{Q}^T$, the discrete-time Lyapunov matrix equation

$$(\widehat{A} - \widehat{B}\widehat{K})^T \widehat{P}\,(\widehat{A} - \widehat{B}\widehat{K}) - \widehat{P} + \widehat{Q} = 0 \tag{10}$$

has a unique solution for $\widehat{P} = \widehat{P}^T$ and this solution is positive definite since $(\widehat{A} - \widehat{B}\widehat{K})$ is Schur stable by assumption (see, e.g., (Vidyasagar, 2002, p. 267)).

Lemma 6. Let $\widehat{Q} = \widehat{Q}^T \succ 0$ be arbitrary and let $\widehat{P} = \widehat{P}^T \succ 0$ be the solution of (10). Let $\underline{\rho}, \overline{\rho} \in \mathbb{R}_+$ with $\underline{\rho} \leq \overline{\rho}$ and let $\hat{x}_0 \in \mathcal{E}(\widehat{P}, \overline{\rho})$. Then, $\hat{\varphi}(k, \hat{x}_0) \in \mathcal{E}(\widehat{P}, \underline{\rho})$ for every $k \in \mathbb{N}$ with $k \geq t_{\max}$, where

$$t_{\max} := \left(\frac{\overline{\rho}^2}{\underline{\rho}^2} - 1\right) \frac{\lambda_{\max}(\widehat{P})}{\lambda_{\min}(\widehat{Q})}. \tag{11}$$

Proof. We first show that $\hat{x}_0 \in \mathcal{E}(\widehat{P}, \overline{\rho})$ implies $\hat{\varphi}(i, \hat{x}_0) \in \mathcal{E}(\widehat{P}, \underline{\rho})$ for at least one $i \in \mathbb{N}_{0,i_{\max}}$, where $i_{\max} := \lceil t_{\max} \rceil$. This claim is shown by assuming $\hat{\varphi}(i, \hat{x}_0) \notin \mathcal{E}(\widehat{P}, \underline{\rho})$ for every $i \in \mathbb{N}_{0,i_{\max}}$ and showing that a contradiction results. Let the positive definite function $\hat{v} : \mathbb{R}^n \to \mathbb{R}_0$ be defined by $\hat{v}(x) = \|x\|^2_{\widehat{P}}$ and note that

$$\hat{v}(\hat{x}) \leq \underline{\rho}^2 \quad \Longleftrightarrow \quad \hat{x} \in \mathcal{E}(\widehat{P}, \underline{\rho}). \tag{12}$$

It follows from (10), that the trajectories of the closed-loop system (5) satisfy the relation

$$\Delta\hat{v}(k, \hat{x}_0) = -\|\hat{\varphi}(k, \hat{x}_0)\|^2_{\widehat{Q}} \tag{13}$$

for every $\hat{x}_0 \in \mathbb{R}^n$ and every $k \in \mathbb{N}$, where $\Delta\hat{v}(k, \hat{x}_0) := \hat{v}(\hat{\varphi}(k+1, \hat{x}_0)) - \hat{v}(\hat{\varphi}(k, \hat{x}_0))$. In the remainder of the proof, we use (13) to show

$$\hat{v}(\hat{\varphi}(i_{\max}, \hat{x}_0)) < \underline{\rho}^2, \tag{14}$$

which results in the desired contradiction, since (14) implies $\hat{\varphi}(i_{\max}, \hat{x}_0) \in \mathcal{E}(\widehat{P}, \underline{\rho})$ with (12). To show (14), first note that

$$\hat{v}(\hat{\varphi}(i, \hat{x}_0)) = \hat{v}(\hat{x}_0) + \textstyle\sum_{k=0}^{i-1} \Delta\hat{v}(k, \hat{x}_0) \tag{15}$$

for every $i \in \mathbb{N}$. Thus, overestimating the l.h.s. in (14) yields

$$\begin{aligned}\hat{v}(\hat{\varphi}(i_{\max}, \hat{x}_0)) &= \hat{v}(\hat{x}_0) - \textstyle\sum_{k=0}^{i_{\max}-1} \|\hat{\varphi}(k, \hat{x}_0)\|_{\widehat{Q}}^2, \\ &\leq \overline{\rho}^2 - \textstyle\sum_{k=0}^{i_{\max}-1} \lambda_{\min}(\widehat{Q}) \, \|\hat{\varphi}(k, \hat{x}_0)\|_2^2, \\ &\leq \overline{\rho}^2 - \textstyle\sum_{k=0}^{i_{\max}-1} \frac{\lambda_{\min}(\widehat{Q})}{\lambda_{\max}(\widehat{P})} \, \|\hat{\varphi}(k, \hat{x}_0)\|_{\widehat{P}}^2, \\ &< \overline{\rho}^2 - \textstyle\sum_{k=0}^{i_{\max}-1} \frac{\lambda_{\min}(\widehat{Q})}{\lambda_{\max}(\widehat{P})} \, \underline{\rho}^2, \\ &\leq \overline{\rho}^2 - t_{\max} \frac{\lambda_{\min}(\widehat{Q})}{\lambda_{\max}(\widehat{P})} \, \underline{\rho}^2 = \underline{\rho}^2,\end{aligned} \tag{16}$$

where the first relation holds according to (15) and (13). To understand the second and the third relation, first note that $\hat{v}(\hat{x}_0) \leq \overline{\rho}^2$ according to (12) since $\hat{x}_0 \in \mathcal{E}(P, \overline{\rho})$ by assumption. Moreover, we have

$$\lambda_{\min}(V)\|\hat{x}\|_2^2 \leq \|\hat{x}\|_V^2 \leq \lambda_{\max}(V)\|\hat{x}\|_2^2$$

for any symmetric matrix $V \in \mathbb{R}^{n\times n}$ (Horn and Johnson, 1985, p. 176). Thus, we obtain the second and the third relation, since Q and P are symmetric positive definite matrices. The fourth relation results with the assumption that $\hat{\varphi}(i, \hat{x}_0) \notin \mathcal{E}(P, \underline{\rho})$ for every $i \in \mathbb{N}_{0,i_{\max}}$, i.e.,

$$\hat{v}(\hat{\varphi}(i, \hat{x}_0)) = \|\hat{\varphi}(i, \hat{x}_0)\|_{\widehat{P}}^2 > \underline{\rho}^2$$

according to (12). Finally, the fifth and the sixth relation hold due to $i_{\max} \geq t_{\max}$ and by definition of $t_{\max}$ in (11). In summary, we obtain (14). Thus, we proved that there exists an $i^* \in \mathbb{N}_{0,i_{\max}}$ such $\hat{\varphi}(i^*, \hat{x}_0) \in \mathcal{E}(\widehat{P}, \underline{\rho})$. Taking into account that $\Delta\hat{v}(k, \hat{x}_0) \leq 0$ for every $k \in \mathbb{N}$ and every $\hat{x}_0 \in \mathbb{R}^n$, it directly follows from (15) that $\hat{\varphi}(i^*, \hat{x}_0) \in \mathcal{E}(\widehat{P}, \underline{\rho})$ implies $\hat{\varphi}(k, \hat{x}_0) \in \mathcal{E}(\widehat{P}, \underline{\rho})$ for every $k \in \mathbb{N}_{i^*}$.

It is obvious how to apply Lem. 6 in order to provide the desired iteration bound. Basically, the ellipsoids $\mathcal{E}(\widehat{P}, \underline{\rho})$ and $\mathcal{E}(\widehat{P}, \overline{\rho})$ should be chosen in such a way that they approximate the set $\widehat{\mathcal{C}}_{\max}$ from inside and outside, respectively. Proposition 7 formalizes this approach. See the discrete-time example in Sect. 4.1 for a geometrical interpretation.

Proposition 7. Let $\widehat{Q} = \widehat{Q}^T \succ 0$ be arbitrary and let $\widehat{P} = \widehat{P}^T \succ 0$ be the solution of (10). Let $\underline{\rho}, \overline{\rho} \in \mathbb{R}_+$ be such that $\mathcal{E}(\widehat{P}, \underline{\rho}) \subseteq \widehat{\mathcal{C}}_{\max} \subseteq \mathcal{E}(\widehat{P}, \overline{\rho})$ and let $t_{\max}$ be defined as in (11). Then, there exists an $i \in \mathbb{N}$ with $i < i_{\max}$ such that $\widehat{\mathcal{C}}_i = \widehat{\mathcal{C}}_{i+1}$, where $i_{\max} := \max\{\lceil t_{\max}\rceil, 1\}$.

Proof. Since $\widehat{\mathcal{C}}_{\max} \subseteq \mathcal{E}(\widehat{P}, \overline{\rho})$, we have $\hat{\varphi}(k, \hat{x}_0) \in \mathcal{E}(\widehat{P}, \underline{\rho}) \subseteq \widehat{\mathcal{C}}_{\max}$ for every $\hat{x}_0 \in \widehat{\mathcal{C}}_{\max}$ and every $k \in \mathbb{N}_{i_{\max}}$ according to Lem. 6. In particular, we find $\hat{\varphi}(i_{\max}, \hat{x}_0) \in \widehat{\mathcal{C}}_{\max}$ for every $\hat{x}_0 \in \widehat{\mathcal{C}}_{\max}$. Thus, we obtain

$$\hat{x}_0 \in \widehat{\mathcal{C}}_{i_{\max}-1} \quad \Longrightarrow \quad \hat{x}_0 \in \widehat{\mathcal{C}}_{i_{\max}} \tag{17}$$

according to the definition of $\widehat{\mathcal{C}}_i$ in (8). Obviously, (17) implies $\widehat{\mathcal{C}}_{i_{\max}-1} \subseteq \widehat{\mathcal{C}}_{i_{\max}}$. Since we also have $\widehat{\mathcal{C}}_{i_{\max}-1} \supseteq \widehat{\mathcal{C}}_{i_{\max}}$ according to (9), we finally find $\widehat{\mathcal{C}}_{i_{\max}-1} = \widehat{\mathcal{C}}_{i_{\max}}$. Thus, there exists at least one $i \in \mathbb{N}$ with $i < i_{\max}$ such that $\widehat{\mathcal{C}}_i = \widehat{\mathcal{C}}_{i+1}$, e.g., $i = i_{\max} - 1$.

Note that it is necessary to use $i_{\max} := \max\{\lceil t_{\max}\rceil, 1\}$ in Prop. 7 in order to guarantee $i_{\max} \geq 1$ for the (rare but possible) exception $t_{\max} = 0$.

3. THE CONTINUOUS-TIME CASE

Consider the continuous-time linear system

$$\dot{x}(t) = A\,x(t) + B\,u(t), \quad x(0) = x_0 \tag{18}$$

with input and state constraints of the form

$$u(t) \in \mathcal{U} \quad \text{and} \quad x(t) \in \mathcal{X} \quad \text{for every } t \in \mathbb{R}_0. \tag{19}$$

Further assume that a linear control law

$$u(t) = -K\,x(t) \tag{20}$$

is given, such that the closed-loop system

$$\dot{x}(t) = (A - BK)\,x(t), \quad x(0) = x_0 \tag{21}$$

is Hurwitz stable.

3.1 State of the art: Outer approximation of $\mathcal{C}_\infty$

Let $\varphi(t, x_0)$ denote the solution of the closed-loop system (21) at time $t \in \mathbb{R}_0$ with initial condition x_0 and define the set

$$\mathcal{C}_{\max} := \{x \in \mathcal{X} \mid -Kx \in \mathcal{U}\}. \tag{22}$$

Then, analogously to the discrete-time case, the largest CAS for continuous-time systems can be expressed as

$$\mathcal{C}_\infty := \{x_0 \in \mathcal{C}_{\max} \,|\, \varphi(t, x_0) \in \mathcal{C}_{\max}, \,\forall t \in \mathbb{R}_0\}. \tag{23}$$

Inspired by (8), we may overestimate $\mathcal{C}_\infty$ by considering only a finite number of points in time. Let

$$\mathcal{C}_i^{\Delta t} := \{x_0 \in \mathcal{C}_{\max} \,|\, \varphi(k\Delta t, x_0) \in \mathcal{C}_{\max}, \,\forall k \in \mathbb{N}_{0,i}\} \tag{24}$$

for an arbitrary but fixed $\Delta t \in \mathbb{R}_+$ and note that

$$\mathcal{C}_\infty \subseteq \mathcal{C}_{i+1}^{\Delta t} \subseteq \mathcal{C}_i^{\Delta t} \subseteq \mathcal{C}_{\max} \tag{25}$$

for every $i \in \mathbb{N}$. With regard to (24), the computation of $\mathcal{C}_i^{\Delta t}$ builds on the discretization of the continuous-time system based on the sample time Δt. To see this, note that $\varphi(k\Delta t, x_0)$ can be written as

$$\varphi(k\Delta t, x_0) = \exp((A - BK)\Delta t)^k x_0.$$

Obviously, the outer approximation of $\mathcal{C}_\infty$ in terms of $\mathcal{C}_i^{\Delta t}$ (generally) becomes "tighter" for larger $i \in \mathbb{N}$ and smaller $\Delta t \in \mathbb{R}_+$, respectively. However, in contrast to the discrete-time case, the limit

$$\mathcal{C}_\infty^{\Delta t} := \lim_{i\to\infty} \mathcal{C}_i^{\Delta t} \tag{26}$$

is, in general, still an overestimation of the largest CAS, i.e., $\mathcal{C}_\infty \subseteq \mathcal{C}_\infty^{\Delta t}$. This is due to the fact that $\mathcal{C}_\infty^{\Delta t}$ is generally not positively invariant [1] for system (21). To solve this problem, it is reasonable to scale $\mathcal{C}_\infty^{\Delta t}$ in such a way that the shrunken set is a subset of $\mathcal{C}_\infty$. Obviously, the underlying scaling factor determines the accuracy of the inner approximation. Thus, if we are interested in a certain user-defined accuracy (see ϵ in (1)), the scaling is limited. We have to choose the remaining parameter Δt in an appropriate way to address these limitations. In fact, Gilbert and Tan (1991) already showed that an appropriate Δt always exists. We summarize the corresponding result in the following theorem which builds on (Gilbert and Tan, 1991, Thm. 6.1).

Theorem 8. For every $\epsilon \in (0, 1)$ there exists a $\Delta t \in \mathbb{R}_+$ such that

$$(1 - \epsilon)\,\mathcal{C}_\infty^{\Delta t} \subseteq \mathcal{C}_\infty \subseteq \mathcal{C}_\infty^{\Delta t}. \tag{27}$$

[1] For discrete-time systems, a $(\widehat{A} - \widehat{B}\,\widehat{K})$-invariant set $\mathcal{T}$ is also positively invariant for the closed-loop system (5). In contrast, for continuous-time systems, a $\exp((A - BK)\Delta t)$-invariant set $\mathcal{T}$ is not necessarily positively invariant for (21). See (Blanchini and Miani, 2008, p. 99 ff.) or Blanco and De Moor (2007) for details on positive invariance.

3.2 Improvement: Appropriate discretization

In principle, Thm. 8 provides the statement that we are looking for. Assume we found a $\Delta t \in \mathbb{R}_+$ such that (27) holds for a given $\epsilon \in (0,1)$, then the choice $\underline{\mathcal{C}}_\infty = (1-\epsilon)\,\mathcal{C}_\infty^{\Delta t}$ fulfills (1). To see this, note that $\underline{\mathcal{C}}_\infty \subseteq \mathcal{C}_\infty$ holds according to the first relation in (27). Moreover, multiplying the second relation in (27) with the factor $(1-\epsilon) > 0$ yields

$$(1-\epsilon)\,\mathcal{C}_\infty \subseteq (1-\epsilon)\,\mathcal{C}_\infty^{\Delta t} = \underline{\mathcal{C}}_\infty.$$

Note that, analogously to the discrete-time case, the set $\mathcal{C}_\infty^{\Delta t}$ can be exactly calculated by a finite number of iterations (see Prop. 15 further below for details). Thus, it only remains to choose $\Delta t \in \mathbb{R}_+$. Unfortunately, neither Thm. 8 nor the corresponding proof in Gilbert and Tan (1991) provide a practical algorithm for the computation of Δt. The main contribution of this paper is a simple criterion (see Eq. (29)) for an appropriate choice of Δt such that (27) (or equivalently (1) with $\underline{\mathcal{C}}_\infty = (1-\epsilon)\,\mathcal{C}_\infty^{\Delta t}$) holds. This criterion will be used to provide a method for the tight inner approximation of $\mathcal{C}_\infty$, which is summarized in Prop. 12. Several steps are required as a preparation to Prop. 12. We begin by formulating conditions on a set $\mathcal{T}$ to be a subset of the largest CAS. Obviously, the following lemma represents the continuous-time counterpart to Lem. 3.

Lemma 9. Let $\Delta t \in \mathbb{R}_+$ and let $\mathcal{T} \subseteq \mathcal{C}_{\max}$ be $\widetilde{A}$-invariant with $\widetilde{A} = \exp((A-BK)\,\Delta t)$. Let $\varphi(\tau, x_0) \in \mathcal{C}_{\max}$ for every $x_0 \in \mathcal{T}$ and every $\tau \in [0, \Delta t)$. Then $\mathcal{T} \subseteq \mathcal{C}_\infty$.

Proof. We have to show that $\varphi(t, x_0) \in \mathcal{C}_{\max}$ for every $x_0 \in \mathcal{T}$ and every $t \in \mathbb{R}_0$. Since $\mathcal{T}$ is $\widetilde{A}$-invariant, we have $\exp((A-BK)\,\Delta t)^k x_0 = \exp((A-BK)\,k\,\Delta t) = \varphi(k\,\Delta t, x_0) \in \mathcal{T}$ for every $x_0 \in \mathcal{T}$ and $k \in \mathbb{N}$ according to Lem. 1. Moreover, we have $\varphi(\tau, x_0) \in \mathcal{C}_{\max}$ for every $x_0 \in \mathcal{T}$ and every $\tau \in [0, \Delta t)$ by assumption. Thus, since $\varphi(k\,\Delta t, x_0) \in \mathcal{T}$, we obtain

$$\varphi(\tau, \varphi(k\,\Delta t, x_0)) = \varphi(k\,\Delta t + \tau, x_0) \in \mathcal{C}_{\max} \tag{28}$$

for every $x_0 \in \mathcal{T}$, every $k \in \mathbb{N}$ and every $\tau \in [0, \Delta t)$. Now let $x_0 \in \mathcal{T}$ and $t \in \mathbb{R}_0$ be arbitrary but fixed. Set $k^* = \lfloor \frac{t}{\Delta t} \rfloor$ and $\tau^* = t - k^*\,\Delta t$ and note that $k^* \in \mathbb{N}$, $\tau^* \in [0, \Delta t)$ and $t = k^*\,\Delta t + \tau^*$. Thus

$$\varphi(t, x_0) = \varphi(k^*\,\Delta t + \tau^*, x_0) \in \mathcal{C}_{\max}$$

according to (28). This completes the proof, since $x_0 \in \mathcal{T}$ and $t \in \mathbb{R}_0$ were arbitrary.

Similar to the discrete-time blueprint, Lem. 9 requires $\mathcal{T}$ to be $\widetilde{A}$-invariant (with $\widetilde{A} = \exp((A-BK)\,\Delta t)$). Analogously to Lem. 4, $\mathcal{C}_i^{\Delta t} = \mathcal{C}_{i+1}^{\Delta t}$ implies $\widetilde{A}$-invariace. This is summarized in the following lemma.

Lemma 10. If $\mathcal{C}_i^{\Delta t} = \mathcal{C}_{i+1}^{\Delta t}$, then $\mathcal{C}_i^{\Delta t}$ is $\widetilde{A}$-invariant with $\widetilde{A} = \exp((A-BK)\,\Delta t)$.

We omit the proof of Lem. 10 since it is a direct consequence of Lem. 4. While Lem. 3 (discrete-time) and Lem. 9 (continuous-time) are similar in terms of $\widehat{A}$-invariace, Lem. 9 differs from Lem. 3 in that it contains additional conditions on $\varphi(\tau, x_0)$. A simple criterion for the fulfillment of these conditions is introduced in Lem. 11.

Lemma 11. Let $\epsilon \in (0,1)$ and let $\mathcal{T}$ be such that $\mathcal{T} \subseteq (1-\epsilon)\,\mathcal{C}_{\max}$. Let $\underline{r}, \overline{r} \in \mathbb{R}_+$ be such that $\mathcal{B}(\underline{r}) \subseteq \mathcal{C}_{\max}$ and $\mathcal{T} \subseteq \mathcal{B}(\overline{r})$. Finally, let $\Delta t \in \mathbb{R}_+$ be such that

$$\Delta t \leq \frac{1}{\mu_2(A-BK)} \ln\left(\frac{\mu_2(A-BK)}{\|A-BK\|_2}\frac{\epsilon\,\underline{r}}{\overline{r}} + 1\right). \tag{29}$$

Then, $\varphi_c(\tau, x_0) \in \mathcal{C}_{\max}$ for every $x_0 \in \mathcal{T}$ and $\tau \in [0, \Delta t)$.

Proof. Assume for a moment

$$\|\varphi(\tau, x_0) - x_0\|_2 \leq \epsilon\,\underline{r} \tag{30}$$

for every $x_0 \in \mathcal{T}$ and every $\tau \in [0, \Delta t)$. Define $\Delta x(\tau, x_0) := \varphi(\tau, x_0) - x_0$. Then, we have

$$\varphi(\tau, x_0) = x_0 + (\varphi(\tau, x_0) - x_0) = x_0 + \Delta x(\tau, x_0) \in \mathcal{C}_{\max}$$

for every $x_0 \in \mathcal{T} \subseteq (1-\epsilon)\,\mathcal{C}_{\max}$ and every $\tau \in [0, \Delta t)$ according to Lem. 2 in combination with (30) and $\mathcal{B}(\underline{r}) \subseteq \mathcal{C}_{\max}$. Thus, it remains to show (30) in order to prove the claim in Lem. 11. To do so, we initially note that the l.h.s. in (30) cannot be larger than the arc length

$$s(\tau, x_0) = \int_0^\tau \|\dot{\varphi}(t, x_0)\|_2\,\mathrm{d}t \tag{31}$$

at time τ of the trajectory emanating from x_0. Thus, we have

$$\|\varphi(\tau, x_0) - x_0\|_2 \leq s(\tau, x_0)$$

for every $x_0 \in \mathcal{T}$ and every $\tau \in [0, \Delta t)$. Since $\varphi(\tau, x_0)$ refers to the solution of (21), the derivative of the trajectory with respect to t is given by

$$\begin{aligned} \dot{\varphi}(t, x_0) &= (A-BK)\varphi(t, x_0), \\ &= (A-BK)\,\exp((A-BK)\,t)\,x_0. \end{aligned} \tag{32}$$

Overestimating the arc length (31) yields

$$\begin{aligned} s(\tau, x_0) &= \int_0^\tau \|(A-BK)\,\exp((A-BK)\,t)\,x_0\|_2\,\mathrm{d}t, \\ &\leq \int_0^\tau \|(A-BK)\|_2 \|\exp((A-BK)\,t)\|_2 \|x_0\|_2\,\mathrm{d}t, \\ &\leq \|(A-BK)\|_2 \|x_0\|_2 \int_0^\tau \exp(\mu_2(A-BK)\,t)\,\mathrm{d}t, \\ &= \tfrac{\|(A-BK)\|_2 \|x_0\|_2}{\mu_2(A-BK)} \left[\exp(\mu_2(A-BK)\,t)\right]_0^\tau, \\ &= \tfrac{\|(A-BK)\|_2 \|x_0\|_2}{\mu_2(A-BK)} (\exp(\mu_2(A-BK)\,\tau) - 1), \end{aligned}$$

where the first relation holds due to (32). The second relation is satisfied since the matrix 2-norm (spectral norm) is compatible with the Euclidean vector norm in the sense that $\|Vx\|_2 \leq \|V\|_2\,\|x\|_2$ for every matrix $V \in \mathbb{R}^{n \times n}$ and every $x \in \mathbb{R}^n$ (Horn and Johnson, 1985, p. 293). The third relation holds, since the norm of the matrix exponential can be overestimated according to $\|\exp(V\,t)\|_2 \leq \exp(\mu_2(V)\,t)$ for every $V \in \mathbb{R}^{n \times n}$ and every $t \in \mathbb{R}_0$. Note that the logarithmic norm was introduced in order to provide tight bounds for the solution of (21) (Söderlind, 2006). Since $x_0 \in \mathcal{T} \subseteq \mathcal{B}(\overline{r})$, i.e., $\|x_0\|_2 \leq \overline{r}$, and since $\tau \in [0, \Delta t)$, we obtain

$$s(\tau, x_0) \leq \frac{\|(A-BK)\|_2\,\overline{r}}{\mu_2(A-BK)} (\exp(\mu_2(A-BK)\,\Delta t) - 1)$$

for every $x_0 \in \mathcal{T}$ and every $\tau \in [0, \Delta t)$. Taking into account that Δt is bounded above according to (29), we easily find $s(\tau, x_0) \leq \epsilon\,\underline{r}$, which finally proves (30).

The combination of Lems. 9, 10 and 11 leads to Prop. 12, which states a method for the tight underestimation of $\mathcal{C}_\infty$ based on an appropriate discretization in terms of Δt.

Proposition 12. Let $\epsilon \in (0,1)$ and let $\underline{r}, \overline{r} \in \mathbb{R}_+$ be such that $\mathcal{B}(\underline{r}) \subseteq \mathcal{C}_{\max}$ and $(1-\epsilon)\,\mathcal{C}_{\max} \subseteq \mathcal{B}(\overline{r})$. Let $i \in \mathbb{N}$ and let $\Delta t \in \mathbb{R}_+$ be such that (29) holds. If $\mathcal{C}_i^{\Delta t} = \mathcal{C}_{i+1}^{\Delta t}$, then $\underline{\mathcal{C}}_\infty := (1-\epsilon)\,\mathcal{C}_i^{\Delta t}$ is such that (1) holds.

Proof. In order to prove the claim, we assume $\mathcal{C}_i^{\Delta t} = \mathcal{C}_{i+1}^{\Delta t}$ and show that the relations (1) are fulfilled for the choice $\underline{\mathcal{C}}_\infty := (1-\epsilon)\,\mathcal{C}_i^{\Delta t}$. The second relation in (1) obviously holds, since we have

$$(1-\epsilon)\,\mathcal{C}_\infty \subseteq (1-\epsilon)\,\mathcal{C}_i^{\Delta t} = \underline{\mathcal{C}}_\infty$$

according to (25) and by definition of $\underline{\mathcal{C}}_\infty$, respectively. In order to prove the first relation in (1), i.e., $\underline{\mathcal{C}}_\infty \subseteq \mathcal{C}_\infty$, we will apply Lem. 9. It remains to show that $\underline{\mathcal{C}}_\infty$ is $\widetilde{A}$-invariant with $\widetilde{A} = \exp((A-BK)\,\Delta t)$ and that

$$\varphi(\tau, x_0) \in \mathcal{C}_{\max} \text{ for every } x_0 \in \underline{\mathcal{C}}_\infty,\ \tau \in [0, \Delta t). \qquad (33)$$

Since $\mathcal{C}_i^{\Delta t} = \mathcal{C}_{i+1}^{\Delta t}$, it follows that $\mathcal{C}_i^{\Delta t}$ is $\widetilde{A}$-invariant according to Lem. 10. Thus, $\underline{\mathcal{C}}_\infty := (1-\epsilon)\,\mathcal{C}_i^{\Delta t}$ is $\widetilde{A}$-invariant according to Lem. 1. In order to show (33), first note that

$$\underline{\mathcal{C}}_\infty = (1-\epsilon)\,\mathcal{C}_i^{\Delta t} \subseteq (1-\epsilon)\,\mathcal{C}_{\max} \subseteq \mathcal{B}(\overline{r})$$

by definition of $\underline{\mathcal{C}}_\infty$, according to (25) and by assumption in Prop. 12, respectively. Moreover, again by assumption, we have $\mathcal{B}(\underline{r}) \subseteq \mathcal{C}_{\max}$ and $\Delta t \in \mathbb{R}_+$ such that (29) holds. Thus, (33) follows from Lem. 11, which completes the proof.

Propositon 12 suggests to implement the following algorithm for the iterative computation of $\underline{\mathcal{C}}_\infty$.

Algorithm 2. Inner approximation of $\mathcal{C}_\infty$ for $\epsilon \in (0,1)$.

(1) Choose $\underline{r}$, $\overline{r}$ and Δt according to Prop. 12.
(2) Set $i = 0$ and $\mathcal{C}_0^{\Delta t} = \mathcal{C}_{\max}$.
(3) Compute $\mathcal{C}_{i+1}^{\Delta t}$.
(4) If $\mathcal{C}_i^{\Delta t} = \mathcal{C}_{i+1}^{\Delta t}$, return $\underline{\mathcal{C}}_\infty := (1-\epsilon)\,\mathcal{C}_i^{\Delta t}$ and terminate, otherwise increase i by one an go to step 3.

The termination of Alg. 2 after a finite number of iterations can be guaranteed analogously to the discrete-time case. In fact, we find Lem. 13 and Prop. 14 as direct counterparts to Lem. 6 and Prop. 7, respectively. As a preparation, note that, for every positive definite matrix $Q = Q^T$, the continuous-time Lyapunov matrix equation

$$(A-BK)^T P + P(A-BK) + Q = 0 \qquad (34)$$

has a unique solution for $P = P^T$ and this solution is positive definite since $(A - BK)$ is Hurwitz stable by assumption (see, e.g., (Vidyasagar, 2002, p. 199)).

Lemma 13. Let $Q = Q^T \succ 0$ be arbitrary and let $P = P^T \succ 0$ be the solution of (34). Let $\underline{\rho}, \overline{\rho} \in \mathbb{R}_+$ with $\underline{\rho} \leq \overline{\rho}$ and let $x_0 \in \mathcal{E}(P, \overline{\rho})$. Then, $\varphi(t, x_0) \in \mathcal{E}(P, \underline{\rho})$ for every $t \in \mathbb{R}_0$ with $t \geq t_{\max}$, where $t_{\max}$ is defined as in (11) with $\widehat{P} = P$ and $\widehat{Q} = Q$.

Proof. Analogously to the proof of Lem. 6 with $\dot{v}(\varphi(t, x_0)) = -\|\varphi(t, x_0)\|_Q^2$ and

$$v(\varphi(t^*, x_0)) = v(x_0) + \int_0^{t^*} \dot{v}(\varphi(t, x_0))\,\mathrm{d}t$$

instead of Eqs. (13) and (15), respectively.

Proposition 14. Let $Q = Q^T \succ 0$ be arbitrary and let $P = P^T \succ 0$ be the solution of (34). Let $\underline{\rho}, \overline{\rho} \in \mathbb{R}_+$ be such that $\mathcal{E}(P, \underline{\rho}) \subseteq \mathcal{C}_{\max} \subseteq \mathcal{E}(P, \overline{\rho})$. Let $\Delta t \in \mathbb{R}_+$ and let $t_{\max}$ be defined as in (11). Then, there exists an $i \in \mathbb{N}$ with $i < i_{\max}$ such that $\mathcal{C}_i^{\Delta t} = \mathcal{C}_{i+1}^{\Delta t}$, where $i_{\max} := \max\{\lceil \frac{t_{\max}}{\Delta t} \rceil, 1\}$.

The proof of Prop. 14 is analogous to that of Prop. 7. Finally, in order to link Thm. 8 and Prop. 12, we state the following trivial result without proof.

Proposition 15. Let $i \in \mathbb{N}$. If $\mathcal{C}_i^{\Delta t} = \mathcal{C}_{i+1}^{\Delta t}$, then $\mathcal{C}_\infty^{\Delta t} = \mathcal{C}_i^{\Delta t}$.

4. NUMERICAL EXAMPLES

We apply Alg. 1 and Alg. 2 to a discrete-time and continuous-time example, respectively. For both examples, we study the shape of the computed (approximation of) the largest CAS. Moreover, we show that the iteration bounds introduced in Props. 7 and 14 hold.

4.1 Discrete-time example

We study the same discrete-time system as in Sznaier and Damborg (1987) with the matrices

$$\widehat{A} = \begin{pmatrix} 1.0000 & 0.2212 \\ 0.0000 & 0.7788 \end{pmatrix}, \quad \widehat{B} = \begin{pmatrix} 0.0288 \\ 0.2212 \end{pmatrix}$$

and the constraints

$$\widehat{\mathcal{X}} = \{\hat{x} \in \mathbb{R}^2 \,|\, |\hat{x}_1| \leq 1.5,\ |\hat{x}_2| \leq 0.3\},$$
$$\widehat{\mathcal{U}} = \{\hat{u} \in \mathbb{R} \,|\, |\hat{u}| \leq 0.5\}.$$

For the choice $\widehat{K} = (\,0.8831\ 0.8811\,)$, the closed-loop system (5) is stable. With the weighting matrix $Q = I$, we obtain

$$\widehat{P} = \begin{pmatrix} 6.5334 & 2.0254 \\ 2.0254 & 2.6003 \end{pmatrix}$$

as a solution of (10). Moreover, it is easy to prove and visualized in Fig. 1 that we have $\mathcal{E}(\widehat{P}, \underline{\rho}) \subseteq \widehat{\mathcal{C}}_{\max} \subseteq \mathcal{E}(\widehat{P}, \overline{\rho})$ for the particular choice $\underline{\rho} = 0.4213$ and $\overline{\rho} = 2.0191$. Thus, corresponding to Lem. 6 and Eq. (11), we obtain $t_{\max} = 162.3200$. According to Prop. 7, this implies that the number of iterations required by Alg. 1 can a priori be limited above by $i_{\max} - 1 = \lceil t_{\max} \rceil - 1 = 162$. In practice, for this example, Alg. 1 terminates after 3 steps. Thus, the computed iteration bound is valid but conservative.

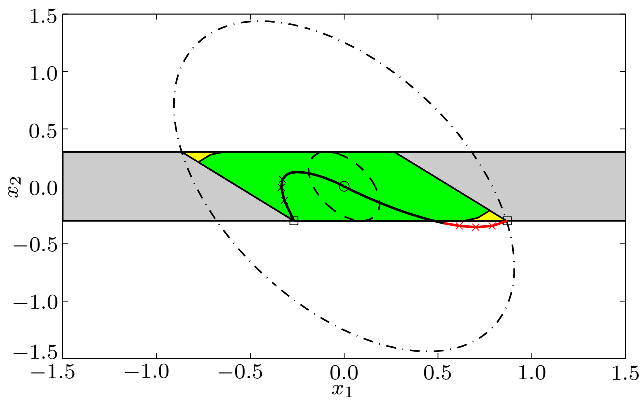

Fig. 1. Visualization of $\widehat{\mathcal{X}}$ (gray), $\widehat{\mathcal{C}}_{\max}$ (yellow) and $\widehat{\mathcal{C}}_\infty$ (green) for the discrete-time example in Sect. 4.1. The dashed and the dashed-dotted elliptic shapes refer to the ellipses $\mathcal{E}(\widehat{P}, \underline{\rho})$ and $\mathcal{E}(\widehat{P}, \overline{\rho})$, respectively. The solid curves represent trajectories of the closed-loop system (5) emanating from $\hat{x}_0 = (\,-0.27\ \ -0.30\,)^T$ and $\hat{x}_0 = (\,0.86\ \ -0.30\,)^T$, respectively. For both trajectories, the first three steps are marked by crosses. The red line segments of the right trajectory highlight the critical steps, where constraints are violated.

Figure 1 visualizes the computed shape of the largest CAS. Moreover, two trajectories of the closed-loop system (5) are shown in order to illustrate the fundamental idea behind Lem. 6. Basically, any trajectory emanating from a state $\hat{x}_0 \in \mathcal{E}(\widehat{P}, \overline{\rho})$ will enter $\mathcal{E}(\widehat{P}, \underline{\rho})$ after $i_{\max}$ steps. However, even trajectories that start in $\widehat{\mathcal{C}}_{\max}$ may violate the input or state constraints during the first $i_{\max} - 1$ steps (see the red highlighted steps in Fig. 1).

4.2 Continuous-time example

We now consider the continuous-time system with the matrices

$$A = \begin{pmatrix} -0.1 & 1.0 \\ -0.2 & 0.1 \end{pmatrix}, \quad B = \begin{pmatrix} -0.9 \\ 0.5 \end{pmatrix}$$

and the constraints

$$\mathcal{X} = \{x \in \mathbb{R}^2 \,|\, \|x\|_\infty \le 2,\} \text{ and } \mathcal{U} = \{u \in \mathbb{R} \,|\, |u| \le 1\}.$$

For the choice $K = (\,-0.1903\ 1.7768\,)$, the closed-loop system (21) is stable. Moreover, with the following weighting matrix Q, we obtain the listed matrix P as a solution of (34):

$$Q = \begin{pmatrix} 1.0 & -0.6 \\ -0.6 & 7.0 \end{pmatrix}, \qquad P = \begin{pmatrix} 1.2458 & 1.5458 \\ 1.5458 & 9.5353 \end{pmatrix}.$$

To compute an inner approximation of $\mathcal{C}_\infty$, we a priori have to fix the maximal "error" by selecting ϵ. Here, we choose $\epsilon = 0.1$ relatively large in order to provide a better illustration. To satisfy the specified error bound, we have to select the sample time Δt such that (29) holds. We initially compute $\underline{r} = 0.5596$ and $\overline{r} = 1.9311$ such that $\mathcal{B}(\underline{r}) \subseteq \mathcal{C}_{\max} \subseteq \frac{1}{1-\epsilon}\mathcal{B}(\overline{r})$ (see Fig. 2). Then, according to Prop. 12, $\Delta t = 0.0105$ results in an appropriate discretization.In order to provide an upper bound for the number of iterations required by Alg. 2, we choose $\underline{\rho} = 1.3352$ and $\overline{\rho} = 3.9425$ such that $\mathcal{E}(P, \underline{\rho}) \subseteq \mathcal{C}_{\max} \subseteq \mathcal{E}(P, \overline{\rho})$. Then, corresponding to Lem. 13, we obtain $t_{\max} = 80.5430$. According to Prop. 14, this implies that the number of iterations can a priori be limited to $i_{\max} - 1 = \lceil \frac{t_{\max}}{\Delta t} \rceil - 1 = 7605$. In practice, Alg. 2 terminates after 89 steps for this example. Again, the computed iteration bound is valid but conservative.

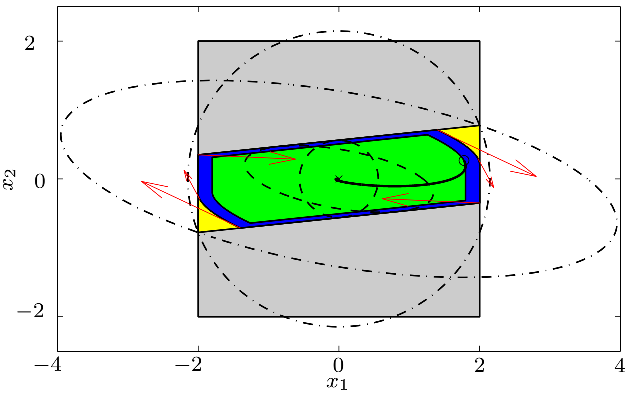

Fig. 2. Visualization of $\mathcal{X}$ (gray), $\mathcal{C}_{\max}$ (yellow), $\mathcal{C}_\infty^{\Delta t}$ (blue) and $\underline{\mathcal{C}}_\infty$ (green) for the continuous-time example in Sect. 4.2. The dashed and the dashed-dotted elliptic shapes refer to the ellipses $\mathcal{E}(P, \underline{\rho})$ and $\mathcal{E}(P, \overline{\rho})$, respectively. The dashed and the dashed-dotted round shapes refer to the circles $\mathcal{B}(\underline{r})$ and $\frac{1}{1-\epsilon}\mathcal{B}(\overline{r})$, respectively. The solid curve represents a trajectory of the closed-loop system (21) emanating from $x_0 = (\,1.76\ \ 0.29\,)^T$. The red vectors visualize the slope field associated with system (21) for some points at the boundary of $\mathcal{C}_\infty^{\Delta t}$.

Figure 2 illustrates the shape of the computed set $\mathcal{C}_\infty^{\Delta t}$ and the scaled set $\underline{\mathcal{C}}_\infty = (1-\epsilon)\,\mathcal{C}_\infty^{\Delta t}$. Moreover, the slope field associated with the differential equation (21) is sketched. Based on the illustrated velocity vectors and Nagumo's invariance theorem (see, e.g., (Blanchini and Miani, 2008, pp. 101-103)), it is immediately clear that $\mathcal{C}_\infty^{\Delta t}$ can not be positively invariant. Obviously, the same conclusion holds for the scaled set $(1-\epsilon)\,\mathcal{C}_\infty^{\Delta t}$. In fact, the illustrated trajectory temporarily leaves[2] $\underline{\mathcal{C}}_\infty$. However, based on the suitable choice of Δt, it is guaranteed (by Prop. 12) that every trajectory that starts in $\underline{\mathcal{C}}_\infty$ remains in $\mathcal{C}_\infty^{\Delta t}$ for all times.

[2] Note that the resolution of Fig. 2 is too low to visualize this excursion.

5. CONCLUSION AND OUTLOOK

We presented methods for the computation of CAS for constrained linear systems. For discrete-time systems, we extended established results on the iterative computation of the largest CAS $\widehat{\mathcal{C}}_\infty$ by providing an upper bound for the number of required iterations. Simulation studies confirm the derived bound is valid but also indicate it is conservative w.r.t. the actually required number of iterations. For continuous-time systems, we showed how to reasonably discretize the system in order to provide an inner approximation of $\mathcal{C}_\infty$ with a certain, user-defined accuracy. In fact, criterion (29), which allows for an appropriate choice of the sampling time Δt, is the main result of the paper. Taking this criterion into account, CAS for continuous-time systems can be computed analogously to the discrete-time case.

We claim criterion (29) is interesting, since it allows for an a priori choice of Δt depending on the desired accuracy. Unfortunately, appropriate sampling times will be very small, in general. Consequently, the number of iterations that are necessary to approximate $\mathcal{C}_\infty$ will be high. One way to reduce the number of iterations will be to start with sampling times larger than Δt in order to derive coarse outer approximations of $\mathcal{C}_\infty$. Subsequently, the required number of steps based on the small sampling time Δt should be significantly smaller. Future work has to investigate such strategies.

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