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Low Complexity Constraint Control Using Contractive Sets ⋆

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Abstract: This paper proposes a simple, non-iterative way of calculating low-complexity controlled contractive sets, and shows how such set can be used for controller design for systems with state and input constraints. The result is a low complexity controller, suitable for implementation in embedded control systems with modest computational power and available memory.

1. INTRODUCTION

Model predictive control (MPC) has found wide application in industry (Qin and Badgwell [2003]). However, as conventional MPC is based on on-line solution of an optimization problem, computational requirements and safety concerns have (practically) limited application to systems with slow dynamics that are not safety-critical. This lead to the development of explicit MPC, where all possible optimization problems are solved off-line in the design phase, and on-line calculations are reduced to a simple table look-up and very simple arithmetic operations (Bemporad et al. [2002], Töndel et al. [2003]). Such simple on-line calculations enable the use of verifiable code for the on-line implementation. Unfortunately, the computational cost of developing the explicit solution grows very rapidly with problem size, as does also the memory required to store the explicit solution. For these reasons, application of explicit MPC is limited to systems with a modest number of states and/or short prediction horizons. This has also motivated a substantial volume of research into finding simpler, approximate solutions to explicit MPC, *e.g.*, (Scibilia et al. [2009], Kvasnica et al. [2010], Bemporad and Filippi [2006], Jones and Morari [2009]).

An alternative approach to constrained control is the so-called *vertex control* (Gutman and Cwikel [1986]) with recent modifications to enhance robustness and performance (Nguyen et al. [2013]). In this approach, knowledge of an admissible input at each vertex of the operating region is used to design the overall controller. However, it is often difficult to find an operating region such that an admissible input exists at all vertices - and the number of vertices may itself be prohibitively high for high-dimensional systems. Accordingly, the study of invariant and contractive sets has a long history in control. An early contribution for discrete time systems is given by Bitsoris [1988]. Hennet and Lasserre [1993] extends these results to the construction of more complex positively-contractive sets. A complete review of the contributions in this area is beyond the scope of this paper, and we instead refer the interested reader to Blanchini [1999] and the references therein.

There is also a substantial body of recent work on the calculation of Robust Positively Invariant (RPI) sets, see, *e.g.*, Tahir and Jaimoukha [2012] and the references therein. These works typically optimize the volume of the RPI set with respect to linear state feedback, and a motivation is to use the RPI set as a terminal set in a robust MPC. However, our aim is to find a low complexity controlled contractive set for design of a simplified MPC-type controller, which implies that the controller will be non-linear (typically Piece-Wise Affine, PWA).

Dorea and Hennet [1999] provide an iterative procedure for constructing successively tighter outer approximations to a contractive set. This (by now classical method) is the starting point of our approach and will be presented in section 2, where disadvantages are pointed out as motivations for the present work. We go on to introduce conservative assumptions which allow a non-iterative procedure to calculate a particularly simple controlled contractive set. In its basic form, this non-iterative procedure requires the system's A-matrix to be diagonalizable with real eigenvalues and -vectors. In section 3, it is shown that oscillatory systems can also be handled, provided the oscillatory dynamics is sufficiently contractive. Section 4 is devoted to identifying an initial set of the required form from which the controlled contractive set can be calculated. Section 5 details how the contractive set can be used to design low-complexity constraint control, and illustrates this with an example. The final section contains conclusions and directions for further work.

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2. FINDING A LOW-COMPLEXITY CONTROLLED CONTRACTIVE SET

Consider the discrete-time system

$$x_{t+1} = Ax_t + Bu_t \tag{1}$$

with state constraints given by $\mathcal{X} = \{x|H_x x \leq h\}$ and input constraints $\mathcal{U} = \{u|H_u u \leq k_u\}$. It is assumed that $h > 0$ and $k_u > 0$, such that $\mathcal{X}$ and $\mathcal{U}$ contain the origin in their respective interiors.

Definition 1. A closed convex set $P \in \mathcal{X}$ with the origin in its interior is called *controlled γ-contractive and admissible* if, for a given $\gamma \in [0, 1)$ and for all $x \in P$ there exists an $u \in \mathcal{U}$ such that $Ax + Bu \in \gamma P$.

Whenever the more compact term *controlled (γ-)contractive set* is used it is assumed that the required input is also admissible, *i.e.*, that the required input also fulfills the relevant constraints.

2.1 The method of Dorea and Hennet

The method of Dorea and Hennet [1999] starts with some initial polytopic set $P_0 = \{H_0 x \leq k_0\}$ containing the origin in its interior. If the initial set is closed, the final set is also guaranteed to be closed. The initial set P_0 could for instance be the state constraints - and if this is not closed it could be intersected with some large 'box' (minimum and maximum constraints on each state). The method then proceeds as follows

(1) Express the initial set, the contractiveness requirement, and the input constraint as a polytope in the space of x and u:

$$\begin{bmatrix} H_0 & 0 \\ H_0 A & H_0 B \\ 0 & H_u \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} k_0 \\ \gamma k_0 \\ k_u \end{bmatrix} \tag{2}$$

(2) Define P_1 as the projection of this polytope onto x (removal of redundant constraints is advised).
(3) If $P_0 = P_1$, the contractive set is identified, terminate. Else, set $P_0 = P_1$ and go to 1.

Example 1:
Consider a spring-mass-damper system with the state space representation

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -7 & -7 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{3}$$

The system is discretized in time with a timestep of 0.01, and the resulting discrete time system has poles at 0.988 and 0.944. Next we run 100 iterations of Dorea and Hennet's algorithm, with a desired contraction factor of 0.99 and input constraints of $u \in [-10\ 10]$. This results in the array of polytopes shown in Fig. 1. We see that the procedure hasn't converged even after 100 iterations, and upon inspection it is found that the final polytope - although apparently of a simple shape - is the intersection of 62 halfspaces.

Changing the desired contraction factor to 0.98, the procedure still hasn't converged after 100 iterations, but this time the final polytope is described by the intersection of only 10 halfspaces.

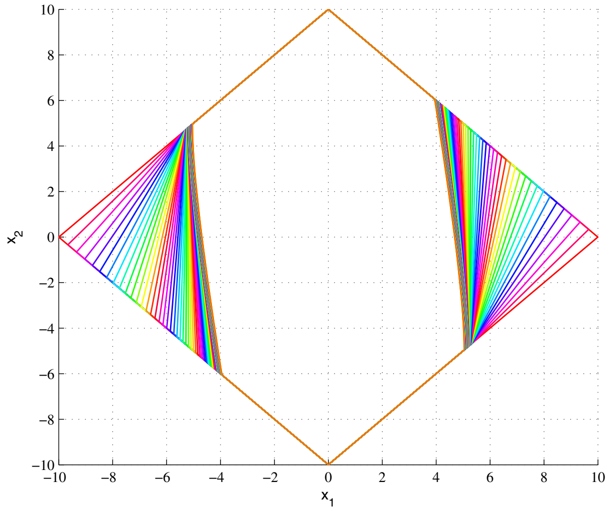

Fig. 1. The result of 100 iterations with Dorea and Hennet's method.

2.2 Lessons from 1d

Consider the system $x_{t+1} = ax_t + bu_t$, with $x \in [-k_x\ k_x]$ and $u \in [-k_u\ k_u]$. Using Dorea and Hennet's method, we therefore get the following polytope in (x, u)-space:

$$\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ -a & -b \\ a & b \\ 0 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} k_x \\ k_x \\ \gamma k_x \\ \gamma k_x \\ k_u \\ k_u \end{bmatrix} \tag{4}$$

In this case the projection is rather trivial, resulting in

$$\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} x \leq \begin{bmatrix} k_x \\ k_x \\ (\gamma k_x + bk_u)/a \\ (\gamma k_x + bk_u)/a \end{bmatrix} \tag{5}$$

Clearly, either the first or the second pair of inequalities will be redundant. We will focus on the case when the first pair of inequalities is redundant (as in the opposite case the procedure has converged). We see that the i'th polytope can be described by

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} x \leq k_{x,i} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{6}$$

where $k_{x,i} = (\gamma/a)k_{x,i-1} + bk_u/a$. That is, the iteration converges as a first order process, and the convergence is very slow if γ and a are close. In this case, it is also trivial to calculate the steady state value of the iteration. Note that if $\gamma/a > 1$, $k_{x,i}$ will diverge - however, this corresponds to the case where the second pair of inequalities is redundant in the result of the projection above.

2.3 Particular choice of initial constraint set

Introduce four assumptions about the system and problem formulation:

(1) The matrix A is diagonalizable.
(2) The matrix A has real eigenvalues.
(3) The matrix H_0 describes 'box constraints' (min/max constraints) in terms of the modes of the matrix A

(4) The matrix H_u describes simple 'box constraints' in the inputs.

The first assumption will be fulfilled for many systems. The second assumption is probably a little more restrictive, and will be relaxed in the next section. The fourth assumption is quite natural - it is rare that the constraints in the inputs depend on each other. The third assumption is clearly restrictive, and would require the initial 'box constraints' in terms of the modes of the matrix A to be fitted to lie within the actual state constraints (i.e., the resulting operating region will typically be restricted by this choice of constraints).

Let V be the eigenvector matrix of A (normalized to have columns of unit length), such that $V^{-1}AV = \Lambda$, where Λ is the (diagonal) eigenvalue matrix. The Dorea and Hennet procedure then yields

$$\begin{bmatrix} \begin{bmatrix} V^{-1} \\ -V^{-1} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} V^{-1}A \\ -V^{-1}A \end{bmatrix} & \begin{bmatrix} V^{-1}B \\ -V^{-1}B \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} I \\ -I \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq \begin{bmatrix} k_0 \\ k_0 \\ \gamma k_0 \\ \gamma k_0 \\ k_u \\ k_u \end{bmatrix} \tag{7}$$

Substituting $\tilde{x} = Vx$, this yields

$$\begin{bmatrix} \begin{bmatrix} I \\ -I \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} \Lambda \\ -\Lambda \end{bmatrix} & \begin{bmatrix} V^{-1}B \\ -V^{-1}B \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} I \\ -I \end{bmatrix} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ u \end{bmatrix} \leq \begin{bmatrix} k_0 \\ k_0 \\ \gamma k_0 \\ \gamma k_0 \\ k_u \\ k_u \end{bmatrix} \tag{8}$$

The inequalities in (8) have been arranged in three groups of inequalities. The Fourier-Motzkin projection algorithm uses (positive) linear combinations of inequalities to remove the dependency on the variable that is to be eliminated in the projection. *Ignoring for now* the possibility of linear combinations involving only the second group of inequalities leading to non-redundant lower-dimensional constraints after the projection, one is left with a similarly simple problem as in the $1d$ case. The orientation of the constraints do not change between iterations, making the number of constraints constant ($= 2n$) for each iteration. This is illustrated in Figure 2, where for the same dynamical system as in example 1 we start with initial constraints aligned with the modes of the system. We see that at each iteration we get the same number of constraints. Here γ is chosen as 0.98, and we see that the iterations make the polytope shrink only along the mode corresponding to the eigenvalue that is larger than 0.98. Convergence is not faster than before. However, in this case it is trivial to calculate the polytope to which the iteration will converge - shown in solid red in the figure. The calculation of the converged (red) polytope in Fig. 2 has been verified by performing an iteration of the Dorea and Hennet algorithm, returning the same polytope as the starting point. For a multiple-input state space model we find that the iterations for the γ-contractive set for the mode described by eigenvector/eigenvalue pair c ($c \in [0, \ldots, n_x]$) are given by

$$k_{c,i} = (\gamma/\lambda_c)k_{c,i-1} + \sum_{r=1}^{n_u} \left(|[V^{-1}B]_{c,r}| k_{u,r} \right) / \lambda_c \tag{9}$$

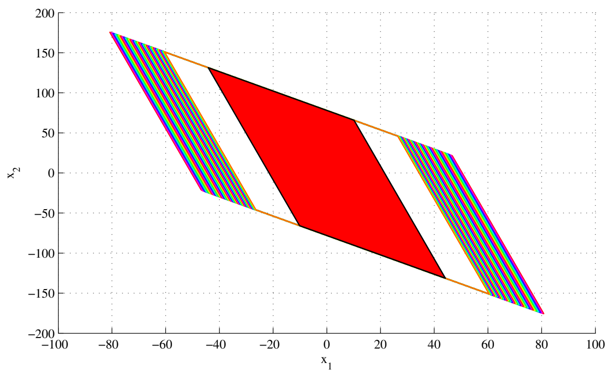

Fig. 2. The result of 100 iterations with initial constraints along system modes - and final polytopic convergence limit set.

and again we find that the steady state value of k_c can be expressed in an analytic form, thus avoiding any iterative procedure.

Above we ignored the case when linear combinations within the second group of inequalities in (8) lead to non-redundant constraints after projection. These are cases where the required input usage for the different modes conflict, and the resulting limitation in contraction factor (if such a simple contractive set is desired) does not result from input constraints. A simple way of testing whether this is a problem, is to perform one iteration of the Dorea-Hennet algorithm, and verify that the set does not change. If this fails, one may calculate the minimum achievable contraction factor for a given set by solving the bi-level optimization problem (Colson et al. [2005]):

$$max_x \gamma^* \tag{10a}$$

$$\begin{bmatrix} V^{-1} \\ -V^{-1} \end{bmatrix} x \leq \begin{bmatrix} k \\ k \end{bmatrix} \tag{10b}$$

$$[\gamma^* \; u^*] = arg\; min_{\gamma,u} 0.5\gamma^2 \tag{10c}$$

$$\begin{bmatrix} V^{-1} \\ -V^{-1} \end{bmatrix} (Ax + Bu) \leq \gamma \begin{bmatrix} k \\ k \end{bmatrix} \tag{10d}$$

$$\begin{bmatrix} I \\ -I \end{bmatrix} u \leq \begin{bmatrix} k_u \\ k_u \end{bmatrix} \tag{10e}$$

3. OSCILLATORY MODES

It was noted previously that the simple approach developed above is not directly applicable to oscillatory modes, corresponding to a complex conjugate pair of eigenvalues. However, if such oscillatory modes are sufficiently contractive, a low-complexity contractive set may nevertheless be found. We will use simple considerations involving the magnitude of the complex-conjugate eigenvalues to find a simple contractive set and a conservative estimate for the corresponding contraction factor. This conservative estimate may afterwards be improved using the results of Bitsoris [1988] or Hennet and Lasserre [1993]. Consider the eigenvalue decomposition of the A matrix, $A = V\Lambda V^{-1}$, which for oscillatory modes yield

$$\Lambda = diag\,(\cdots \;\; \lambda_r + i\lambda_i \;\; \lambda_r - i\lambda_i \;\; \cdots) \tag{11a}$$

$$V = [\cdots \;\; v_r + iv_i \;\; v_r - iv_i \;\; \cdots] \tag{11b}$$

where λ_r and λ_i are scalars, and v_r and v_i are column vectors of length n_x. Instead of fully diagonalizing A, we choose instead to use the similarity transform resulting in a 2×2 block for each oscillatory mode, *i. e.*,

$$A = V_b \Lambda_b V_b^{-1} \tag{12a}$$

$$\Lambda_b = \begin{bmatrix} \lambda_1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \lambda_r & \lambda_i & \ddots \\ \vdots & \ddots & \ddots & -\lambda_i & \lambda_r & \vdots \\ 0 & \ddots & \ddots & \cdots & \cdots & \ddots \end{bmatrix} \tag{12b}$$

$$V_b = \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_1 & v_2 & \vdots & v_r & v_i & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{12c}$$

Consider next Figure 3. A contractive oscillatory mode corresponding to a pair of complex conjugate eigenvalues of magnitude $|\lambda|$ starting inside the outer circle will by necessity in the next timestep be inside the inner circle. Consequently, a mode starting inside the outer square will in the next timestep end up inside the inner square. Simple geometric considerations then show that there exists a contraction factor

$$\gamma \geq |\lambda|\sqrt{2} \tag{13}$$

for the square-shaped set. Clearly, selecting a square contractive set is conservative, a less conservative bound could be derived for polytopic shapes (typically in the shape of a regular polygon) that more closely approximate the circle.

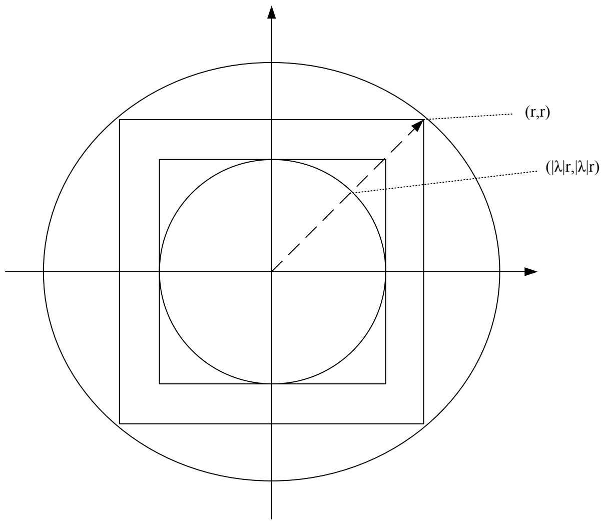

Fig. 3. Finding a polytopic contractive set for a contractive oscillatory mode.

4. CALCULATING AN INITIAL SET ALIGNED WITH THE MODES OF THE SYSTEM

This section addresses the fulfillment of assumption 3 above. The methods of sections 2-3 make it straight forward to find a low-complexity controlled contractive set (if this low-complexity set exists for the specified contraction factor), with calculations that are decoupled for each mode (or pair of oscillatory modes). However, even though the magnitude of each mode is acceptable with respect to the original constraints, it does not necessarily follow that the overall state fulfills the constraints. This is illustrated in Fig. 4.

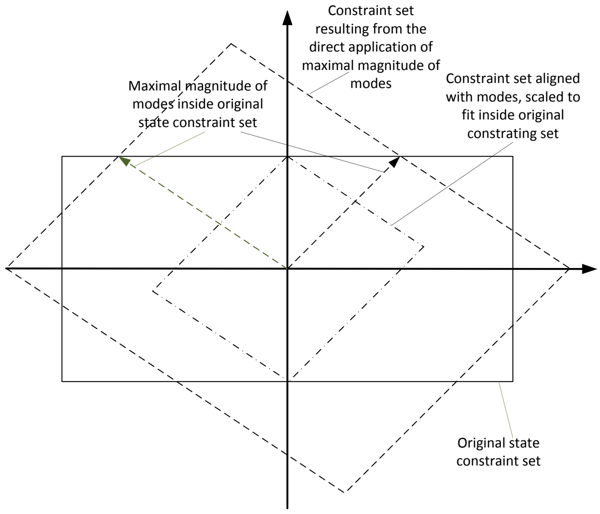

Fig. 4. Original state constraint set and constraint on individual modes.

Therefore, an initial set aligned with the modes of the system that do not violate the state constraints is needed. It is of course desirable that this initial set is as large as possible, since the final set (in which the controller is defined) will be a subset of this initial set. An algorithm to calculate such an initial set is proposed next.

Given a system (1) with diagonalizable A-matrix $A = V\Lambda V^{-1}$, and state constraints $Hx \leq h$. If the system contains (sufficiently contractive) oscillatory modes, use the matrix V_b from (12c) instead of V. It is assumed that the state constraints describe a bounded set, otherwise the set can be intersected with a large 'box' to make it bounded. In the following, it is assumed that desired initial set is also symmetric, and thus can be represented as

$$\begin{bmatrix} V^{-1} \\ -V^{-1} \end{bmatrix} x \leq \begin{bmatrix} k_x \\ k_x \end{bmatrix}. \tag{14}$$

Minor modifications are necessary if that is not the case. A set aligned with the modes of the system can be calculated by solving then following bi-level optimization problem:

$$\max_k J(k) = \log\left(\prod_j k_{x,j}\right) \tag{15a}$$

$$y_i - h_i \leq 0 \forall i \tag{15b}$$

$$y_i = \max_x H_i x \tag{15c}$$

$$\begin{bmatrix} V^{-1} \\ -V^{-1} \end{bmatrix} x \leq \begin{bmatrix} k_x \\ k_x \end{bmatrix} \tag{15d}$$

Here, j identifies the element of the vector k_x, i refers to row i of the inequality set $Hx \leq h$, and there are thus as many lower-level problems as there are such inequalities. However, if the inequality set is symmetric, only half of them need to be considered. The common re-formulations of bi-level programming (Colson et al. [2005])

apply. However, this will be practical only for cases with few state constraints and few states.

Instead, careful examination of the geometry of the problem and the KKT conditions of the lower-level problems will be employed. Assume that none of the constraints in (15d) are parallel to H_i. If this assumption is violated, a constraint on the corresponding element of k can be set directly, and subproblem i can be removed. Consider an initial parameter guess k_x^*, and let y_i^* be the corresponding optimal solution to lower level problem i, and W_i^a the matrix collecting the left-hand-sides of the corresponding set of active constraints. From the geometry of the problem one can observe that W_i^a is independent of the value of k_x^*. Each lower-level problem i thus imposes the following constraint on the upper level problem

$$H_i \left[W_i^a\right]^{-1} (k_x - k_x^*) \leq (h_i - y_i^*) \tag{16}$$

The first and second order gradients of the objective function are trivial to calculate:

$$\frac{dJ(k)}{dk} = \left[\cdots \ \frac{1}{k_{x,j}} \ \cdots\right] \tag{17a}$$

$$\frac{d^2J(k)}{dk^2} = \operatorname{diag}\left(\left[\cdots \ \frac{-1}{k_{x,j}^2} \ \cdots\right]\right) \tag{17b}$$

It is thus simple to formulate a quadratic approximation to the upper-level problem, where only the gradient and Hessian of the upper-level optimization problem changes between iterations.

Note (see, *e.g.,* Fig. 4) that the scaling to achieve a maximal size set aligned with system modes that is inside the original state constraint set will depend on the weight given to each of the modes. The objective function in (15) maximizes the product of the elements of the vector k_x. If different weights are given to different modes, a linear objective function may be chosen instead. In this case the overall problem becomes a single LP problem. Clearly, if the system has oscillatory modes, and a contractive set as described in section 3 is desired, the two $k_{x,j}$'s associated with the same oscillatory mode must be equal.

Example 2
Consider the following system, taken from Grieder et al. [2004]:

$$x_{t+1} = \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix} x_t + \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix} u_t$$

with the constraints $-2 \leq u_t \leq 2$ and $-100 \leq x_i \leq 100$, $i = \{1, 2\}$. The largest set (in terms of the product of the elements of k) inside the state constraints and aligned with the modes of the system is shown in Fig. 5. Shown for comparison are also the Maximal Output Admissible Set (MOAS) for an LQ controller designed with unity weights on inputs and states, and the corresponding feasible region for an MPC controller with horizon $N = 10$ and the MOAS as terminal set.

5. CONSTRAINT CONTROL DESIGN

Having a controlled contractive set, there are several possible controller design formulations possible that can utilize the availability of such a set. Assume that a controlled contractive set as defined by (14) has been found, and that using the techniques of sections 2 and 3 this has been found

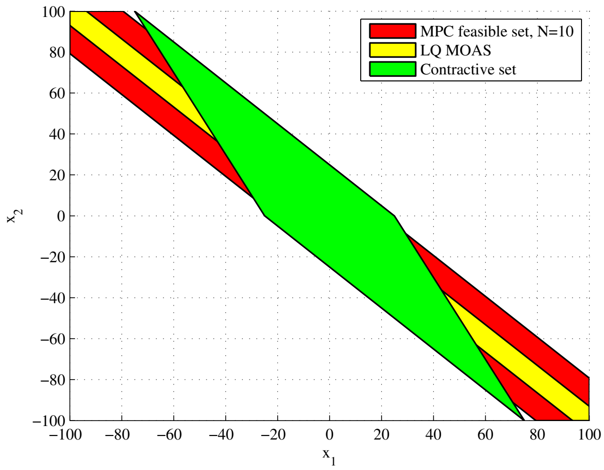

Fig. 5. Maximal set aligned with system modes inside original state constraints, maximal output admissible set for LQ controller, and feasible region for MPC controller.

to be controlled contractive with a contraction factor γ^*. One possible control problem formulation is then

$$\min_{u_t} x_{t+1}^T Q x_{t+1} + u_t^T R u_t \tag{18a}$$

$$x_{t+1} = Ax_t + Bu_t \tag{18b}$$

$$\begin{bmatrix} V^{-1} \\ -V^{-1} \end{bmatrix} x_{t+1} \leq \gamma\alpha \begin{bmatrix} k_x \\ k_x \end{bmatrix} \tag{18c}$$

where

$$\alpha = \max_j \frac{k_{x,j}}{|[V^{-1}x_t]_j|}$$

and $\gamma^* \leq \gamma < 1$. From the problem formulation and the construction of the controlled contractive set, it follows trivially that the optimization problem is recursively feasible (feasible if it is feasible for the initial state) and exponentially stable. Note that we have here chosen an objective function that weights the predicted state against the input usage, under the contraction constraint. Clearly, an alternative would be a straight forward maximization of the rate of contraction $(1 - \gamma)$.

Example 3.
An 82-state distillation column model is considered ('Column A', Skogestad [Accessed August 2013]). With level and pressure loops closed, the model has two controlled variables (the top and bottom compositions) and two control inputs (manipulated variables, the reflux flowrate and the bottom boilup). The model is scaled such that the largest acceptable control offset is 1 (for each output), and the largest control input is 1 (for each input). The model is discretized with a timestep of 4 minutes (it is a slow process). Due to the exceptionally strong coupling between the states, the eigenvector matrix is virtually singular, and the techniques of this paper therefore do not apply directly. However, the model may be reduced without sacrificing much accuracy. Accordingly, the model is reduced to 18 states, resulting in a stable model whose A-matrix has 6 real-valued eigenvalues and 6 pairs of complex-conjugate eigenvalues. The complex-valued eigenvalues are of small magnitude, and thus cause no problem (according to section 3) for ensuring a reasonable contraction factor.

First an initial set aligned with the modes of the system is found, according to section 4. This is calculated with the minimum and maximum constraints on each output, and in addition specifying rather wide minimum and maximum constraints on each state (to ensure a bounded optimization criterion). The resulting initial set allows for a variation in output 2 of ± 1, while for output 1 the largest variation inside the initial set is ± 0.66. It is thus clear that aligning the operating region with the modes of the system to some extent limits the allowable operating region.

The two largest open-loop eigenvalues of the system are $\lambda_1 = 0.98$ and $\lambda_2 = 0.72$. A contraction factor of 0.8 is chosen, and it is easily verified that this contraction factor can be achieved for the initial set calculated above. Note that the chosen contraction factor corresponds to a decrease in the dominant closed loop time constant by a factor of around 10. A parametric solution to (18) is then found, with x_t and α as parameters. This parametric solution partitions the operating region into 218 partitions, and for each of these partitions and affine state feedback is found. Figure 6 shows the closed loop output response, starting from $y = [\,-0.1347\ \ -1.0000\,]^T$.

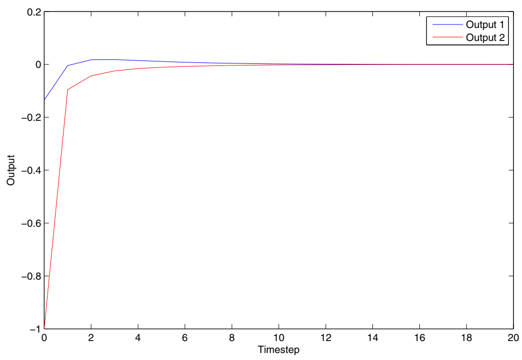

Fig. 6. Controlled outputs of distillation column A.

The figure demonstrates good control with a relatively simple explicit controller (218 partitions for an 18 state system). This is achieved with acceptable input usage (not shown due to space constraints).

6. CONCLUSION

A method for calculating a simple controlled contractive set aligned with the modes of a system is described, and a controller design based on the controlled contractive set is proposed. These methods are applied to the controller design for an 18-state distillation column model, resulting in an explicit design with only 218 partitions. The proposed method seems most appropriate for systems where a single mode is significantly slower than the other modes. The simple shape and simple calculation of the contractive set are the main advantages of the proposed approach - there may exist considerably larger contractive sets.

Further work is needed to consider disturbance handling (with a guaranteed contraction factor), robustness issues, unstable oscillatory modes, as well as repeated poles in series (resulting in non-diagonalizable A-matrices).

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