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On the $\mathcal{H}_\infty$ Norm of 2D Mixed Continuous-Discrete-Time Systems via Rationally-Dependent Complex Lyapunov Functions

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Abstract: This paper addresses the problem of determining the $\mathcal{H}_\infty$ norm of 2D mixed continuous-discrete-time systems. A novel approach is proposed based on the use of a class of complex Lyapunov functions with rational dependence on a parameter, which provides upper bounds on the sought norm via linear matrix inequalities (LMIs). It is also shown that the provided upper bounds are nonconservative by using rational functions in the chosen class with degree sufficiently large. Some numerical examples illustrate the proposed approach.

1. INTRODUCTION

An important area of control systems is represented by 2D mixed continuous-discrete-time systems. Indeed, such systems contain both continuous-time and discrete-time dynamics, which mutually influence each other. Their study has a long history, the reader is referred to Roesser [1975], Fornasini and Marchesini [1978] for the introduction of basic models and fundamentals properties. 2D mixed continuous-discrete-time systems can be found in a number of applications, including repetitive processes Rogers and Owens [1992], disturbance propagation in vehicle platoons Fornasini and Valcher [1997], and irrigation channels Li et al. [2005], Knorn and Middleton [2013].

Fundamental problems in 2D mixed continuous-discrete-time systems include stability analysis, which has been considered in a number of works, see for instance Rogers and Owens [2002], Galkowski [2002], Kar and Singh [2003], Galkowski et al. [2003], Bouagada and Van Dooren [2013], Chesi and Middleton [2014]. As in typical 1D systems, another fundamental problem in 2D mixed continuous-discrete-time systems is performance analysis. In particular, the computation of the $\mathcal{H}_\infty$ norm has been investigated in the literature in order to compute the $\mathcal{L}_2$ gain of the system. Existing works include Paszke et al. [2008, 2011] which propose sufficient conditions based on linear matrix inequalities (LMIs) for establishing upper bounds on the $\mathcal{H}_\infty$ norm.

This paper addresses the problem of determining the $\mathcal{H}_\infty$ norm of 2D mixed continuous-discrete-time systems. A novel approach is proposed based on the use of a class of complex Lyapunov functions with rational dependence on a parameter, which provides upper bounds on the sought norm via LMIs. It is also shown that the provided upper bounds are nonconservative by using rational functions in the chosen class with degree sufficiently large. Some numerical examples illustrate the proposed approach. This paper extends our previous work Chesi and Middleton [2014] which investigates the use of complex Lyapunov functions with polynomial dependence on a parameter, and where asymptotical non-conservatism is not guaranteed.

The paper is organized as follows. Section 2 provides the problem formulation and some preliminaries about SOS matrix polynomials. Section 3 describes the proposed results. Section 4 presents some illustrative examples. Lastly, Section 5 concludes the paper with some final remarks.

2. PRELIMINARIES

2.1 Problem Formulation

Notation:

- $\mathbb{R}, \mathbb{C}$: real and complex number sets;
- j: imaginary unit, i.e. $j^2 = -1$;
- I: identity matrix (of size specified by the context);
- $\Re(A)$, $\Im(A)$: real and imaginary parts of A;
- $\bar{A}$: complex conjugate of A;
- A^T, A^H: transpose and complex conjugate transpose of A;
- $adj(A)$: adjoint of A;
- $\det(A)$: determinant of A;
- $trace(A)$: trace of A;
- $\lambda_i(A)$: i-th eigenvalue of A;
- $\|A\|_2$: Euclidean norm of A;
- $|a|$: magnitude of a;
- Hermitian matrix A: a complex square matrix satisfying $A^H = A$;
- $\star$: corresponding block in symmetric or Hermitian matrices;
- $A > 0$, $A \geq 0$: Hermitian positive definite and Hermitian positive semidefinite matrix A.

Let us consider the 2D mixed continuous-discrete-time system described by

$$\begin{cases} \dfrac{d}{dt}x_c(t,k) &= A_{cc}x_c(t,k) + A_{cd}x_d(t,k) \\ &\quad +B_c u(t,k) \\ x_d(t,k+1) &= A_{dc}x_c(t,k) + A_{dd}x_d(t,k) \\ &\quad +B_d u(t,k) \\ y(t,k) &= C_c x_c(t,k) + C_d x_d(t,k) \\ &\quad +Du(t,k) \end{cases} \tag{1}$$

where $x_c \in \mathbb{R}^{n_c}$ and $x_d \in \mathbb{R}^{n_d}$ are the continuous and discrete states, respectively, the scalars t and k are independent variables, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^{n_y}$ are the input and output, respectively, and $A_{cc} \in \mathbb{R}^{n_c\times n_c}$, $A_{cd} \in \mathbb{R}^{n_c\times n_d}$, $A_{dc} \in \mathbb{R}^{n_d\times n_c}$, $A_{dd} \in \mathbb{R}^{n_d\times n_d}$, $B_c \in \mathbb{R}^{n_c\times n_u}$, $B_d \in \mathbb{R}^{n_d\times n_u}$, $C_c \in \mathbb{R}^{n_y\times n_c}$, $C_d \in \mathbb{R}^{n_y\times n_d}$ and $D \in \mathbb{R}^{n_y\times n_u}$ are given matrices.

The system (1) is said to be exponentially stable (see, e.g., Pandolfi [1984], Yeganefar et al. [2013]) if there exist $\beta, \delta \in \mathbb{R}$ such that

$$\left\| \begin{pmatrix} x_c(t,k) \\ x_d(t,k) \end{pmatrix} \right\|_2 \leq \beta\varrho e^{-\delta \min\{t,k\}} \tag{2}$$

for all $t \geq 0$, $k \geq 0$ and initial conditions $x_c(0,k)$ and $x_d(t,0)$, where

$$\begin{array}{c} \varrho = \max\{\varrho_1, \varrho_2\} \\ \varrho_1 = \sup_{t\geq 0} \|x_d(t,0)\|_2, \quad \varrho_2 = \sup_{k\geq 0} \|x_c(0,k)\|_2. \end{array} \tag{3}$$

The $\mathcal{H}_\infty$ norm of the system (1) is defined as

$$\gamma_\infty = \sup_{u:\ \|u\|_{\mathcal{L}_2}\neq 0} \frac{\|y\|_{\mathcal{L}_2}}{\|u\|_{\mathcal{L}_2}} \tag{4}$$

and $\|\cdot\|_{\mathcal{L}_2}$ is the $\mathcal{L}_2$ norm defined as

$$\|u\|_{\mathcal{L}_2} = \sqrt{\sum_{k=0}^{\infty} \int_0^\infty \|u(t,k)\|_2^2 dt}. \tag{5}$$

Problem. The problem addressed in this paper consists of determining the $\mathcal{H}_\infty$ norm of the system (1), i.e., γ_∞ in (4). □

2.2 SOS Matrix Polynomials

Here we provide some information about establishing whether a matrix polynomial is SOS via an LMI feasibility test.

Let us consider a Hermitian matrix polynomial $M : \mathbb{R} \to \mathbb{R}^{n\times n}$ of degree $2d$. The matrix polynomial $M(\omega)$, $\omega \in \mathbb{R}$, is said to be SOS if there exist matrix polynomials $M_i : \mathbb{R} \to \mathbb{R}^{n\times n}$, $i = 1, \ldots, k$, such that

$$M(\omega) = \sum_{i=1}^{k} M_i(\omega)^T M_i(\omega). \tag{6}$$

A necessary and sufficient condition for establishing whether $M(\omega)$ is SOS can be obtained via an LMI feasibility test.

Indeed, $M(\omega)$ can be expressed as

$$M(\omega) = (b(\omega)\otimes I)^T (K + L(\alpha))(b(\omega)\otimes I) \tag{7}$$

where $b(\omega) \in \mathbb{R}^c$ is a vector whose entries are the monomials in ω of degree less than or equal to d, and c is the number of these monomials given by

$$c = d + 1, \tag{8}$$

$K \in \mathbb{R}^{cn\times cn}$, $K = K^T$, satisfies

$$M(\omega) = (b(\omega)\otimes I)^T K (b(\omega)\otimes I), \tag{9}$$

$L : \mathbb{R}^\tau \in \mathbb{R}^{cn\times cn}$ is a linear parametrization of the linear subspace

$$\mathcal{L} = \left\{ L = L^T :\ (b(\omega)\otimes I)^T L (b(\omega)\otimes I) = 0 \right\} \tag{10}$$

and $\alpha \in \mathbb{R}^\tau$ is a free vector. The quantity τ is the dimension of $\mathcal{L}$ given by

$$\tau = \frac{1}{2} n \left(c(cn+1) - (n+1)(2d+1)\right). \tag{11}$$

The representation (7) is known as square matrix representation (SMR) Chesi et al. [2003] and extends the Gram matrix method for (scalar) polynomials to the matrix case. One has that $M(\omega)$ is SOS if and only if there exists α satisfying the LMI

$$K + L(\alpha) \geq 0. \tag{12}$$

See for instance Chesi [2010] and references therein for details on SOS matrix polynomials.

3. PROPOSED RESULTS

In this section we address the problem of determining the $\mathcal{H}_\infty$ norm of the system (1), i.e., γ_∞ in (4).

Let us start by introducing the following assumption, which is a necessary condition for exponential stability of the system (1).

Assumption 1. The matrix A_{cc} is Hurwitz (i.e., all its eigenvalues have negative real parts) and the matrix A_{dd} is Schur (i.e., all its eigenvalues have magnitude less than one). □

The fact that Assumption 1 is a necessary condition for exponential stability of the system (1) can be easily verified by considering $u(t,k) = 0$ and $x_d(t,k) = 0$, since one would get $x_c(t,0) = \exp(A_{cc}t)x_c(0,0)$, or $u(t,k) = 0$ and $x_c(t,k) = 0$, since one would get $x_d(0,k) = A_{dd}^k x_d(0,0)$.

For $S \in \mathbb{C}^{n\times n}$, let us define the function

$$\Phi(S) = \begin{pmatrix} S_R & S_I \\ -S_I & S_R \end{pmatrix} \tag{13}$$

where $S_R, S_I \in \mathbb{R}^{n\times n}$ are the real and imaginary parts of S, i.e., $S = S_R + jS_I$. Let us observe that

$$S \text{ is Hermitian} \iff \Phi(S) = \Phi(S)^T. \tag{14}$$

Let us denote with $U_L(s,k)$ and $Y_L(s,k)$ the Laplace transforms of $u(t,k)$ and $y(t,k)$, respectively, where $s \in \mathbb{C}$. Let us denote with $U_{LZ}(s,z)$ and $Y_{LZ}(s,z)$ the Z-transforms of $U_L(s,k)$ and $Y_L(s,k)$, respectively, where $z \in \mathbb{C}$. The transfer function from $u(t,k)$ and $y(t,k)$ can be expressed as

$$F(s,z) = \frac{Y_{LZ}(s,z)}{U_{LZ}(s,z)} \tag{15}$$

and standard manipulations lead to

$$F(s,z) = F_3(s)\left(zI - F_1(s)\right)^{-1} F_2(s) + F_4(s) \quad (16)$$

where

$$\begin{cases} F_1(s) &= A_{dc}(sI - A_{cc})^{-1}A_{cd} + A_{dd} \\ F_2(s) &= A_{dc}(sI - A_{cc})^{-1}B_c + B_d \\ F_3(s) &= C_c(sI - A_{cc})^{-1}A_{cd} + C_d \\ F_4(s) &= C_c(sI - A_{cc})^{-1}B_c + D. \end{cases} \quad (17)$$

We express $F_i(s)$, $i = 1,\ldots,4$, as

$$F_i(s) = \frac{G_i(s)}{g(s)} \quad (18)$$

where $G_i(s)$, $i = 1,\ldots,4$, are matrix polynomials of suitable size, and $g(s)$ is defined as

$$g(s) = \det(sI - A_{cc}). \quad (19)$$

Let us define

$$F_\omega(z) = F(j\omega, z). \quad (20)$$

The $\mathcal{H}_\infty$ norm of the system (1) can be written as

$$\gamma_\infty = \sup_{\omega\in\mathbb{R}} \|F_\omega\|_{\mathcal{H}_\infty} \quad (21)$$

where

$$\|F_\omega\|_{\mathcal{H}_\infty} = \sup_{\theta\in[-\pi,\pi]} \left\|F_\omega(e^{j\theta})\right\|_2. \quad (22)$$

For a matrix function $M : \mathbb{R} \to \mathbb{C}^{n_1\times n_2}$, we say that $M(\omega)$ is *symmetric with respect to* ω if

$$M(-\omega) = \overline{M(\omega)} \quad \forall\omega \in \mathbb{R} \quad (23)$$

and we say that $M(\omega)$ is *anti-symmetric with respect to* ω if

$$M(-\omega) = -\overline{M(\omega)} \quad \forall\omega \in \mathbb{R}. \quad (24)$$

The next result provides a property of Hermitian matrix functions $M(\omega)$ that are positive semidefinite for all $\omega \in \mathbb{R}$.

Theorem 1. Let $M : \mathbb{R} \to \mathbb{C}^{n\times n}$ be a Hermitian matrix function, and decompose $M(\omega)$ as

$$M(\omega) = M_s(\omega) + M_a(\omega) \quad (25)$$

where $M_s, M_a : \mathbb{R} \to \mathbb{C}^{n\times n}$ are Hermitian matrix functions that are symmetric and anti-symmetric, respectively, with respect to ω, and given by

$$\begin{cases} M_s(\omega) &= \dfrac{M(\omega) + \overline{M(-\omega)}}{2} \\ M_a(\omega) &= \dfrac{M(\omega) - \overline{M(-\omega)}}{2}. \end{cases} \quad (26)$$

Then,

$$M(\omega) \geq 0 \quad \forall\omega \in \mathbb{R} \quad (27)$$

implies

$$M_s(\omega) \geq 0 \quad \forall\omega \in \mathbb{R}. \quad (28)$$

Proof. Suppose that (27) holds and, for contradiction, that (28) does not. This means that there exists $\hat\omega \in \mathbb{R}$ such that

$$M_s(\hat\omega) \not\geq 0. \quad (29)$$

Hence, there exists $b \in \mathbb{C}^n$ such that

$$b^H M_s(\hat\omega) b < 0. \quad (30)$$

At this point there are two possibilities: either $b^H M_a(\hat\omega) b \leq 0$ or $b^H M_a(\hat\omega) b > 0$. If the first possibility holds, then

$$\begin{aligned} b^H M(\hat\omega) b &= b^H M_s(\hat\omega) b + b^H M_a(\hat\omega) b \\ &< 0 \end{aligned} \quad (31)$$

which means that $M(\hat\omega) \not\geq 0$ hence contradicting (27). If the second possibility holds, then one has

$$\begin{cases} \bar b^H M_s(-\hat\omega)\bar b &= b^H M_s(\hat\omega) b \\ \bar b^H M_a(-\hat\omega)\bar b &= -b^H M_a(\hat\omega) b \end{cases} \quad (32)$$

and, consequently,

$$\begin{aligned} \bar b^H M(-\hat\omega)\bar b &= \bar b^H M_s(-\hat\omega)\bar b + \bar b^H M_a(-\hat\omega)\bar b \\ &< 0 \end{aligned} \quad (33)$$

which similarly contradicts (27). Therefore, (28) holds. □

Theorem 1 states that, if a Hermitian matrix function $M(\omega)$ is positive semidefinite for all $\omega \in \mathbb{R}$, then also its part $M_s(\omega)$ enjoys the same property.

Let us define the set

$$\begin{aligned} \mathcal{S}(n) = \{&M : \mathbb{R} \to \mathbb{C}^{n\times n},\ M(\omega) \text{ is a} \\ &\text{Hermitian matrix polynomial} \\ &\text{symmetric with respect to } \omega\}. \end{aligned} \quad (34)$$

In order to compute the $\mathcal{H}_\infty$ norm of the system (1), we introduce Lyapunov function candidates defined by

$$V_{RAT}(\omega) = \frac{V(\omega)}{v(\omega)} \quad (35)$$

where $V \in \mathcal{S}(n_d)$ has degree $2d$, with d integer, and

$$v(\omega) = (1+\omega^2)^d. \quad (36)$$

For $\xi \in \mathbb{R}$ let us define

$$Q(\omega) = \begin{pmatrix} q_1 & q_2 \\ \star & q_3 \end{pmatrix} \quad (37)$$

where

$$\begin{cases} q_1 &= |g(j\omega)|^2 V(\omega) - G_1(j\omega)V(\omega)G_1(j\omega)^H \\ &\quad -v(\omega)G_2(j\omega)G_2(j\omega)^H \\ q_2 &= -G_1(j\omega)V(\omega)G_3(j\omega)^H - v(\omega)G_2(j\omega)G_4(j\omega)^H \\ q_3 &= \xi v(\omega)\,|g(j\omega)|^2 I - G_3(j\omega)V(\omega)G_3(j\omega)^H \\ &\quad -v(\omega)G_4(j\omega)G_4(j\omega)^H. \end{cases} \quad (38)$$

Since $G_i(\omega)$ is symmetric with respect to ω for all $i = 1,\ldots,4$, it follows that

$$Q \in \mathcal{S}(n_q) \quad (39)$$

where

$$n_q = n_d + n_u. \quad (40)$$

The following result provides an upper bound on the $\mathcal{H}_\infty$ norm of the system (1) via a semidefinite program.

Theorem 2. Suppose that there exist $V \in \mathcal{S}(n_d)$ of degree $2d$ and $\xi, \varepsilon \in \mathbb{R}$ such that

$$\begin{cases} \Phi(Q(\omega) - \varepsilon v(\omega)\,|g(j\omega)|^2 I) \text{ is SOS} \\ \varepsilon > 0. \end{cases} \quad (41)$$

Then,

$$\sqrt{\xi} > \gamma_\infty. \quad (42)$$

Proof. Suppose that there exist $V \in \mathcal{S}(n_d)$ and $\xi, \varepsilon \in \mathbb{R}$ such that (41) holds. From the definition of SOS matrix polynomials in Section 2.2, the first constraint in (41) implies that

$$\Phi\left(Q(\omega) - \varepsilon v(\omega)\,|g(j\omega)|^2 I\right) \geq 0 \quad \forall\omega \in \mathbb{R}. \quad (43)$$

From (13) it follows that

$$Q(\omega) \geq \varepsilon v(\omega)\,|g(j\omega)|^2 I \quad \forall\omega \in \mathbb{R}. \quad (44)$$

Since Assumption 1 implies that

$$|g(j\omega)| > 0 \quad \forall \omega \in \mathbb{R}, \tag{45}$$

one can write

$$Q(\omega) = v(\omega)\,|g(j\omega)|^2\,\hat{Q}(\omega) \tag{46}$$

where

$$\hat{Q}(\omega) = \begin{pmatrix} \hat{q}_1 & \hat{q}_2 \\ \star & \hat{q}_3 \end{pmatrix} \tag{47}$$

and

$$\begin{cases} \hat{q}_1 = V_{RAT}(\omega) - G_1(j\omega)V_{RAT}(\omega)G_1(j\omega)^H \\ \qquad -G_2(j\omega)G_2(j\omega)^H \\ \hat{q}_2 = -G_1(j\omega)V_{RAT}(\omega)G_3(j\omega)^H - G_2(j\omega)G_4(j\omega)^H \\ \hat{q}_3 = \xi I - G_3(j\omega)V_{RAT}(\omega)G_3(j\omega)^H \\ \qquad -G_4(j\omega)G_4(j\omega)^H. \end{cases} \tag{48}$$

Hence, (44)–(45) imply

$$\hat{Q}(\omega) \geq \varepsilon \quad \forall \omega \in \mathbb{R}. \tag{49}$$

Since $\varepsilon > 0$ due to the second constraint in (41), from the bounded real lemma and Schur complement it follows that (see, e.g., de Oliveira et al. [2002])

$$\sqrt{\xi} \geq \|F_\omega\|_{\mathcal{H}_\infty} \quad \forall \omega \in \mathbb{R}. \tag{50}$$

Hence, (42) holds since this implies

$$\begin{aligned} \sqrt{\xi} &\geq \sup_{\omega} \|F_\omega\|_{\mathcal{H}_\infty} \\ &= \gamma_\infty. \end{aligned} \tag{51}$$

□

Theorem 2 provides a condition for establishing an upper bound on the $\mathcal{H}_\infty$ norm of the system (1), γ_∞. This condition is based on the search for a matrix polynomial $V(\omega)$ of degree $2d$ in the set $\mathcal{S}(n_d)$ and scalars ξ, ε such that (41) holds. Hence, the condition provided by Theorem 2 is equivalent to an LMI feasibility test as explained in Section 2.2 since $\Phi(Q(\omega)-\varepsilon I)$ is affine linear in the variables $V(\omega)$, ξ and ε. Let us observe that $V_{RAT}(\omega)$ defines a complex Lyapunov function candidate with rational dependence on ω of degree $2d$ and structure defined by the set $\mathcal{S}(n_d)$.

It is possible to show that the conservatism of the condition provided by Theorem 2 is monotonically non-increasing with $2d$, i.e., (41) holds with $2d+2$ if it holds with $2d$.

The number of LMI scalar variables in the condition provided by Theorem 2 is given by the number of free coefficients in the matrix polynomial $V(\omega)$, plus two (for the scalars ξ, ε), plus the length of the vector α required to establish whether $\Phi(Q(\omega)-\varepsilon I)$ is SOS according to Section 2.2. The following results states an important property of the condition provided by Theorem 2, namely that this condition is nonconservative by using $V_{RAT}(\omega)$ of degree sufficiently large.

Theorem 3. Let $\xi \in \mathbb{R}$ be such that $\sqrt{\xi} > \gamma_\infty$. Then, there exists a sufficiently large integer d such that (41) holds for some $V \in \mathcal{S}(n_d)$ of degree $2d$ and $\varepsilon \in \mathbb{R}$.

Proof. Suppose that $\sqrt{\xi} > \gamma_\infty$. Then, there exists a scalar $\hat{\varepsilon} > 0$ and a Hermitian matrix function $\hat{V} : \mathbb{R} \to \mathbb{C}^{n_d \times n_d}$ such that (49) holds with $V_{RAT}(\omega)$ and ε replaced by $\hat{V}(\omega)$ and $\hat{\varepsilon}$, respectively. The limit for ω that tends to infinity of such a matrix function $\hat{V}(\omega)$ does exist, i.e.,

$$\lim_{\omega \to \infty} \hat{V}(\omega) = \hat{V}_\infty \tag{52}$$

for some symmetric matrix $\hat{V}_\infty \in \mathbb{R}^{n_d \times n_d}$. Since the matrices of the system (1) are real, one has

$$\left.\begin{aligned} G_i(j\omega) &= \overline{G_i(-j\omega)} \;\; \forall i = 1, \ldots, 4 \\ g(j\omega) &= \overline{g(-j\omega)} \end{aligned}\right\} \quad \forall \omega \in \mathbb{R}. \tag{53}$$

From Theorem 1, this implies that $\hat{V}(\omega)$ can be assumed symmetric with respect to ω without loss of generality.

Let us define

$$\hat{V}_R(\omega) = \Re(\hat{V}(\omega)). \tag{54}$$

Since $\hat{V}(\omega)$ is symmetric with respect to ω, it follows that $\hat{V}_R(\omega)$ can be rewritten as

$$\hat{V}_R(\omega) = \hat{V}_1(\omega^2) \tag{55}$$

where $\hat{V}_1 : \mathbb{R} \to \mathbb{R}^{n_d \times n_d}$ is a symmetric matrix function. Let us define

$$\begin{cases} m_1(\psi) = \dfrac{\psi}{1-\psi} \\ m_2(\omega) = \dfrac{\omega^2}{1+\omega^2} \end{cases} \tag{56}$$

and

$$\hat{V}_2(\psi) = \hat{V}_1(m_1(\psi)). \tag{57}$$

It follows that $\hat{V}_1(\omega^2)$ and $\hat{V}_1(\psi)$ are the same function defined on different domains, i.e.,

$$\forall \omega \in \mathbb{R},\ \exists \psi = m_2(\omega) \in [0,1) :\ \hat{V}_1(\omega^2) = \hat{V}_2(\psi). \tag{58}$$

Since $\hat{V}_2(\psi)$ is continuous and the limit for ψ that tends to 1 of $\hat{V}_2(\psi)$ does exist, in particular

$$\lim_{\psi \to 1} \hat{V}_2(\psi) = \hat{V}_\infty, \tag{59}$$

it follows that $\hat{V}_2(\psi)$ can be approximated arbitrarily well over $[0,1]$ by a symmetric matrix polynomial $\hat{V}_3 : \mathbb{R} \to \mathbb{R}^{n_d \times n_d}$. Hence, let us define

$$\hat{V}_4(\omega) = \hat{V}_3(m_2(\omega)). \tag{60}$$

It follows that $\hat{V}_4(\omega)$ is a symmetric rational function that approximates arbitrarily well $\hat{V}_R(\omega)$. Moreover, from (56) and since $\hat{V}_4(\omega)$ is symmetric with respect to ω, it follows that $\hat{V}_4(\omega)$ has the form

$$\hat{V}_4(\omega) = \frac{V_R(\omega)}{v(\omega)} \tag{61}$$

where $v(\omega)$ is as in (36) for a suitable integer d and $V_R(\omega)$ is a symmetric matrix polynomial of degree $2d$ in the set $\mathcal{S}(n_d)$.

Next, let us define

$$\hat{V}_I(\omega) = \Im(\hat{V}(\omega)). \tag{62}$$

Since $\hat{V}(\omega)$ is symmetric with respect to ω, it follows that $\hat{V}_I(\omega)$ can be rewritten as

$$\hat{V}_I(\omega) = \omega \hat{V}_5(\omega^2) \tag{63}$$

where $\hat{V}_5 : \mathbb{R} \to \mathbb{R}^{n_d \times n_d}$ is an anti-symmetric matrix function. Similarly to $\hat{V}_1(\omega^2)$, $\hat{V}_5(m_1(\psi))$ can be approximated arbitrarily well by an anti-symmetric matrix polynomial $\hat{V}_6 : \mathbb{R} \to \mathbb{R}^{n_d \times n_d}$ over $[0,1]$, and hence

$$\hat{V}_7(\omega) = \omega \hat{V}_6(m_2(\omega)) \tag{64}$$

is an anti-symmetric rational function that approximates arbitrarily well $\hat{V}_I(\omega)$ of the form

$$\hat{V}_7(\omega) = \frac{V_I(\omega)}{v(\omega)} \tag{65}$$

where $V_I(\omega)$ is an anti-symmetric matrix polynomial of degree $2d$, with $jV_I(\omega)$ in the set $\mathcal{S}(n_d)$.

Lastly, let us define $V_{RAT}(\omega)$ as in (35) with $V(\omega)$ given by

$$V(\omega) = V_R(\omega) + jV_I(\omega). \tag{66}$$

Due to the continuity of $\hat{Q}(\omega)$ with $V_{RAT}(\omega)$, it follows that the degree $2d$ can be chosen such that (49) holds for some $\varepsilon > 0$. This implies that (43) holds. Since $\Phi\left(Q(\omega) - \varepsilon v(\omega)\,|g(j\omega)|^2 I\right)$ is a symmetric matrix polynomial and in a scalar variable, it follows that (43) holds if and only if (41) holds. The proof is concluded by observing that $V \in \mathcal{S}(n_d)$. $\square$

Theorem 3 states that the condition provided by Theorem 2 is nonconservative by choosing an integer d sufficiently large, where d defines the degree of $V_{RAT}(\omega)$ given by $2d$.

Let us define the quantity

$$\hat{\gamma}_\infty = \sqrt{\hat{\xi}} \tag{67}$$

where $\hat{\xi}$ is the solution of the semidefinite program

$$\begin{aligned} \hat{\xi} = & \inf_{\substack{V\in\mathcal{S}(n_d)\\ \xi,\varepsilon\in\mathbb{R}}} \xi \\ & \text{s.t.} \quad \begin{cases} \Phi(Q(\omega) - \varepsilon v(\omega)\,|g(j\omega)|^2 I) \text{ is SOS} \\ \varepsilon > 0. \end{cases} \end{aligned} \tag{68}$$

From Theorem 2 it follows that

$$\hat{\gamma}_\infty \geq \gamma_\infty, \tag{69}$$

and $\hat{\gamma}_\infty$ is the best upper bound on the $\mathcal{H}_\infty$ norm of the system (1) provided by Theorem 2 for a chosen degree $2d$ of $V_{RAT}(\omega)$. The computation of this upper bound amounts to solving the optimization problem (68), which is a semidefinite program since the cost function is linear and the constraints are LMIs.

4. EXAMPLES

In this section we present two illustrative examples of the proposed results. The LMI problems are solved with the toolbox SeDuMi Sturm [1999] for Matlab.

4.1 Example 1

Let us consider the problem of determining the $\mathcal{H}_\infty$ norm of the system (1) with

$$A_{cc} = -2,\ A_{cd} = \begin{pmatrix} 0.6 & -0.6 & 0.4 \end{pmatrix}$$

$$A_{dc} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ A_{dd} = \begin{pmatrix} 0 & -0.2 & 0 \\ 0 & 0 & 0.4 \\ 0.3 & 0 & 0 \end{pmatrix}$$

$$B_c = 1,\ B_d = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$C_c = 0.5,\ C_d = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix},\ D = -1.$$

Hence, it turns out that $n_d = 3$ and $n_c = n_u = n_y = 1$. Let us observe that the matrices A_{cc} and A_{dd} are Hurwitz and Schur, respectively.

Let us use compute the upper bound $\hat{\gamma}_\infty$ on the $\mathcal{H}_\infty$ norm γ_∞. We solve the semidefinite program (68) by using $V_{RAT}(\omega)$ as in (35) with degree $2d = 0$. We find $\hat{\xi} = 5.445$ and, hence,

$$\hat{\gamma}_\infty = 2.333.$$

The found $V_{RAT}(\omega)$ is

$$V_{RAT}(\omega) = \begin{pmatrix} 3.991 & 2.742 & -3.019 \\ \star & 5.676 & -4.689 \\ \star & \star & 4.474 \end{pmatrix}.$$

Brute force search shows that this upper bound is tight, i.e., $\hat{\gamma}_\infty = \gamma_\infty$. In particular, $\left\|F_\omega(e^{j\theta})\right\|_2 = \hat{\gamma}_\infty$ for $\omega = 6.000$ rad/s and $\theta = -1.209$ rad.

4.2 Example 2

Let us consider the problem of determining the $\mathcal{H}_\infty$ norm of the system (1) with

$$A_{cc} = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix},\ A_{cd} = \begin{pmatrix} 0.5 & 0.4 \\ -0.7 & 0 \end{pmatrix}$$

$$A_{dc} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},\ A_{dd} = \begin{pmatrix} 0.4 & -0.5 \\ 0.3 & 0.6 \end{pmatrix}$$

$$B_c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ B_d = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$C_c = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ C_d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, it turns out that $n_c = n_d = n_u = n_y = 2$. Let us observe that the matrices A_{cc} and A_{dd} are Hurwitz and Schur, respectively.

Let us use compute the upper bound $\hat{\gamma}_\infty$ on the $\mathcal{H}_\infty$ norm γ_∞. We solve the semidefinite program (68) by using $V_{RAT}(\omega)$ as in (35) with degree $2d = 0$. We find $\hat{\xi} = 3050.077$ and, hence,

$$\hat{\gamma}_\infty = 55.228.$$

The found $V_{RAT}(\omega)$ is

$$V_{RAT}(\omega) = \begin{pmatrix} 146.879 & -39.210 \\ \star & 119.378 \end{pmatrix}.$$

This upper bound can be improved by increasing the degree of $V_{RAT}(\omega)$. Indeed, with $2d = 2$ we find $\hat{\xi} = 155.276$ and, hence,

$$\hat{\gamma}_\infty = 12.461.$$

The found $V_{RAT}(\omega)$ is $\Re(V_{RAT}(\omega)) + j\Im(V_{RAT}(\omega))$ where

$$\Re(V_{RAT}(\omega)) = \frac{\begin{pmatrix} 37.830 + 27.557\omega^2 & -8.238 - 6.291\omega^2 \\ \star & 19.006 + 30.910\omega^2 \end{pmatrix}}{1+\omega^2}$$

$$j\Im(V_{RAT}(\omega)) = \frac{\begin{pmatrix} 0 & j14.450\omega \\ \star & 0 \end{pmatrix}}{1+\omega^2}.$$

Brute force search shows that this upper bound is tight, i.e., $\hat{\gamma}_\infty = \gamma_\infty$. In particular, $\left\|F_\omega(e^{j\theta})\right\|_2 = \hat{\gamma}_\infty$ for $\omega = 1.040$ rad/s and $\theta = 2.148$ rad.

We have also investigated the use of complex Lyapunov function candidates with polynomial dependence instead of rational, i.e., setting the denominator $v(\omega)$ to 1. Interesting, by using the degree $2d = 2$ as before, one finds only the upper bound 54.981.

Lastly, we have tested the method in Paszke et al. [2008] for comparison, which provides the upper bound 55.228.

5. CONCLUSION

We have proposed a novel approach for determining the $\mathcal{H}_\infty$ norm of 2D mixed continuous-discrete-time systems. The approach is based on the use of a class of complex Lyapunov functions with rational dependence on a parameter, and provides upper bounds on the sought norm via LMIs. It has also been shown that the provided upper bounds are nonconservative by using rational functions in the chosen class with degree sufficiently large.

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