

Efficient Convergence Rate Analysis of Multi-Agent Positive Systems Under Formation Control^{*}

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Abstract: This paper addresses the formation control problem of multi-agent systems whose dynamics are all positive. Recently, as a byproduct of the analysis of interconnected positive systems, we have shown an effective way for designing a communication scheme (i.e., an interconnection matrix) over the positive agents so that a prescribed formation can be achieved. For the convergence rate analysis of such multi-agent positive systems under formation control, we propose an efficient algorithm to compute the dominant pole of interconnected positive systems by actively using the positive property of each agent. We illustrate by numerical examples that the proposed algorithm is definitely efficient particularly when the number of agents gets larger.

Keywords: positive system

1. INTRODUCTION

Recently, the analysis and synthesis of positive systems have been active in conjunction with the development of convex-optimization-based control theory [Mason and Shorten (2007); Rantzer (2011, 2012); Tanaka and Langbort (2011); Blanchini et al. (2012); Ebihara et al. (2014)]. A linear time-invariant (LTI) system is said to be positive if its state and output are both nonnegative for any nonnegative initial state and nonnegative input [Farina and Rinaldi (2000); Kaczorek (2001)]. This property can be seen naturally in biology, network communications, economics and probabilistic systems. Moreover, simple dynamical systems such as integrator and first-order lag and their series/parallel connections are also all positive. Even though their dynamics are relatively simple, large-scale systems constructed from those subsystems exhibit complicated behavior and deserve investigation in the area of multi-agent systems [Olfati-Saber et al. (2007); Tanner et al. (2003a,b)]. Motivated by these facts, we made continuing efforts toward establishing effective methods for the analysis and synthesis of interconnected positive systems [Ebihara et al. (2011, 2013a)]. In particular, by exploiting the positive property of each agent, we showed an efficient method to design a communication scheme (i.e., the interconnection matrix) over the agents to achieve prescribed formation [Ebihara et al. (2013a)].

The convergence performance of such multi-agent positive systems under formation control can be determined by the dominant pole of the corresponding interconnected

positive systems (the dominant pole is often called algebraic connectivity in the study area of multi-agent systems [Olfati-Saber et al. (2007)]). Even though we can compute the dominant pole by directly computing all of the eigenvalues of the coefficient matrices of the interconnected system, such straightforward method is computationally demanding especially when the total number of agents gets larger. Moreover, such method does not give any insights on how we can accelerate the convergence by appropriately designing the interconnection matrices. Motivated by these facts, in this paper, we propose an efficient algorithm for the computation of the dominant pole by decomposing the original large size problem into small size problems and exploiting the positive property of each agent. We illustrate by numerical examples that the proposed algorithm is definitely efficient.

Notations:

\mathbb{R} (\mathbb{C}):	the set of real (complex) numbers.
\mathbb{D} :	$\mathbb{D} := \{\nu \in \mathbb{C} : \nu \leq 1\}$.
\mathbb{R}^n :	the set of real vectors of size n .
\mathbb{R}_+^n :	the set of nonnegative vectors of size n .
\mathbb{R}_{++}^n :	the set of strictly positive vectors of size n .
$\mathbb{R}^{n \times m}$:	the set of real matrices of size $n \times m$.
$\mathbb{R}_+^{n \times m}$:	the set of nonnegative matrices of size $n \times m$.
\mathbb{H}^n :	the set of Hurwitz stable matrices of size $n \times n$.
\mathbb{M}^n :	the set of Metzler matrices of size $n \times n$.
$\sigma(A)$:	the set of the eigenvalues of $A \in \mathbb{C}^{n \times n}$.
$\rho(A)$:	the spectral radius of $A \in \mathbb{C}^{n \times n}$.
$\kappa(A)$:	the maximal real part of $\sigma(A)$.
$\kappa_2(A)$:	the second maximal real part of $\sigma(A)$.
\mathbb{Z}_N :	$\mathbb{Z}_N := \{1, \dots, N\}$.

^{*} This work is supported in part by JSPS KAKENHI Grant Number 25420436.

2. INTERCONNECTED POSITIVE SYSTEMS

Consider the linear system described by

$$G: \begin{cases} \dot{x} = Ax + Bw, \\ z = Cx + Dw \end{cases} \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_w}$, $C \in \mathbb{R}^{n_z \times n}$, and $D \in \mathbb{R}^{n_z \times n_w}$. The definition and a basic result of positive systems are given in the following.

Definition 1. (Farina and Rinaldi (2000)). The linear system (1) is said to be *positive* if its state and output are both nonnegative for any nonnegative initial state and nonnegative input.

Proposition 2. (Farina and Rinaldi (2000)). The system (1) is positive if and only if $A \in \mathbb{M}^n$, $B \in \mathbb{R}_+^{n \times n_w}$, $C \in \mathbb{R}_+^{n_z \times n}$, and $D \in \mathbb{R}_+^{n_z \times n_w}$.

In this paper, we deal with multi-agent systems where the dynamics of each agent, or say, subsystem, is positive. Consider the positive subsystem G_i ($i \in \mathbb{Z}_N$) given by

$$G_i: \begin{cases} \dot{x}_i = A_i x_i + B_i w_i, \\ z_i = C_i x_i + D_i w_i, \end{cases} \quad (2)$$

$$A_i \in \mathbb{M}^{n_i} \cap \mathbb{H}^{n_i}, \quad B_i \in \mathbb{R}_+^{n_i \times n_{w_i}},$$

$$C_i \in \mathbb{R}_+^{n_{z_i} \times n_i}, \quad D_i \in \mathbb{R}_+^{n_{z_i} \times n_{w_i}}.$$

As clearly shown in (2), we assumed that G_i ($i \in \mathbb{Z}_N$) are all stable. With these positive subsystems, we define positive and stable system \mathcal{G} by $\mathcal{G} := \text{diag}(G_1, \dots, G_N)$. The state space realization of \mathcal{G} is given by

$$\mathcal{G}: \begin{cases} \dot{\hat{x}} = \mathcal{A}\hat{x} + \mathcal{B}\hat{w}, \\ \hat{z} = \mathcal{C}\hat{x} + \mathcal{D}\hat{w} \end{cases} \quad (3)$$

where

$$\mathcal{A} := \text{diag}(A_1, \dots, A_N), \quad \mathcal{B} := \text{diag}(B_1, \dots, B_N),$$

$$\mathcal{C} := \text{diag}(C_1, \dots, C_N), \quad \mathcal{D} := \text{diag}(D_1, \dots, D_N), \quad (4)$$

$$\hat{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^{n_{\hat{x}}}, \quad n_{\hat{x}} := \sum_{i=1}^N n_i,$$

$$\hat{w} := \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix} \in \mathbb{R}^{n_{\hat{w}}}, \quad n_{\hat{w}} := \sum_{i=1}^N n_{w_i},$$

$$\hat{z} := \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix} \in \mathbb{R}^{n_{\hat{z}}}, \quad n_{\hat{z}} := \sum_{i=1}^N n_{z_i}.$$

For a given interconnection matrix $\Omega \in \mathbb{R}_+^{n_{\hat{w}} \times n_{\hat{z}}}$, we are interested in the analysis of interconnected system $\mathcal{G} \star \Omega$ defined by $\hat{w} = \Omega \hat{z}$. In relation to the well-posedness of this interconnection, we give the next definition.

Definition 3. (Ebihara et al. (2011)). Interconnected system $\mathcal{G} \star \Omega$ is said to be *admissible* if Metzler matrix $\mathcal{D}\Omega - I$ is Hurwitz stable.

In the sequel, we require the admissibility of interconnected system $\mathcal{G} \star \Omega$ whenever we analyze its properties. The meaning of this presupposition, and its rationality as well, can be explained as follows. If $\det(\mathcal{D}\Omega - I) \neq 0$,

then the interconnection is well-posed, and the state-space description of the interconnected system is represented by

$$\dot{\hat{x}} = \mathcal{A}_{\text{cl}} \hat{x}, \quad \mathcal{A}_{\text{cl}} := \mathcal{A} + \mathcal{B}\Omega(I - \mathcal{D}\Omega)^{-1}\mathcal{C}. \quad (5)$$

Thus, if the admissibility is ensured, we see that

- (i) interconnection $\mathcal{G} \star \Omega$ is well-posed;
- (ii) Metzler matrix $\mathcal{D}\Omega - I$ is Hurwitz and hence $(I - \mathcal{D}\Omega)^{-1} \geq 0$ holds [Farina and Rinaldi (2000); Kaczorek (2001); Mason and Shorten (2007)]. Therefore \mathcal{A}_{cl} given by (5) is Metzler. It follows that the interconnected system $\mathcal{G} \star \Omega$ inherits the positive nature of G_i ($i \in \mathbb{Z}_N$), i.e., the nonnegativity of states x_i ($i \in \mathbb{Z}_N$) for any nonnegative initial states is still preserved under the interconnection.

3. FORMATION CONTROL OF MULTI-AGENT POSITIVE SYSTEMS

The goal of this paper is to give an efficient method to analyze the convergence rate performance of multi-agent positive systems under formation control. In this section, we review our preceding results on formation control of multi-agent positive systems and show that a practical formation control of moving agents can be cast as a formation control of multi-agent positive systems.

3.1 Basic Results for Formation Control

The next result concerns the formation control of the interconnected positive system $\mathcal{G} \star \Omega$ with respect to the output \hat{z} .

Theorem 4. (Ebihara et al. (2013a)). Consider the case where every stable positive subsystem G_i represented by (2) is SISO. Suppose G_i ($i \in \mathbb{Z}_N$) and given interconnection matrix $\Omega \in \mathbb{R}_+^{N \times N}$ satisfy the following conditions:

- (i) (A_i, B_i) is controllable and (A_i, C_i) is observable for all $i \in \mathbb{Z}_N$.
- (ii) $G_1(0) = \dots = G_N(0) =: \gamma (> 0)$ holds for $G_i(s) := C_i(sI - A_i)^{-1}B_i + D_i$ ($i \in \mathbb{Z}_N$).
- (iii) $\Omega \in \mathbb{R}_+^{N \times N}$ is irreducible (i.e., the directed graph of Ω , denoted by $\Gamma(\Omega)$, is strongly connected).
- (iv) $\Omega v_{\text{obj}} = (1/\gamma)v_{\text{obj}}$ holds for given $v_{\text{obj}} \in \mathbb{R}_+^N$.

Then, for the interconnected system $\mathcal{G} \star \Omega$, the next results hold.

- (I) $\mathcal{G} \star \Omega$ is admissible, i.e., the Metzler matrix $\mathcal{D}\Omega - I$ is Hurwitz stable.
- (II) The output of interconnected system $\mathcal{G} \star \Omega$ satisfies $\hat{z}_{\infty} = \gamma \alpha(\hat{x}(0))v_{\text{obj}}$, $\hat{z}_{\infty} := \lim_{t \rightarrow \infty} \hat{z}(t)$ (6)

where

$$\alpha(\hat{x}(0)) := \xi_{\text{L}}^T \hat{x}(0) / \xi_{\text{L}}^T \xi_{\text{R}} \in \mathbb{R},$$

$$\xi_{\text{R}} := -\mathcal{A}^{-1} \mathcal{B} v_{\text{obj}} \in \mathbb{R}_+^{n_{\hat{x}}},$$

$$\xi_{\text{L}} := -\mathcal{A}^{-T} \mathcal{C}^T v_{\text{L}} \in \mathbb{R}_+^{n_{\hat{x}}}, \quad (7)$$

and $v_{\text{L}} \in \mathbb{R}_+^N$ is a left-eigenvector of Ω with respect to the eigenvalue $1/\gamma$, i.e., v_{L} satisfies $v_{\text{L}}^T \Omega = (1/\gamma)v_{\text{L}}^T$. Namely, for any initial state $\hat{x}(0) \in \mathbb{R}_+^{n_{\hat{x}}} \setminus \{0\}$, we can achieve the convergence of output $\hat{z}(t) = [z_1(t) \dots z_N(t)]^T$ to $\gamma \alpha(\hat{x}(0))v_{\text{obj}} \in \mathbb{R}_+^N$.

This theorem implies that, for a given $v_{\text{obj}} \in \mathbb{R}_+^N$ that represents the desired formation of N -agents, we can

enforce their outputs (positions) to converge to (a positive scalar multiple of) v_{obj} by designing the interconnection matrix Ω to satisfy conditions (iii) and (iv).

3.2 Synthesis of Interconnected Matrices

It is meaningful to show a concrete way to design a desired $\Omega \in \mathbb{R}_+^{N \times N}$ satisfying $\Omega v_{\text{obj}} = (1/\gamma)v_{\text{obj}}$ and $\Gamma(\Omega) = \Gamma$ for a prescribed vector $v_{\text{obj}} \in \mathbb{R}_+^N$ and a graph structure Γ . For illustration, consider the cases where Γ is schematically shown in Figs.1 and 2 for $N = 3$.

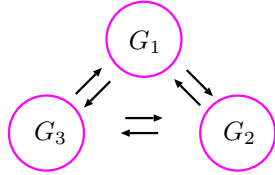


Fig. 1. Graph structure Γ_A .



Fig. 2. Graph structure Γ_B .

For the graph structure Γ_A , any interconnection matrix $\Omega \in \mathbb{R}_+^{N \times N}$ satisfying $\Omega v_{\text{obj}} = (1/\gamma)v_{\text{obj}}$ and $\Gamma(\Omega) = \Gamma_A$ can be parametrized by

$$\Omega = \frac{1}{\gamma} \Omega(v_{\text{obj}}, p) \in \mathbb{R}_+^{N \times N} \quad (8)$$

where

$$\Omega(v_{\text{obj}}, p)_{i,j} = \begin{cases} (1-p_1) \frac{v_{\text{obj},1}}{v_{\text{obj},N}} & (i,j) = (1,N), \\ p_i \frac{v_{\text{obj},i}}{v_{\text{obj},j}} & (1 \leq i \leq N, j = i+1), \\ (1-p_i) \frac{v_{\text{obj},i}}{v_{\text{obj},j}} & (1 \leq i \leq N, j = i-1), \\ p_N \frac{v_{\text{obj},N}}{v_{\text{obj},1}} & (i,j) = (N,1), \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

Here, the vector of parameters $p \in \mathbb{R}_+^N$ can be chosen arbitrarily over $0 < p < \mathbf{1}_N$. On the other hand, for the graph structure Γ_B , any interconnection matrix $\Omega \in \mathbb{R}_+^{N \times N}$ satisfying $\Omega v_{\text{obj}} = (1/\gamma)v_{\text{obj}}$ and $\Gamma(\Omega) = \Gamma_B$ can be parametrized again by (8) and (9) where the vector of parameters $p \in \mathbb{R}_+^N$ can be chosen such that $p_1 = 1$, $p_N = 0$, and $0 < p_i < 1$ ($i \in \mathbb{Z}_N \setminus \{1, N\}$). In both cases, we can confirm that the resulting interconnection matrix is irreducible (since Γ_A and Γ_B are both strongly connected). We finally note that the eigenvalues of $\Omega(v_{\text{obj}}, p) \in \mathbb{R}_+^{N \times N}$ depend solely on $p \in \mathbb{R}_+^N$ and do not depend on $v_{\text{obj}} \in \mathbb{R}_+^N$. This can be easily confirmed if we note $\Omega(v_{\text{obj}}, p) = \text{diag}(v_{\text{obj},1}, \dots, v_{\text{obj},N}) \Omega(\mathbf{1}_N, p) \text{diag}(v_{\text{obj},1}, \dots, v_{\text{obj},N})^{-1}$.

3.3 Concrete Examples of Formation Control

In this section, we show that a practical formation control problem can be cast as a formation control problem of

multi-agent positive systems as discussed in the preceding subsection. Consider the formation control problem of N agents where the i -th agent can move on the (x, y) -plane. We denote by $(z_{i,x}(t), z_{i,y}(t))$ the position of agent i . Furthermore, we define $\hat{z}_j := [z_{1,j} \ \dots \ z_{N,j}]^T$ ($j = x, y$) by stacking the coordinates of all agents.

We assume that agent i has independent dynamics along the x - and y -axes, denoted by $P_{i,x}(s)$ and $P_{i,y}(s)$, respectively, and independent control inputs $u_{i,x}(t)$ and $u_{i,y}(t)$. Suppose $P_{i,x}(s)$ and $P_{i,y}(s)$ are typical dynamics of moving agents given by

$$\begin{aligned} Z_{i,j}(s) &= P_{i,j}(s)U_{i,j}(s), \\ P_{i,j}(s) &= \frac{k_{i,j}}{s(s+a_{i,j})} \quad (i \in \mathbb{Z}_N, j = x, y) \end{aligned}$$

where $k_{i,j}, a_{i,j} > 0$. Roughly speaking, our goal here is to design communication schemes (interconnection matrices) over N -agents along x - and y -axes independently so that prescribed formation can be achieved asymptotically.

Before designing interconnection matrices over the agents, we apply the following local feedback

$$u_{i,j}(t) = -f_{i,j}(z_{i,j}(t) - w_{i,j}(t)) \quad (i \in \mathbb{Z}_N, j = x, y)$$

with $0 < f_{i,j} \leq a_{i,j}^2/4k_{i,j}$, where $w_{i,j}$ ($i \in \mathbb{Z}_N, j = x, y$) is the exogenous input kept for the interconnection. Then we have

$$\begin{aligned} Z_{i,j}(s) &= G_{i,j}(s)W_{i,j}(s), \\ G_{i,j}(s) &= \left[\begin{array}{cc|c} -b_{i,j} & 1 & 0 \\ 0 & -c_{i,j} & b_{i,j}c_{i,j} \\ \hline 1 & 0 & 0 \end{array} \right], \\ b_{i,j} + c_{i,j} &= a_{i,j}, \quad b_{i,j}c_{i,j} = f_{i,j}k_{i,j}. \end{aligned} \quad (10)$$

It follows that $G_{i,j}$ ($i \in \mathbb{Z}_N, j = x, y$) are stable positive systems (in the realization (10)) with $G_{i,j}(0) = 1$ ($i \in \mathbb{Z}_N, j = x, y$). The latter is a natural consequence from the fact that each open-loop transfer function $P_{i,j}(s)$ ($i \in \mathbb{Z}_N, j = x, y$) includes an integrator. For description simplicity, we define $\hat{w}_j := [w_{1,j} \ \dots \ w_{N,j}]^T$ ($j = x, y$).

Suppose each agent can communicate its position to other agents. More specifically, we assume that N -agents independently communicate their x and y positions. Then, we can restate our goal as in designing Ω_x and Ω_y such that, under the interconnection with Ω_x and Ω_y for (\hat{z}_x, \hat{w}_x) and (\hat{z}_y, \hat{w}_y) , respectively, the following formation can be achieved:

$$\begin{aligned} \lim_{t \rightarrow \infty} [\hat{z}_x(t) \ \hat{z}_y(t)] \\ = [\alpha_x(\hat{x}_x(0))v_{\text{obj},x} \ \alpha_y(\hat{x}_y(0))v_{\text{obj},y}]. \end{aligned} \quad (11)$$

Here, $(v_{\text{obj},x}, v_{\text{obj},y})$ are given vectors that specify the desired formation. On the other hand, $\hat{x}_x(0)$ and $\hat{x}_y(0)$ stand for the initial states of the corresponding interconnected systems. It is obvious that we can readily solve this problem by following Theorem 4.

For a concrete and concise illustration, consider the case where the dynamics are uniform over the agents and (x, y) -coordinates as well by letting $a_{i,j} = a = 50$, $k_{i,j} = k = 1$ and $f_{i,j} = 0.8 \times a^2/4k$ ($i \in \mathbb{Z}_N, j = x, y$). We let $N = 20$ and

$$[v_{\text{obj},x} \ v_{\text{obj},y}]_i = [2 + \cos(2\pi i/N) \quad 2 + \sin(2\pi i/N)].$$

Namely, the vectors $(v_{\text{obj},x}, v_{\text{obj},y})$ are chosen to form a circle. Under this setting, we designed two pairs of $(\Omega_{A,x}, \Omega_{A,y})$ and $(\Omega_{B,x}, \Omega_{B,y})$ detailed below:

$$(A) \quad \Gamma(\Omega_{A,x}) = \Gamma(\Omega_{A,y}) = \Gamma_A, \\ \Omega_{A,x} = \Omega(v_{\text{obj},x}, p_{A,x}), \quad \Omega_{A,y} = \Omega(v_{\text{obj},y}, p_{A,y}).$$

$$(B) \quad \Gamma(\Omega_{B,x}) = \Gamma(\Omega_{B,y}) = \Gamma_B, \\ \Omega_{B,x} = \Omega(v_{\text{obj},x}, p_{B,x}), \quad \Omega_{B,y} = \Omega(v_{\text{obj},y}, p_{B,y}).$$

Here we choose parameters $p_{A,x} \in \mathbb{R}_+^N$ and $p_{B,x} \in \mathbb{R}_+^N$ so that there exists $v_{L,x} \in \mathbb{R}_+^N$ satisfying

$$v_{L,x}^T \Omega(v_{\text{obj},x}, p_{A,x}) = v_{L,x}^T \Omega(v_{\text{obj},x}, p_{B,x}) = v_{L,x}^T.$$

Similarly, we choose parameters $p_{A,y} \in \mathbb{R}_+^N$ and $p_{B,y} \in \mathbb{R}_+^N$ so that there exists $v_{L,y} \in \mathbb{R}_+^N$ satisfying

$$v_{L,y}^T \Omega(v_{\text{obj},y}, p_{A,y}) = v_{L,y}^T \Omega(v_{\text{obj},y}, p_{B,y}) = v_{L,y}^T.$$

Such parameters can be computed by solving LPs along the line in [Ebihara et al. (2013a)]. From (7), we can confirm that the output $[\hat{z}_x(t) \quad \hat{z}_y(t)]$ converges to the same value under the interconnections $(\Omega_{A,x}, \Omega_{A,y})$ and $(\Omega_{B,x}, \Omega_{B,y})$.

In Figs. 3–10 given at the last page, we show the plots of $[\hat{z}_x(t) \quad \hat{z}_y(t)]$ under the settings (A) and (B). In both cases, we took exactly the same initial states as implied by Figs. 3 and 7. In both cases, we successfully achieved exactly the same circular formation (scaled along x - and y -axes independently) as expected. However, the convergence performance is quite different between (A) and (B): it is clear from Figs. 4 and 8 that the convergence of case (A) is much faster than case (B).

The real part of the dominant pole of the system $\mathcal{G} \star \Omega$, i.e., the second largest real part of the eigenvalues of $\mathcal{A} + \mathcal{B}\Omega\mathcal{C}$ denoted by $\kappa_2(\mathcal{A} + \mathcal{B}\Omega\mathcal{C})$ is a reasonable measure for the convergence rate performance. Of course the computation of the dominant pole can be done by simply computing all the eigenvalues of $\mathcal{A} + \mathcal{B}\Omega\mathcal{C}$ and comparing their real parts. By this simple procedure, we know in the above numerical examples that

$$\begin{aligned} (\kappa_2(\mathcal{A} + \mathcal{B}\Omega_{A,x}\mathcal{C}), \kappa_2(\mathcal{A} + \mathcal{B}\Omega_{A,y}\mathcal{C})) &= (-0.3719, -0.4890) \\ (\kappa_2(\mathcal{A} + \mathcal{B}\Omega_{B,x}\mathcal{C}), \kappa_2(\mathcal{A} + \mathcal{B}\Omega_{B,y}\mathcal{C})) &= (-0.0951, -0.1501). \end{aligned} \quad (12)$$

These results surely validates the fact that the convergence for case (A) is much faster than case (B).

Even though we obtained (12) by directly computing all the eigenvalues of $\mathcal{A} + \mathcal{B}\Omega\mathcal{C}$, such straightforward method is computationally demanding especially when the total number of agents gets larger. Moreover, such method does not give any insights on how we can accelerate the convergence by appropriately designing Ω . Motivated by these facts, in the next section, we propose an efficient algorithm for the computation of $\kappa_2(\mathcal{A} + \mathcal{B}\Omega\mathcal{C})$ by decomposing the original large size problem into small size problems and exploiting the positive property of each agent. Note that the real part of the dominant pole is often called algebraic connectivity in the study area of multi-agent systems [Fujita and Hatanaka (2008); Olfati-Saber and Murray (2004); Olfati-Saber et al. (2007); Tanner et al. (2003a,b)]. In the case where the dynamics of each agent are simple such as

in the case of an integrator, the algebraic connectivity can be determined solely by the interconnection matrix but this is not the case in general. Our objective is to establish an efficient algorithm by exploiting the positive property of each agent.

From now on we assume that the dynamics of all agents are uniform. This assumption is indispensable in constructing the efficient algorithm.

4. EFFICIENT ALGORITHM FOR THE COMPUTATION OF $\kappa_2(\mathcal{A} + \mathcal{B}\Omega\mathcal{C})$

In this section, we propose an efficient algorithm for the computation of (the real part of) the dominant pole of multi-agent positive systems under formation control. As noted previously, we assume that the dynamics of all agents are uniform and thus

$$\begin{aligned} A_i &= A \in \mathbb{M}_n \cup \mathbb{H}_n, \quad B_i = B \in \mathbb{R}_+^{n \times 1}, \\ C_i &= C \in \mathbb{R}_+^{1 \times n}, \quad D_i = 0 \quad (i \in \mathbb{Z}_N). \end{aligned} \quad (13)$$

holds in (2). In the following we further assume that $G_i = G$ ($i \in \mathbb{Z}_N$) and $\Omega \in \mathbb{R}_+^{N \times N}$ satisfies the conditions (i)-(iv) in Theorem 4 with $\gamma = 1$. From now on all proofs for technical results are omitted due to limited space.

4.1 Basic Algorithm by Problem Decomposition

The next lemma is instrumental for the decomposition of the computation of $\kappa_2(\mathcal{A} + \mathcal{B}\Omega\mathcal{C})$ into smaller size problems.

Lemma 5. For given $\Omega \in \mathbb{R}^{N \times N}$ and $\mathcal{P}, \mathcal{Q} \in \mathbb{R}^{m \times m}$, we have

$$\sigma(I_n \otimes \mathcal{P} + \Omega \otimes \mathcal{Q}) = \{\lambda \in \sigma(\mathcal{P} + \nu\mathcal{Q}) : \nu \in \sigma(\Omega)\}. \quad (14)$$

The next result readily follows from Lemma 5.

Proposition 6. For given $\Omega \in \mathbb{R}^{N \times N}$ and $\mathcal{A} \in \mathbb{M}^{Nn}$, $\mathcal{B} \in \mathbb{R}_+^{Nn \times N}$, $\mathcal{C} \in \mathbb{R}_+^{N \times Nn}$ given by (2), (4) and (13), we have

$$\sigma(\mathcal{A} + \mathcal{B}\Omega\mathcal{C}) = \{\lambda \in \sigma(A + \nu BC) : \nu \in \sigma(\Omega)\}. \quad (15)$$

Recall that Proposition 6 holds only if G is SISO. This proposition implies that we can compute the eigenvalues of $\mathcal{A} + \mathcal{B}\Omega\mathcal{C} \in \mathbb{R}^{Nn \times Nn}$ by computing the eigenvalues of $A + \nu_i BC \in \mathbb{R}^{n \times n}$ repeatedly over $\nu_i \in \sigma(\Omega)$. Based on this fact, we can conceive the following algorithm that is expected to be efficient if N is large.

Algorithm I (Basic Algorithm)

Step 0: Sort the distinct eigenvalues of Ω with non-negative imaginary part in descending order with respect to their absolute values and denote them by $\lambda_1, \dots, \lambda_M$ ($M \leq N$). From the underlying assumption on Ω we have $\lambda_1 = 1$.

Step 1: Let $\kappa^* = \kappa_2(A + \lambda_1 BC) = \kappa_2(A + BC)$ and $i = 2$.

Step 2: Compute $\kappa(\lambda_i) := \kappa(A + \lambda_i BC)$ and let $\kappa^* := \max(\kappa(\lambda_i), \kappa^*)$.

Step 3: If $i = M$, exit. Else, let $i := i+1$ and go to Step 2.

In this algorithm, we need to compute the eigenvalues of $A + \nu_i BC$ repeatedly over $i \in \mathbb{Z}_M$. However, if we focus on the positivity of G , i.e., if we exploit the fact that $A \in \mathbb{M}^n$,

$B \in \mathbb{R}_+^n$ and $C \in \mathbb{R}_+^{1 \times n}$, we can drastically reduce the number of eigenvalues $\nu_i \in \sigma(\Omega)$ to be examined. This is the key idea for conceiving a much efficient algorithm as detailed in the next subsection.

4.2 Efficient Algorithm by Exploiting Positivity

The next theorem is very important for enhancing the efficiency of Algorithm I.

Theorem 7. For given $A, B \in \mathbb{R}_+^{n \times n}$, we have

$$\rho(A + \nu B) \leq \rho(A + B) \quad \forall \nu \in \mathbb{D}. \quad (16)$$

Next corollary follows directly from this theorem.

Corollary 8. For given $A \in \mathbb{M}^n$ and $B \in \mathbb{R}_+^{n \times n}$, we have

$$\kappa(A + \nu B) \leq \kappa(A + B) \quad \forall \nu \in \mathbb{D}. \quad (17)$$

In relation to the computation of $\kappa_2(\mathcal{A} + \mathcal{B}\Omega\mathcal{C})$ along the line of Algorithm I, Corollary 8 shows that we can reduce the number of eigenvalues $\nu_i \in \sigma(\Omega)$ to be examined. Indeed, by paying attention to the fact that $\lambda_1, \dots, \lambda_M$ ($M \leq N$) are sorted in descending order with respect to their absolute values, we can include the following stopping conditions to Algorithm I:

- (i) If $\bar{\kappa}(\lambda_i) := \kappa(A + |\lambda_i|BC)$ is not larger than the tentative value κ^* , we can let $\kappa_2(\mathcal{A} + \mathcal{B}\Omega\mathcal{C}) = \kappa^*$ and exit the algorithm.
- (ii) If $\lambda_i \in \mathbb{R}_{++}$, we can let $\kappa_2(\mathcal{A} + \mathcal{B}\Omega\mathcal{C}) = \max(\kappa^*, \kappa(A + \lambda_i BC))$ and exit the algorithm.

We are now ready to give the efficient algorithm for the computation of $\kappa_2(\mathcal{A} + \mathcal{B}\Omega\mathcal{C})$.

Algorithm II (Efficient Algorithm)

Steps 0 and 1: The same as Algorithm I.

Step 2: Compute $\bar{\kappa}(\lambda_i) := \kappa(A + |\lambda_i|BC)$. If $\bar{\kappa}(\lambda_i) \leq \kappa^*$, exit. If $\lambda_i \in \mathbb{R}_{++}$, let $\kappa^* := \max(\bar{\kappa}(\lambda_i), \kappa^*)$ and exit. Else, go to Step 3.

Step 3: Compute $\kappa(\lambda_i) := \kappa(A + \lambda_i BC)$ and let $\kappa^* := \max(\kappa(\lambda_i), \kappa^*)$.

Step 4: If $i = M$, exit. Else, let $i := i + 1$ and go to Step 2.

Algorithm II is efficient particularly when $\Omega \in \mathbb{R}_+^{N \times N}$ has only real eigenvalues (ex., this happens when Ω is symmetric). In such a case, if Ω has a positive eigenvalue λ , then we do not need to test the eigenvalues whose absolute values are less than or equal to λ . By following this line, we can analytically write down $\kappa_2(\mathcal{A} + \mathcal{B}\Omega\mathcal{C})$ for some specific interconnection matrices.

For example, consider the interconnection matrix $\Omega_{A,0} := \Omega(\mathbf{1}_N, p_{A,0})$ where $p_{A,0} := \frac{1}{2}\mathbf{1}_N$. In this case, $\gamma(\Omega_{A,0}) = \Gamma_A$ holds and

$$\sigma(\Omega_{A,0}) = \left\{ \cos\left(\frac{2\pi(i-1)}{N}\right) : i \in \mathbb{Z}_N \right\}. \quad (18)$$

It follows that

$$\kappa_2(\mathcal{A} + \mathcal{B}\Omega_{A,0}\mathcal{C}) = \quad (19)$$

$$\begin{cases} \max(\kappa_2(A + BC), \kappa(A + \lambda_2 BC), \kappa(A + \lambda_{K+1} BC)) & N = 2K + 1 \\ \max(\kappa_2(A + BC), \kappa(A + \lambda_2 BC), \kappa(A - BC)) & N = 2K \end{cases}$$

where $K \in \mathbb{Z}_+$ and

$$\lambda_2 := \cos\left(\frac{2\pi}{N}\right), \quad \lambda_{K+1} := \cos\left(\frac{2K\pi}{2K+1}\right).$$

On the other hand, we see that $\Omega_{B,0} := \Omega(\mathbf{1}_N, p_{B,0})$ with $p_{B,0} := [1 \ \mathbf{1}_{N-2}^T \ 0]^T$ satisfies $\gamma(\Omega_{B,0}) = \Gamma_B$ and

$$\sigma(\Omega_{B,0}) = \left\{ \cos\left(\frac{\pi(i-1)}{N-1}\right) : i \in \mathbb{Z}_N \right\} \quad (20)$$

for $N \geq 3$. Therefore we have

$$\kappa_2(\mathcal{A} + \mathcal{B}\Omega_{B,0}\mathcal{C}) = \max\left(\kappa_2(A + BC), \kappa(A - BC), \kappa\left(A + \cos\left(\frac{\pi}{N-1}\right)BC\right)\right). \quad (21)$$

It follows that in the cases $\Omega = \Omega_{A,0}$ and $\Omega = \Omega_{B,0}$ we can compute $\kappa_2(\mathcal{A} + \mathcal{B}\Omega\mathcal{C})$ very efficiently irrespective of N .

Remark 9. One may intuitively think that $\kappa(A - \nu BC) \leq \kappa(A)$ holds for any $A \in \mathbb{M}^n \cup \mathbb{H}^n$, $B \in \mathbb{R}_+^{n \times 1}$, $C \in \mathbb{R}_+^{1 \times n}$ and $\nu \in \mathbb{R}_{++}$ and hence we can omit $\kappa(A + \lambda_{K+1} BC)$ and/or $\kappa(A - BC)$ in the evaluation $\kappa_2(\mathcal{A} + \mathcal{B}\Omega_{A,0}\mathcal{C})$ and $\kappa_2(\mathcal{A} + \mathcal{B}\Omega_{B,0}\mathcal{C})$. However, this is not true in general. Indeed, for

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = [1 \ 0 \ 0]$$

where (A, B) is controllable, (A, C) is observable and $G(0) = -CA^{-1}B = 1$, we see that $\kappa(A) = -1$ whereas $\kappa(A - \nu BC) = -1/2$ for $\nu = 1$. Due to this reason, we have to leave $\kappa(A + \lambda_{K+1} BC)$ and $\kappa(A - BC)$ for the max evaluation stated above.

5. NUMERICAL EXAMPLES

In this section, we demonstrate the efficiency of Algorithms I and II by numerical examples. For given n and N , we randomly generated $A_i \in \mathbb{M}^n \cup \mathbb{H}^n$, $B_i \in \mathbb{R}_+^{n \times 1}$ and $C_i \in \mathbb{R}_+^{1 \times n}$ ($i \in \mathbb{Z}_N$) satisfying $G_i(0) = -C_i A_i^{-1} B_i = 1$ ($i \in \mathbb{Z}_N$) and irreducible $\Omega \in \mathbb{R}_+^{N \times N}$ satisfying $\lambda_F(\Omega) = 1$. Then, we computed $\kappa_2(\mathcal{A} + \mathcal{B}\Omega\mathcal{C})$ by direct computation of $\sigma(\mathcal{A} + \mathcal{B}\Omega\mathcal{C})$ (this method is denoted by Algorithm 0 for simplicity) and Algorithms I and II. The average computation times over 10 tested cases for each setting $N = 10, 100, 1000$ with $n = 2$ and 3 are shown in Tables 1 and 2, respectively.

Table 1. Average CPU time ($n = 2$).

N	Algorithm 0	Algorithm I	Algorithm II
10	0.3907×10^{-3}	0.2983×10^{-3}	0.2541×10^{-3}
100	0.0319	0.0074	0.0065
1000	8.4881	1.2354	1.2269

Table 2. Average CPU time ($n = 3$).

N	Algorithm 0	Algorithm I	Algorithm II
10	0.5488×10^{-3}	0.3709×10^{-3}	0.2798×10^{-3}
100	0.0756	0.0076	0.0067
1000	27.9230	1.2339	1.2230

These tables show that Algorithms I and II are much more efficient than Algorithms 0. This is due to the fact that in Algorithms I and II we can avoid the eigenvalue computation of matrices of large size nN .

On the other hand, from these tables we cannot see clearly the efficiency of Algorithm II over Algorithm I. This is due to the fact that in both algorithms most of computation time is consumed by the common steps, Steps 0 and 1, i.e., the computation of the eigenvalues of Ω . To compare the efficiency of Algorithms I and II in more detail, we show in Tables 3 and 4 the computation times of Algorithm I needed for Steps 0 and 1, and for the remaining steps, Steps 3 and 4. Similarly for Algorithm II. From these tables, it is clear that Algorithm II consumes very little time for the Steps 2, 3, and 4, i.e., the computation of the eigenvalues $A + \lambda_i BC$ over $\lambda_i \in \sigma(\Omega)$ that have to be examined. To highlight this point, we show in Tables 5 and 6 the average number of the eigenvalues of Ω examined in Algorithms I and II. It is clear that Algorithm II successfully reduce the number of eigenvalues to be examined by including stopping conditions (i) and (ii) stated in Subsection 4.2.

Table 3. Average CPU time of Algorithms I and II ($n = 2$).

N	Steps 0,1	Algorithm I Steps 2, 3	Algorithm II Steps 2, 3, 4
10	0.1647×10^{-3}	0.1336×10^{-3}	0.0894×10^{-3}
100	0.0064	0.9208×10^{-3}	0.1205×10^{-3}
1000	1.2266	0.0088	0.0003

Table 4. Average CPU time of Algorithms I and II ($n = 3$).

N	Steps 0, 1	Algorithm I Steps 2, 3	Algorithm II Steps 2, 3, 4
10	0.1786×10^{-3}	0.1923×10^{-3}	0.1012×10^{-3}
100	0.0067	0.0012	0.0002
1000	1.2225	0.0114	0.0004

Table 5. Average number of eigenvalues of Ω examined in Algorithms I and II ($n = 2$).

N	Algorithm I	Algorithm II
10	7.4	3.2
100	54.6	7.9
1000	514.2	13.3

Table 6. Average number of eigenvalues of Ω examined in Algorithms I and II ($n = 3$).

N	Algorithm I	Algorithm II
10	7.0	3.4
100	54.8	7.3
1000	512.6	12.2

6. CONCLUSION

In this paper, we proposed an efficient algorithm for the convergence rate analysis of multi-agent positive systems under formation control. Assuming that the dynamics of all agents are positive and uniform, we first show that the dominant pole of the overall interconnected positive system can be computed by evaluating eigenvalues of matrices whose sizes are equal to the dimension of each subsystem. Then, by actively using positive property of each subsystem, we showed that we can drastically decrease the number of matrices to be examined. We illustrated by numerical examples that the proposed algorithm is definitely efficient.

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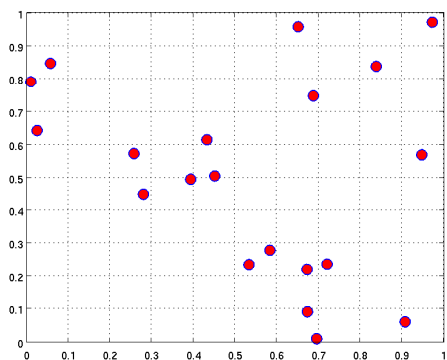


Fig. 3. $[\hat{z}_x(t) \hat{z}_y(t)]$ for $t = 0$ [sec] under $(\Omega_{A,x}, \Omega_{A,y})$.

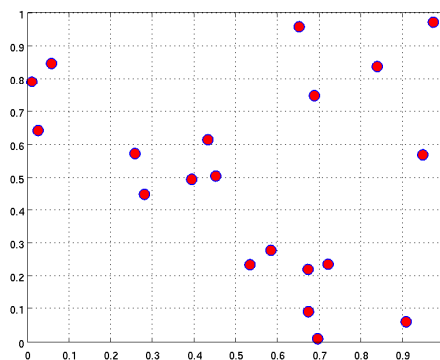


Fig. 7. $[\hat{z}_x(t) \hat{z}_y(t)]$ for $t = 0$ [sec] under $(\Omega_{B,x}, \Omega_{B,y})$.

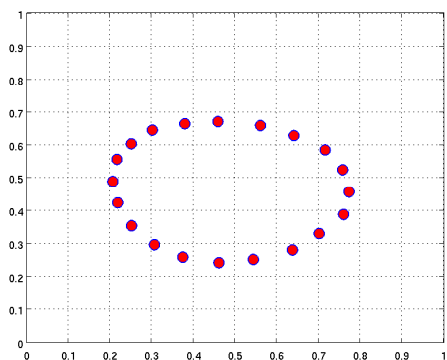


Fig. 4. $[\hat{z}_x(t) \hat{z}_y(t)]$ for $t = 10$ [sec] under $(\Omega_{A,x}, \Omega_{A,y})$.

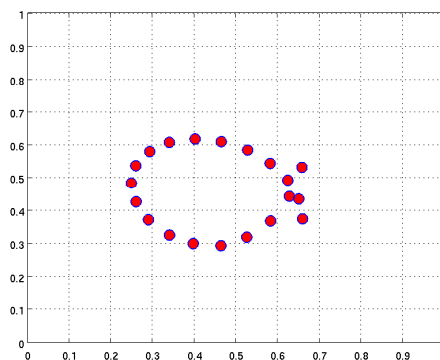


Fig. 8. $[\hat{z}_x(t) \hat{z}_y(t)]$ for $t = 10$ [sec] under $(\Omega_{B,x}, \Omega_{B,y})$.

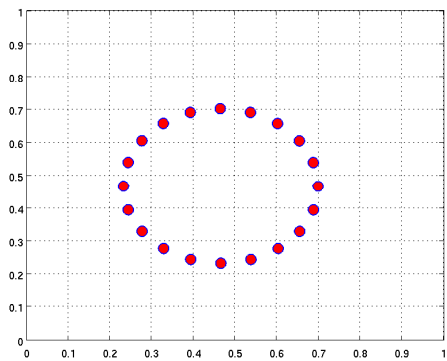


Fig. 5. $[\hat{z}_x(t) \hat{z}_y(t)]$ for $t = 20$ [sec] under $(\Omega_{A,x}, \Omega_{A,y})$.

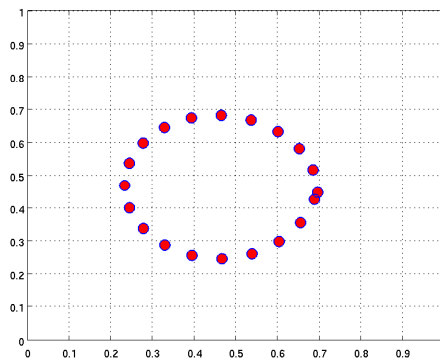


Fig. 9. $[\hat{z}_x(t) \hat{z}_y(t)]$ for $t = 20$ [sec] under $(\Omega_{B,x}, \Omega_{B,y})$.

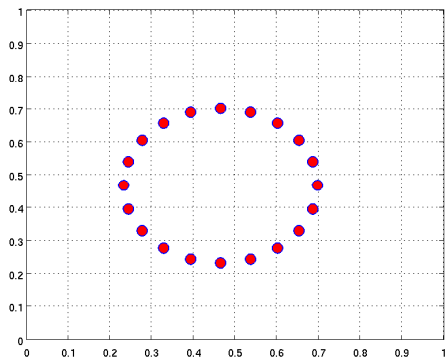


Fig. 6. $[\hat{z}_x(t) \hat{z}_y(t)]$ for $t = 30$ [sec] under $(\Omega_{A,x}, \Omega_{A,y})$.

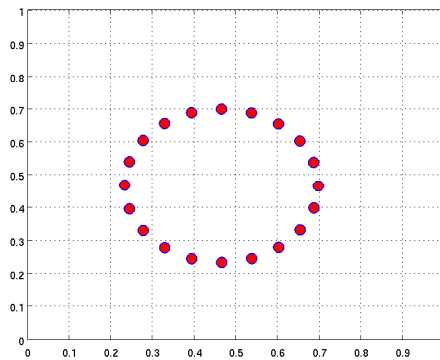


Fig. 10. $[\hat{z}_x(t) \hat{z}_y(t)]$ for $t = 30$ [sec] under $(\Omega_{B,x}, \Omega_{B,y})$.