Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

On bounds of Input-Output systems. Reachability set determination and polyhedral constraints verification

C. Moussaoui [*,**] R. Abbou [*] JJ. Loiseau [*]

[*] *LUNAM Université, Institut de Recherche en Communications et Cybernétique de Nantes. UMR CNRS 6597, École Centrale de Nantes, 1 rue de la Noë, BP 92101, 44321 NANTES Cedex 3, France. (charifa.moussaoui/rosa.abbou /jean-jacques.Loiseau@irccyn.ec-nantes.fr)*

[**] *Laboratoire d'Informatique, de Modélisation et d'Optimisation des Systèmes. UMR CNRS 6158, Université d'Auvergne, Campus Scientifique des Cézeaux, 63173 AUBIÈRE Cedex, France.(charifa.moussaoui@udamail.fr)*

Abstract: In this paper some results about constrained Input-Output systems are presented. The considered framework is the Callier-Desoer Class of convolution systems, which covers both finite dimensional systems and infinite dimensional ones such as systems with time delays. Based on bounds calculus for the input-output gain of the convolution kernel, and convex sets properties, a characterization of the output reachable set is first presented. Necessary and sufficient conditions are then derived to warrant the constraints meeting, and some invariance properties, when the system is subject to polyhedral constraints on both its inputs and outputs. A numerical example is given to illustrate the proposed approach on a time delayed system.

Keywords: Time-invariant systems; Infinite-dimensional systems; Systems with time-delays; Input-Output systems; Reachable sets; Polyhedral constraints.

1. INTRODUCTION

Reachable set estimation and constrained systems dynamics are related topics, which are commonly encountered in designing and controlling issues of many practical applications. The constraints can be tracked back to some physical limitations of the considered system, or to specific requirements such as safety instructions and operating modes. Roughly speaking, for any given dynamical system, reachability space is defined as the set of all the trajectories that the system can generates, starting from a given point or set. The exact characterization of those spaces is of high importance for designing and controlling constrained systems, since it describes the evolution of their trajectories with respect to the time, which allows a direct checking of the imposed constraints. The exact characterization of such spaces is known to be non-trivial, and thus, polyhedral and ellipsoidal approximations of the reachable set are often proposed. Many works investigates the topic for the case of finite dimensional systems, such like Gilbert and Tan (1991); De Santis (1994); Miani and Savorgnan (2005); Blanchini and Miani (2007) and reference therein, where reachable sets approximations are computed using polytopic and ellipsoidal inclusions, using some step algorithms as in Miani and Savorgnan (2005); Blanchini and Miani (2007). For the constraints verification, invariant set theory received a great deal of attention, and was shown to be an efficient tool for solving control and conditioned problems, see, for instance, Basile and Marro (1969); Bitsoris (1988); Blanchini (1999); Dórea and Hennet (1999); Blanchini and Miani (2007).

For infinite dimensional systems in general, and continuous time delay ones in particular, the literature is much more restrictive, since the presence of transcendental terms due to the delays makes the set computation task more complex. In the reference work of Fridman and Shaked (2003), a Lyapunov-Razumikhin function was used to give a delay dependent condition, for bounding the reachable space by an ellipsoidal set. These conditions were improved in Kim (2008), using a Lyapunov-Krasovskii functional and convex hull properties. Lyapunov-Krasovskii functional was also used in the works of Zuo, Kwon, Nam and their co-autors in Zuo et al. (2010); Kwon et al. (2011) and Phan and Pathirana (2011), where the ellipsoidal bounding of the reachable states was considered, for delayed systems with polytopic uncertainties. In Shen and Zhong (2011) the same problem was tackled for neutral systems, while in Zuo et al. (2012) some distributed delays were also considered. The invariance principle was investigated in Goubet-Bartholomeus et al. (1995); Olaru and Niculescu (2008) and Lombardi et al. (2011) for discrete time delay systems. In Tarbouriech and Hennet (1997) and Hennet and Tarbouriech (1998), linear delayed difference equation was considered, and the invariance conditions where computed using the extended Farkas's Lemma.

In the present paper, the general framework of the Callier Desoer Class of convolution systems Desoer and Vidyasagar (1975) is considered, which involves both finite dimensional systems and infinite dimensional ones, such that fractional systems, distributed systems and time delay ones. This general framework offers a concise representation of the system considered, where the dynamic is mainly described by a convolution kernel, which offers a good handling of system properties such as the stability and the boundedness. Two main issues are investigated. A first on is the characterization of reachable output sets for such con-

volution systems, when the input is subjected to polytopic constraints. It is shown that under some convexity set assumptions, and using Input-Output properties, it is possible to characterize the reachable output space precisely. A second issue is about the constraints fulfilment, when both system inputs and outputs are constrained, particular attention is paid to the invariance conditions for time delayed systems.
The paper is organized as follows. In Section 2, necessary background about polyhedra and convex sets is introduced, and some preliminary results about these sets are given. In section 3, Input-Output systems and their properties are introduced, and the main results about bounds calculus are presented. The application of these results is illustrated in section 4, by the characterization of the output reachable set. Necessary and sufficient conditions for constraints meeting are also formulated and the invariance property of time delayed systems is presented. In section 5, a numerical example of a time delay system is given, to illustrate the effectiveness of the proposed approach. A final discussion with some perspectives concludes the paper.

Notations

The function sgn returns the sign of a real number x such that $\operatorname{sgn}(x)$ is equal to $1, -1$ or 0 for x positive, negative or equal to zero, respectively. The function $\max(\cdot)$ returns the greatest element in a given set or expression. When this latter does not exists we talk about a least upper bound which is given by the $\sup(\cdot)$ function. For any vector x, x_i denotes its i^{th}component, and x^T its transpose. By convention, inequalities between vectors are componentwise, and the notation $\langle x, v\rangle$ denotes the inner/scalar product of two vectors x and v. In this paper, are denoted by Γ, Ω and Δ the sets defined such as: $\Gamma = \{v \in \mathbb{R}^r | v \geq 0 \text{ and } \sum_{i=1}^r v_i = 1\}$, $\Omega = \{v \in \mathbb{R}^r | v \geq 0 \text{ and } \sum_{i=1}^r v_i \leq 1\}$ and $\Delta = \{v \in \mathbb{R}^r : \sum_{i=1}^r |v_i| \leq 1\}$.

2. POLYHEDRA AND CONVEX SETS: BACKGROUNDS AND PRELIMINARY RESULTS

In this section, a brief recall of the main definitions and properties about polyhedra and convex sets used in this work, is first presented. Then, some preliminary results about specific properties of those sets are given. For a detailed lecture on this topic we refer to Henk et al. (1997) and Blanchini (1999).

2.1 Definitions and properties

Definition 1. (Polyhedron). A convex polyhedron $\mathscr{P}(P,\pi)$ on $\mathbb{R}^p$, characterized by a matrix $P \in \mathbb{R}^{q\times p}$ and a vector $\pi \in \mathbb{R}^q$, with $p, q \in \mathbb{Z}^+$ is defined as the intersection of a finite number of half-spaces, by

$$\mathscr{P}(P,\pi) = \{x \in \mathbb{R}^p | Px \leq \pi\}. \tag{1}$$

Such a set contains the origin 0 if and only if the vector π is non-negative. If the polyhedron is symmetric about the origin, it takes the following form

$$\overline{\mathscr{P}}(Q,\rho) = \{x \in \mathbb{R}^p || Qx| \leq \rho \text{ , with } \rho \geq 0\}, \tag{2}$$

where Q is a $q \times p$ matrix and $\rho \in \mathbb{R}^q$.

Definition 2. (Polytope). A convex polytope say $\mathscr{C}(C)$, is a bounded convex polyhedron. It can be seen as convex hull, and admits a vertex representation characterized by a matrix $C \in \mathbb{R}^{p\times r}$, with $p \in \mathbb{Z}^+$, such that

$$\mathscr{C}(C) = \{x \in \mathbb{R}^p | \exists v \in \Gamma, x = Cv\}. \tag{3}$$

If the polytope contains the origin 0, it can be characterized such that

$$\mathscr{C}(M) = \{x \in \mathbb{R}^p | \exists v \in \Omega, x = Mv\}, \tag{4}$$

where $M \in \mathbb{R}^{p\times r}$. If the convex polyhedron is symmetric about the origin, it takes the following form

$$\overline{\mathscr{C}}(S) = \{x \in \mathbb{R}^p | \exists v \in \Delta, x = Sv\}, \tag{5}$$

S being a $p \times r$ matrix.

2.2 Preliminary results

In this section, some results about convex sets properties are presented. Those preliminary results are fundamental in this work since they represent a starting point to establish the bounds presented in the next section.

Lemma 1. For a given vector $x \in \mathbb{R}^r$ where r is a positive integer, the following properties hold.

(i) $\max\limits_{v\in\Gamma}\{x^T v\} = \max\limits_{i=1,..,r}\{x_i\}$.
(ii) $\max\limits_{v\in\Delta}\{x^T v\} = \max\limits_{i=1,..,r}\{|x_i|\}$.

Proof. By definition we have: $x^T v = \sum_{i=1}^r x_i v_i$. Since $x_i \leq \max\limits_i\{x_i\}$ for $i = 1,..,r$, it follows that $x^T v \leq \max\limits_i\{x_i\}\sum_{i=1}^r v_i$, which clearly leads to: $\max\limits_{v\in\Gamma}\{x^T v\} \leq \max\limits_i\{x_i\}$, for $i = 1,..,r$. Thus, consider an index j for which $x_j = \max\limits_i\{x_i\}$. By choosing a vector v such that $v_j = 1$ for $i = j$ and $v_j = 0$ for $i \neq j$, it is seen that $x^T v = \max\limits_i\{x_i\}$, which shows that the maximum is reached and completes the proof for the assertion (i) of the lemma.

Now, considering $v \in \Delta$, it is seen that $x^T v \leq \sum_{i=1}^r |x_i|\ |v_i|$. Provided that $|x_i| \leq \max\limits_i\{|x_i|\}$ for $i = 1,..,r$, it follows that $\max\limits_{v\in\Delta}\{x^T v\} \leq \max\limits_i\{|x_i|\}, \forall i = 1,..,r$. It is shown that this maximum is reached by choosing a suitable vector v such as: $v_j = \operatorname{sgn}(x_j)$ for $i = j$ and $v_j = 0$ for $i \neq j$, with $|x_j| = \max\limits_i\{|x_i|\}$ for $i = 1,..,r$, that completes the proof.

The origin point 0 plays an important role in many application, thus, particular attention is paid to convex sets containing the origin 0, for which the following proposition is formulated.

Proposition 2. Being given a polytope $\mathscr{C}(M)$ containing 0, the following properties hold true, for $i = 1,..,p$

$$\min_{j=1,..,p}\{M_{ij}\} \leq 0 \qquad \text{and} \qquad \max_{j=1,..,p}\{M_{ij}\} \geq 0.$$

Proof. If the origin 0 belongs to the convex $\mathscr{C}(M)$, it exists a vector v which verifies $Mv = 0$, thus $\sum_{i=0}^p M_{ij}v_j = 0$. This vector has at least one non-null component. If all terms of the product $M_{ij}v_j$ are zero, it follows that at least one coefficient M_{ij} is null, and the assertions of Proposition 2 are true. Otherwise, the product $M_{ij}v_j$ leads to both positive and negative terms, and provided that v_j is positive, it follows that the coefficients M_{ij} verify Proposition 2.

3. INPUT-OUTPUT SYSTEMS: DEFINITIONS AND MAIN RESULTS

3.1 Input-output or convolution systems

Input-output systems are of the form $y(t) = (h * u)(t)$, where y is the system output, u the input, h the convolution kernel and the operator $*$ denotes the convolution product. The output $y(t)$ is defined as

$$y(t) = \int_0^t h(\tau)u(t-\tau)d\tau\,. \tag{6}$$

Note that for causal systems, the convolution kernel is a function with positive support. An important family of systems is characterized by convolution kernels of the form

$$h(t) = \begin{cases} h_a(t) + \sum_{i=0}^{\infty} h_i \delta(t-t_i) & \text{for } t \geq 0\,, \\ 0 & \text{for } t < 0\,, \end{cases} \tag{7}$$

where $h_a(t)$ is a measurable function verifying $\int_0^\infty |h_a(t)|dt < \infty$, h_i is such that $\sum_{i=0}^{\infty} |h_i| < \infty$, and $\delta(t)$ is the Dirac distribution. This set of kernels is denoted by $\mathscr{A}$ which forms a commutative Banach algebra (closed under addition, multiplication, and convolution) with unitary element δ and the normed derived by

$$\|h\|_{\mathscr{A}} = \int_0^\infty |h_a(t)|dt + \sum_{n=1}^{\infty} |h_i|.$$

Such systems are known to belong to the Callier-Desoer class, introduced in Desoer and Vidyasagar (1975), which covers finite dimensional systems and infinite dimensional ones as well, such as time delayed systems, fractional systems and many distributed input-output systems. In the multi-variable case, input-output systems are characterized by a matrix kernel such like

$$y(t) = (H * u)(t) \tag{8}$$

where $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$ and H is a $p \times m$ matrix which norm on $\mathscr{A}^{p\times m}$ is defined as follows.

$$\|H\|_{\mathscr{A}} = \max_{i=1,..,p} \{\sum_{j=1}^{m} \|h_{ij}\|_{\mathscr{A}}\}.$$

In Desoer and Vidyasagar (1975), the authors showed that the following equality holds

$$\sup_{u\neq 0} \frac{\|y\|_\infty}{\|u\|_\infty} = \|h\|_{\mathscr{A}}$$

.

Thus, every system with kernel defined on $\mathscr{A}$, is said Bounded Input Bounded Output (BIBO) stable. That means for every bounded input to the system results in a bounded output over the time interval $[t_0,\infty)$, and this must hold for all initial times t_0. These concepts and properties have been established a long time ago. They are powerful tools to investigate some bound properties of linear systems, and thus to solve some fundamental problems, such as reachable sets characterization and constraints meeting for complex systems such like systems with delays, as presented in the sequel.

3.2 Bounds for Input-Output systems

Using the above mentioned properties of the Input-Output systems of the class $\mathscr{A}$, and the preliminary results given in Section 2.2, we propose to formulate the bounds of a given Input-Output system subject to polyhedral constraints on his inputs, as follows.

Theorem 3. For a given Input-Output system of the form (8), with input $u(t)$ evolving in a given convex polyhedron $\mathscr{C}(C)$ of the form (3), the output $y(t)$ is bounded as follows

$$\int_0^t \min_{j=1,..,r} \{(H(\tau)C)_{ij}\}\, d\tau \leq y_i(t) \leq \int_0^t \max_{j=1,..,r} \{(H(\tau)C)_{ij}\}\, d\tau\,.$$

For all $t \geq 0$ and $i = 1,..,p$.

Moreover, those inequalities give exact bounds for the system output, since there exists, for each index i and time t, an input $u(t)$ for which the strict equality is obtained.

Proof. Provided that $u(t) \in \mathscr{C}(C)$, it exist a $v(t) \in \Gamma$, such that $u(t) = Cv(t)$. The expression of $y(t)$ is then rewritten

$$y(t) = \int_0^t H(\tau)\cdot C \cdot v(t-\tau)d\tau\,. \tag{9}$$

Assertion (i) of Lemma 1 allows to upper-bound the integrand part, so that $y_i(t) \leq \int_0^t \max_{i=1,..,q} \{(H(\tau)\cdot C)_{ij}\}d\tau$, which establishes the right part of Theorem 3 inequality. For establishing the left part, one can note that the integrand terms can be upper-bounded, provided that

$$-(H(\tau)Cv(t-\tau))_i \leq \max_{j=1,..,r} \{(-H(\tau)C)_{ij}\}$$

and

$$\max_{j=1,..,r} \{(-H(\tau)C)_{ij}\} = -\min_{j=1,..,r} \{(H(\tau)C)_{ij}\}\,.$$

Those bounds are reachable, since the bounds obtained in Lemma 1 were shown to be exact. Indeed, by considering an index $j(\tau)$ which verifies $(H(\tau)C)_{ij(\tau)} = \max_{k=1,..,r} \{(H(\tau)C)_{ik}\}$, one can see that the upper bound is reachable for a chosen input u such that $u(\tau) = Cv(\tau)$, with $v_i(\tau) = 1$, if $i = j(\tau)$, and $v_i(\tau) = 0$ otherwise. The reachability of the lower bound can be demonstrated in the same manner.

The results presented in this part are quite interesting since they give exact bounds for a large variety of systems belonging to the class $\mathscr{A}$. They can be used for solving many questions raised in constrained control problems, typically such that output reachability sets computation under constrained inputs, and invariance properties. From a practical point of view however, finding the upper and the lower bounds described by the inequalities of Theorem 3 at each moment t is quite complex. Actually, the input space being a general polyhedron of the form (3), the integrand term in Theorem 3 is not a monotonic function of time, and both its maximum and minimum, given by $max(\cdot)$ and $min(\cdot)$ respectively, do not have a constant sign with respect to time. This practical complexity is relaxed when the considered polyhedron contains the origin, as it is demonstrated by Proposition 2. The latter proposition, together with the BIBO stability property of the considered Input-Output system, leads to easier computations and simplifications for the bounds given in Theorem 3. In the rest of the paper, all the polyhedra are assumed to contain the origin 0.

4. CONSTRAINED INPUT-OUTPUT SYSTEMS

In this section, the use of Theorem 3 is illustrated by two main applications for constraints systems. The first issue is related to the characterization of the output reachable space of systems with some polyhedral constraints on their inputs. The second one is about the verification of the constraints warranties, when both the system inputs and outputs are constrained.

4.1 Reachable set

In this section, we aim to characterize the reachable set of an Input-Output system of the form (8), with constrained inputs given in a convex polytope containing the origin of the form (4). We shall make use of the results presented in Section 3.2, to propose a procedure to construct a polyhedral approximation of the reachable set, this latter being defined as follows.

Definition 3. (Reachable set). Being given a system of the form (8), the reachable space, say $\mathscr{R}(t)$, is the set of all the trajectories of the output space that are reachable by the system in a finite time, starting from the origin.

The proposed procedure consists on constructing a convex hull, which encloses the output reachable space. In this purpose, let us consider a variable vector, say α, defined as $\alpha^T = \left[\cos\frac{k}{N}\pi, \sin\frac{k}{N}\pi\right]$, with $k = 0..N$, and $N \in \mathbb{N}$. Using Theorem 3, it is shown that the inner product $\alpha^T.y(t)$ can be bounded at any moment t such that

$$\gamma \leq \alpha^T.y \leq \Gamma, \qquad (10)$$

where

$$\Gamma = \int_0^\infty \max_{j=1,..,r}\{\alpha^T.(H(\tau)M)_j\}\,d\tau,$$

and

$$\gamma = \int_0^\infty \min_{j=1,..,r}\{\alpha^T.(H(\tau)M)_j\}\,d\tau.$$

Provided that the bounds described in Theorem 3 are exact, it follows that Γ and γ are also exact reachable bounds for the inner product $\alpha^T.y$, which form the edges of the polyhedral approximation of $\mathscr{R}(t)$.

Using an appropriate integration program, it is possible to compute and plot iteratively all the edges, corresponding to the different values of α obtained for $k = 0..N$. N is then the final number of the edges used for the approximation of $\mathscr{R}(t)$. The accuracy of the approximation is enhanced with great values of N, such that when N tends to infinity, the obtained approximation of $\mathscr{R}(t)$ tends to be the exact reachable space.

It is important to note that such reachable space is a convex set. Indeed, a complete representation of a convex body $\mathscr{C}$ is given by $\mathscr{C} = \{x \in \mathbb{R}^n : \langle x, \alpha\rangle \leq \Lambda_{\mathscr{C}}(\alpha), \text{for every } \alpha \in S^{n-1}\}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$ with center at the origin, and the function $\Lambda_{\mathscr{C}}(\alpha)$ is the support function defined as $\Lambda_{\mathscr{C}}(\alpha) = \sup_{x\in\mathscr{C}}(x^T.\alpha)$. One can see that the bound Γ is a support function for the obtained set $\mathscr{R}(t)$, and thus, this latter is convex (see Grunbaum et al. (1967); Ghosh and Kumar (1998)).

4.2 Constrained Input, constrained output

In this part, we consider that the system (8) is subject to some constraints on both its inputs and its outputs. The following theorem gives necessary and sufficient conditions under which the system (8) meets some linear constraints on its outputs, for every input defined on a convex polyhedra.

Theorem 4. Being given an Input-Output system of the form (8) from the class $\mathscr{A}$, with input $u(t)$ constrained in a convex polyhedron containing the origin $\mathscr{C}(M)$ of the form (4), and two vectors $\lambda, \mu \in \mathbb{R}^p$, the system output $y(t)$ verifies the constraint

$$\lambda \leq y(t) \leq \mu$$

for all $t \geq 0$ and every input u defined on $\mathscr{C}(M)$ if and only if the following conditions hold true.

$$\lambda_i \leq -\|\min_{j=1,..,r}\{(H(\tau)M)_{ij}\}\|_{\mathscr{A}}$$

and

$$\mu_i \geq \|\max_{j=1,..,r}\{(H(\tau)M)_{ij}\}\|_{\mathscr{A}}\ .$$

Proof. As shown by Proposition 2, if 0 is contained in the polytope $\mathscr{C}(M)$, the integrand terms in Theorem 3 are monotonic, so that the lower bounds are negative, and the upper ones are positive. Provided that the system is from the class $\mathscr{A}$, it follows that the terms, $H(\tau)M$, $\min_{j=1,..,r}\{(H(\tau)M)_{ij}\}$ and $\max_{j=1,..,r}\{(H(\tau)M)_{ij}\}$ are all elements of $\mathscr{A}$, and thus the modulus of the integrands in Theorem 3 are monotonic increasing functions of time. The necessity of the Theorem conditions come from the fact that the bounds

$$\inf_{t\geq 0}\{\int_0^t \min_{j=1,..,r}\{(H(\tau)M)_{ij}\}\,d\tau\} = -\|\min_{j=1,..,r}\{(H(\tau)M)_{ij}\}\|_{\mathscr{A}}$$

and

$$\sup_{t\geq 0}\{\int_0^t \max_{j=1,..,r}\{(H(\tau)M)_{ij}\}\,d\tau\ \} = \|\max_{j=1,..,r}\{(H(\tau)M)_{ij}\}\|_{\mathscr{A}}$$

are well defined for such BIBO-systems. The sufficiency is due to the fact that these bounds are exact ones as shown previously in Theorem 3.

Theorem 4 can be extended for cases where the output constraints are polyhedral too. Indeed, being given a convolution system of the form (8), with polytopic bounds on the input u i.e. $u(t) \in \mathscr{C}(M)$, and polyhedral constraints on the output y i.e. $y(t) \in \mathscr{P}(P,\pi)$, Theorem 5, gives necessary and sufficient condition under which, for every input u of $\mathscr{C}(M)$, the output y will remain in the polyhedron $\mathscr{P}(P,\pi)$.

Theorem 5. Being given a convolution system of the form (8), a polytope of the input space containing the origin, say $\mathscr{C}(M)$, and a polyhedron of the output space $\mathscr{P}(P,\pi) \subset \mathbb{R}^q$. The output $y(t)$ remains in $\mathscr{P}(P,\pi)$, for all $t \geq 0$, and every input $u(t)$ evolving in $\mathscr{C}(M)$, if and only if the following condition holds true

$$\mu \leq \pi, \qquad (11)$$

where μ is a q-vector which components are defined by

$$\mu_i = \|\max_{j=1,..,r}\{(PH(t)M)_{ij}\}\|_{\mathscr{A}}$$

for $i = 1,..,q$.

Proof. Provided that $Py(t)$ is expressed by

$$Py(t) = \int_0^t P\cdot H(\tau)\cdot u(t-\tau)\,d\tau\ , \qquad (12)$$

it can be noted that $Py(t)$ is the output of a convolution system which kernel is given by $PH(t)$. Thus using Theorem 4, it is seen that $Py(t)$ verifies the inequality $Py(t) \leq \mu$. The sufficiency of condition (11) is clear: under Theorem 3 assumptions, it is seen that $Py(t)$ verifies the inequality $Py(t) \leq \pi$. Moreover, this is an exact bound, since for each value of i, there exists an increasing sequence of instants t_k and inputs $u^{(k)}(t_k)$, for which the corresponding outputs generated, say $y_i^{(k)}(t_k)$, converge to μ_i. Now, if the condition $Py_i^{(k)}(t_k) \leq \pi_i$ is verified, the necessity of the condition (11) is thus revealed when t tends to infinity.

In many practical applications, the constraints applied to the system are often symmetric. In such case, we propose the following corollary which stands for Input-Output systems under polyhedral constraints that are symmetrical about the origin.

Corollary 6. Being given a convolution system of the form (8), a 0-symmetric polyhedron of the input space, $\overline{\mathscr{C}}(S)$ as defined in (5) and a 0-symmetric polyhedron of the output space $\overline{\mathscr{P}}(Q,\rho)$ as defined in (2), the output $y(t)$ remains in $\overline{\mathscr{P}}(Q,\rho)$, for all $t \geq 0$, and every input $u(t)$ evolving in $\overline{\mathscr{C}}(S)$, if and only if the following condition holds true

$$\eta \leq \rho\,, \tag{13}$$

where η is an q-vector which components are given by

$$\eta_i(t) = \| \max_{j=1,..,r} \{|(Q \cdot H(\tau) \cdot S)_{ij}|\} \|_{\mathscr{A}}\,,$$

for $i = 1,..,q$.

Proof. The proof of the corollary is quite similar to Theorem 3 demonstration, provided that the amount

$$|Qy(t)| = |\int_0^t Q \cdot H(\tau) \cdot u(t-\tau) d\tau|\,,$$

can be upper bounded, as shown in Theorem 3, using assertion (ii) of Lemma 1 such that:

$$|Qy(t)| \leq \int_0^t \max_{j=1,..,r} \{|(Q \cdot H(\tau) \cdot S)_{ij} d\tau|\}.$$

Thus, the sufficiency and necessity of condition (13) are then obtained as in the proof of Theorem 3.

The results obtained can be extended considering other configurations of the constraints, for example, the case where the input constraints are symmetrical while the output ones are not. Using the corresponding assertion of Lemma 1, and the bounds given by Theorem 3, one can easily compute the corresponding conditions.

4.3 Polyhedral invariance of time delay systems

The particular issue of the polyhedral invariance for delayed system is now considered. Delayed systems are a part of the Input-Output class of systems under consideration. Indeed, when the system is issued from the integration of a differential equation with finite delays, its expression is completed by a term that comes from initial conditions, in the form

$$x(t) = (G * \phi)(t) + (H * u)(t), \tag{14}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, H and G are respectively $n \times m$ and $n \times n$ matrices over $\mathscr{A}$, and $\phi(t) \in \mathbb{R}^n$ is a function with support over $[0, \theta]$, the positive number θ being the maximal delay of the considered differential equation. For such systems, the above results about constrained systems can be reformulated in terms of polyhedral invariance which is defined as follows.

Definition 4. ($\mathscr{C}$-Invariance). Two polytopes $\mathscr{C}(M) \subset \mathbb{R}^m$ and $\mathscr{P}(P, \pi)$ being given, we say that $\mathscr{P}$ is $\mathscr{C}$-invariant for system (14) if the system trajectories remains in $\mathscr{P}$, for every input $u(t) \in \mathscr{C}$, for $t \geq 0$, and every initial condition $\phi(t) \in \mathscr{P}$ for $0 \leq t \leq \theta$.

For a delayed system of the form (14), the explicit expression of the solution $x(t)$ in terms of $\phi(t)$ and $u(t)$ reads

$$x(t) = \int_{t-\theta}^t G(\tau)\phi(t-\tau)d\tau + \int_0^t H(\tau)u(t-\tau)d\tau\,, \tag{15}$$

for $t \geq 0$. Since $\phi(t)$ and $u(t)$ are independent, the estimation of the maximal and minimal values reached by the components of $x(t)$ comes down to the estimation of the maximal and minimal values reached by the components of both terms of this equality respectively. The verification of the $\mathscr{C}$-invariance property of $\mathscr{P}$ hence comes down to the verification of the inequality $\sup_{t\geq 0}\{Px\}_i(t) \leq \pi_i$, which, using the ideas already developed, leads to the following result.

Theorem 7. The system (14) being given, together with two polytopes $\mathscr{C}(M)$ and $\mathscr{P}(P, \pi)$ of the form (4) and (1) respectively, the polytope $\mathscr{P}$ is $\mathscr{C}$-invariant for the system (14) if and only if the upper bound

$$\sup_{t\geq 0} \left(\int_{t-\theta}^t \max_{j=1,..,m} (PG(\tau)M)_{ij} d\tau + \int_0^t \max_{j=1,..,m} (PH(\tau)M)_{ij} d\tau \right)$$

is less than or equal to π_i, for $i = 1,..,n$.

Proof. The proof of Theorem 7 is directly derived from the demonstration of Theorem 3. In this case however, the upper bound of the system output can not be simplified using the norm expression as it was done previously, since the finite integral term, appearing in expression (15) due to the initial conditions function, does not necessarily define a monotonic function.

5. NUMERICAL EXAMPLE

Let us consider the following time delay system

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -0.9 \end{bmatrix} x(t-h) + \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} u(t)\,, \tag{16}$$

where the delay h is taken equal to 1, $x(t) = 0$ for $t \in [-h, 0]$ is the initial state of the system, and $u(t)$ is the system input, which verifies $u(t) \in \mathscr{U} = \{-1 \leq u(t) \leq 1\}$, for $t \geq 0$. One can check that the system is stable independently of the delay. For such configuration, we propose to characterize the set of reachable states, as described in Section 4.1. In order to ensure this, the system 16 is first rewritten in the form (8), by solving the delayed differential equation, where the kernel $H^T = [H_{11}(t), H_{21}(t)]^T$ is depicted in Fig. 1. Using a numerical implementation of the procedure described in Section 4.1, the output reachable set is computed using $N = 4000$, which corresponds to the number of edges used, as shown in Fig. 2.

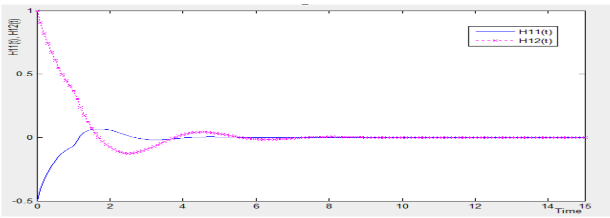

Fig. 1. Time representation of corresponding Kernel of system (16).

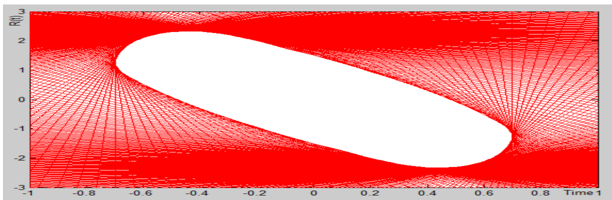

Fig. 2. Output reachable set (uncoloured space) for the system (16) .

6. DISCUSSION AND CONCLUSION

In this work, some original results about bounding Input-Output systems are presented, by exploiting convolution and convex sets properties. These results are quite important since they give exact bounds for a large variety of systems belonging to the class $\mathscr{A}$, such as time delay systems. They can be used for solving many questions raised in constrained problems, not only for output reachability sets computation under constrained inputs, and invariance properties as illustrated in this paper, but also in solving constrained command problems. From a practical point of view however, the various results are expressed as inequalities involving functions of the time, which implies the computation of their upper bounds. This computation may turn out to be fastidious analytically, but still feasible with accurate numerical methods. In the particular case of positive systems, some explicit formulae can be derived as pointed out in Shen and Lam (2013), since in this case the impulse response is positive and non-decreasing. This may be very interesting computationally, and could greatly simplify, for certain systems, the verification of the conditions presented in this paper. In Theorem 7 appears a more complicated condition, that can rarely be simplified, since there appears a finite integral (that does not define an increasing function, even if the integrated is non-negative). Thus, further developing of numerical methods to evaluate with precision upper bounds of functions defined via integrals, remains a very promising subject in view of using the proposed methods in real practical problems.

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