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Set-based Disturbance Attenuation in Economic Model Predictive Control ⋆

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Abstract: In this paper, we develop a tube-based economic MPC framework for nonlinear systems subject to unknown but bounded disturbances. We show that just transferring the design procedure of tube-based stabilizing MPC to an economic MPC framework might not be the optimal choice in terms of the achievable asymptotic average performance. Instead, the asymptotic average performance can possibly be improved by considering the influence of the disturbance explicitly within the design of the control input. This will be done by using a specifically defined integral stage cost, which is the key feature of our proposed robust economic MPC algorithm. Furthermore, we show that for this algorithm, similar results as in nominal economic MPC (i.e., without disturbances) can be established, in particular with respect to bounds on the asymptotic average performance of the resulting closed-loop system as well as stability.

1. INTRODUCTION

Over the last decades, Model Predictive Control (MPC) has become a widely used control method, in theory as well as in practice (Rawlings and Mayne [2009]). Reasons for that are the possibility to handle not only a certain performance criterion (or cost), but also since MPC can deal with hard input and state constraints.

Most of the available MPC approaches in the literature are formulated within a stabilizing framework (see, e.g., Mayne et al. [2000]), which means that the control objective is to stabilize a given setpoint or trajectory to be tracked. In such a setting, the employed stage cost is usually assumed to be positive definite with respect to the considered steady state. On the other hand, driven by applications in the process industry, the idea of using a stage cost which is economically related to the problem has evolved more and more in the last five years. This economic relation can, for example, include aspects like maximization of an output or minimization of energy consumption, and hence the stage cost is not necessarily related to (and hence in particular also not positive definite with respect to) any specific setpoint, which is the major difference to stabilizing (or tracking) MPC. Such a framework has been presented by the name of economic MPC (see Angeli et al. [2012]), and various aspects and different settings have recently been studied (see, e.g., Angeli et al. [2012], Amrit et al. [2011], Heidarinejad et al. [2012], Grüne [2013], Müller et al. [2013]).

Since most real systems are affected by disturbances and since the models might be inaccurate, a wide number of literature within the framework of stabilizing MPC deals with such disturbed or uncertain systems (see, e.g., Chisci et al. [2001], Mayne et al. [2005]). In the following, we refer to this robust stabilizing (or tracking) MPC as robust MPC. Most of these approaches aim to find an invariant set for either the disturbed system or for the difference between the disturbed system and an - artificially introduced - nominal system, which is not affected by the disturbance. Robust MPC approaches of the second kind, especially in the framework of linear systems, are usually referred to as "Tube MPC", since the real system state is kept in a set around the nominal system state. On the other hand, only few publications can be found for disturbed systems in the context of economic MPC. In this respect, a stability result for robust economic MPC is presented in Huang et al. [2012]; these results are, however, obtained in a formulation related to tracking MPC. In Hovgaard et al. [2011], a scenario based approach is used for uncertain systems to minimize the energy consumption taking also probabilistic constraints into account. The authors focus on the special class of linear cost functions, and within their application driven approach, they do not provide any stability or optimality results. Another idea is presented in Müller and Allgöwer [2012], where the robustness of steady state optimality under disturbed constraints is considered. In this reference, however, no disturbances are considered within the system dynamics, but the constraints are assumed to be uncertain.

In this paper, we develop a tube-based economic MPC framework for nonlinear systems subject to unknown but bounded disturbances. Our first contribution (see Section 2.2) is to demonstrate that just transferring the design procedure of tube-based stabilizing MPC to an economic MPC framework, i.e., calculating the control input by predicting the cost for the nominal system and then applying an additional error feedback, might not be the optimal choice in terms of the achievable asymptotic average performance. Instead, the asymptotic average performance can possibly be improved by considering the influence of the disturbance explicitly within the design of the control input. This will be done by using a specifically defined integral stage cost (see Section 2.3 for a detailed description), which is the key feature of our proposed robust economic MPC algorithm. Furthermore, we show that for this algorithm, similar results as in nominal economic MPC (i.e., without disturbances) can be established, in particular with respect to bounds

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on the asymptotic average performance of the resulting closed-loop system as well as stability.

The remainder of this paper is structured as follows. In Section 2, the problem setup, the determination of invariant error sets, and a motivating example introducing our proposed algorithm are given. A stability analysis for the proposed algorithm is provided in Section 3. A further example is presented and discussed in Section 4. Finally, some open problems are stated and a conclusion is given in Section 5.

Notation: The distance of a point $x \in \mathbb{R}^n$ to a set $\Omega \subseteq \mathbb{R}^n$ is defined by $|x|_\Omega = \inf_{y \in \Omega} |x - y|$. We denote by $\mathbb{I}_{\geq 0}$ the set of all non-negative integers, and by $\mathbb{I}_{[a,b]}$ the set of all integers in the interval $[a, b] \subseteq \mathbb{R}$.

2. PROBLEM SETUP AND MOTIVATION

We want to control the time-invariant disturbance-affected discrete-time system

$$x(t+1) = f(x(t), u(t), w(t)), \ \ x(0) = x_0, \tag{1}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^n$ is continuous, $x(t) \in \mathbb{X} \subset \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{U} \subset \mathbb{R}^m$ is the input to the system, and $w(t) \in \mathbb{W} \subset \mathbb{R}^q$ is a disturbance acting on the system. We assume possibly coupled state and input constraints, that is,

$$(x(t), u(t)) \in \mathbb{Z} \subseteq \mathbb{X} \times \mathbb{U}$$

for all $t \in \mathbb{I}_{\geq 0}$, where the set $\mathbb{Z}$ is compact. Moreover, $\mathbb{W}$ is a compact set containing zero.

Our objective is to find an input sequence such that the system remains feasible and such that the asymptotic average performance

$$\lim_{T \to \infty} \frac{1}{T} \sum_{k=0}^{T-1} \ell(x(k), u(k)) \tag{2}$$

is minimized, where $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is the continuous stage cost function. In stabilizing MPC, one usually assumes that the stage cost ℓ is positive definite with respect to the setpoint (x_s, u_s) to be stabilized, i.e., $0 = \ell(x_s, u_s) < \ell(x, u)$ for all $(x, u) \in \mathbb{Z}$. In the following considerations, the stage cost function does not need to satisfy any conditions of this kind, but can be chosen arbitrary. This is one of the key features within economic MPC (Angeli and Rawlings [2010]).

2.1 Invariant Error Sets

Due to the unknown disturbances in the system dynamics (1), it is impossible to predict the exact system states at future time instants. Thus, one wants to determine possibilities to keep the influence of the disturbance on the real system state within known bounds. One of the most widely used approaches it to determine an invariant set for the error between the real system and the associated *nominal* system

$$z(t+1) = f(z(t), v(t), 0), \ \ z(0) = z_0. \tag{3}$$

Here, z is the state of the nominal system, whereas v is the nominal input. We introduce the error by $e(t) = x(t) - z(t)$ and the error dynamics by

$$e(t+1) = f(x(t), u(t), w(t)) - f(z(t), v(t), 0). \tag{4}$$

The following definition from set based control can be used to derive bounds on the error.

Definition 1. (Kerrigan [2000], Yu et al. [2013]) A set $\Omega \subseteq \mathbb{R}^n$ is robust control invariant (RCI) for the error system (4) if and only if there exists a feedback control law $u(t) = \varphi(v(t), x(t), z(t))$ such that for all $x(t), z(t) \in \mathbb{X}$ resulting in $e(t) \in \Omega$, all $v(t) \in \mathbb{U}$, and all $w(t) \in \mathbb{W}$, it holds that $e(t+1) \in \Omega$.

In case of linear systems with additive disturbance, a broad literature is available to provide RCI sets, see, e.g., Chisci et al. [2001] and Raković et al. [2005]. Namely, given a linear system of the form

$$x(t+1) = Ax(t) + Bu(t) + w(t)$$

and using the control law

$$u(t) = \varphi(v(t), x(t), z(t)) = v(t) + K(x(t) - z(t)), \tag{5}$$

where K is chosen such that $A + BK$ is Hurwitz, the error dynamics results in

$$e(t+1) = (A + BK)e(t) + w(t). \tag{6}$$

This means that by considering the special error feedback (5), an RCI set as in Definition 1 is given by a robustly positively invariant (RPI) set for system (6) (see, e.g., Kerrigan [2000]). In the literature, many methods exist determining approximations of the minimal RPI set (see, e.g., Raković et al. [2005]). Within the framework of robust MPC, the open-loop optimization will be performed for the nominal system and the sequence of nominal inputs $\boldsymbol{v}(\cdot)$ is the optimization variable. Accordingly, by means of the input chosen in (5), we can guarantee that the state x of the "real" system will always be within a compact set Ω around the state z of the nominal system. However, when optimizing over the nominal system, one still wants to guarantee the constraints for the real system. Thus, we must tighten the constraints for the nominal system according to $(z(t), v(t)) \in \overline{\mathbb{Z}}$, for all $t \in \mathbb{I}_{\geq 0}$, where $\overline{\mathbb{Z}} := \mathbb{Z} \ominus (\Omega \times K\Omega)$.

For general nonlinear systems, it is rather difficult to find invariant sets. Thus, most of the approaches available in the literature rely on a special class of systems (see, e.g., Limon et al. [2002], Yu et al. [2013], and Bayer et al. [2013]), since an appropriate feedback of the form $u(t) = \varphi(v(t), x(t), z(t))$ must be provided and, additionally, a possibility to determine the reasonably tightened set $\overline{\mathbb{Z}} = \{(z, v) \in \mathbb{Z} : (x, \varphi(v, x, z)) \in \mathbb{Z}$ for all $x \in \{z\} \oplus \Omega\}$ for the nominal states and inputs.

2.2 Motivating Example

The disturbances acting on the system can - as we will see in the following - have an influence on the asymptotic average performance (2), and hence, should be considered within the setup of the economic MPC approach. To understand the influence, let us have a look at the following motivating example, a non-symmetric, but positive definite, quadratic cost

$$\ell(x, u) = \begin{cases} 4x^2 & \text{for } x < 0 \\ \frac{1}{4}x^2 & \text{for } x \geq 0, \end{cases}$$

with a scalar linear system

$$x(k+1) = 0.9x(k) + u(k) + w(k).$$

For the constraints, we state $\mathbb{Z} = \{(x, u) \in \mathbb{R}^2 | \, |x| \leq 10, |u| \leq 1\}$ and $\mathbb{W} = \{w \in \mathbb{R} | |w| \leq 0.1\}$. Since we deal with a linear system, we can apply a control as given in (5) with $K = -0.9$. For the associated error system, the minimal RPI set is given by $\Omega = \mathbb{W}$. The most simple idea to control this system would be to apply an economic MPC scheme to the nominal system (3), and the then use the error feedback (5) to keep the nominal system in a tube around the nominal closed-loop system as, for

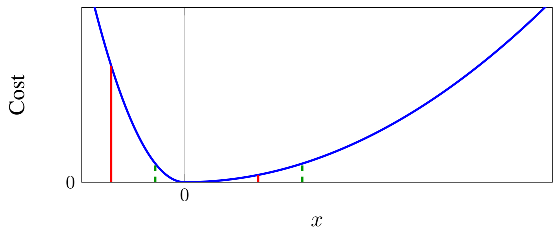

Fig. 1. Cost function of the motivating example in Section 2.2 together with two RPI sets of same cross section but with different center points.

example, introduced in Chisci et al. [2001]. This corresponds to solving the following optimization problem.

$$(P_{\text{class}}) \begin{cases} \min_{\boldsymbol{v}(t)} \sum_{k=t}^{N+t-1} \ell(z(k|t), v(k|t)) \\ \text{s.t. } z(k+1|t) = f(z(k|t), v(k|t), 0), \\ \quad (z(k|t), v(k|t)) \in \overline{\mathbb{Z}} \ \forall k \in \mathbb{I}_{[t,N+t-1]}, \\ \quad z(N+t|t) = \bar{z}_s, \\ \quad z(t|t) = z(t). \end{cases}$$

Here, $N > 0$ is the prediction horizon, and $\bar{z}_s$ is the optimal steady state of the nominal system, i.e.

$$(\bar{z}_s, \bar{v}_s) = \underset{(z,v) \in \overline{\mathbb{Z}}, z=f(z,v,0)}{\arg\min} \ell(z, v).$$

which is usually used as a terminal constraint in economic MPC (Angeli and Rawlings [2010]). In the present example, $(z_s, v_s) = (0, 0)$ and the nominal closed-loop system resulting from (P_{class}) is asymptotically stable at $z_s = 0$. Thus, the real system will stay within an RPI set centered at the origin. Yet, due to the disturbance, one cannot determine a priori the exact state of the system at future iterations. In other words, it may lay everywhere within the RPI set.

In Figure 1, the borders of the associated RPI set are denoted in red. One can see that by driving the nominal system to the origin, the disturbances may result the real system to end up in a point with significantly high cost. As we do not assume any a priori knowledge about the distribution of the disturbance, with respect to the asymptotic average performance (2) it would instead be much more beneficial to drive the nominal system to the middle of the green (dashed) RPI set. Obviously, the cost for the nominal state is higher in this case, however, the overall performance might be better, since larger parts of the green RPI set are on the "cheaper" right half side.

Let us have a look at some numbers from simulating this example. We compare the asymptotic average performance (2) determined with a robust MPC based on the optimization (P_{class}) to an approach which will be presented later that provides the real system to stay within the green, "shifted" RPI set. This approach will take the disturbance into account within the optimization. The average performance $(\frac{1}{T}\sum_{k=0}^{T} \ell(x(k), u(k)))$ of the real closed-loop system is given averaged over 10 simulations with $T = 1000$, with w being a random number in $\mathbb{W}$ (uniformly distributed) and with the initial state $x(0) = 0$.

MPC with (P_{class})	"Shifted approach"
0.0072	0.0022

For this simple example, it is highly beneficial to keep the RPI set - and hence the real system states - not around the origin, but around another nominal state.

Note that there might exists some special specifications of the disturbance, for example $w(k) \equiv 0$ for all k, where the MPC approach with (P_{class}) provides a better average performance; however, as shown above, on average the latter approach will results in a better average performance. Hence, as we assume that the disturbance is unknown a priori, we propse to explicitly incorporate the disturbance into the economic MPC algorithm, which will be formalized in the following section.

2.3 *The Robust Economic MPC Algorithm*

We now formally introduce the economic MPC algorithm used in the "shifted approach" in the motivating example. To this end, let $u = \varphi(v, x, z)$ be an appropriate control law providing an RCI set Ω for the error system (4), as described in Section 2.1, and consider the following definition.

Definition 2. The *robust optimal steady state* (z_s, v_s) of a disturbance-affected system (1) for a given cost function $\ell(x, u)$ is introduced as

$$(z_s, v_s) = \underset{(z,v) \in \overline{\mathbb{Z}}, z=f(z,v,0)}{\arg\min} \int_{x \in \{z\} \oplus \Omega} \ell(x, \varphi(v, x, z)) dx.$$

As shown in Section 2.1, for a linear system the control law (5) can be used, i.e., $\varphi(v, x, z) = v + K(z - x)$. For the motivating example, the robust optimal steady state is located at $(z_s, v_s) = (0.0602, 0.0060)$

As could be seen in the motivating example, it was beneficial to center the RPI set around the robust optimal steady state. This means that some knowledge of the disturbance is incorporated into the cost function. To formalize this idea, the integrated cost function ℓ^{int} is defined as

$$\ell^{\text{int}}(z, v) = \int_{x \in \{z\} \oplus \Omega} \ell(x, \varphi(v, x, z)) dx, \tag{7}$$

where Ω is the associated RCI set. We now propose to use this cost function within the following robust economic MPC algorithm.

$$(P_{\text{REMPC}}) \begin{cases} \min_{\boldsymbol{v}(t)} V^{\text{int}}(z(t), \boldsymbol{v}(t)) \\ \text{s.t. } z(k+1|t) = f(z(k|t), v(k|t), 0), \\ \quad (z(k|t), v(k|t)) \in \overline{\mathbb{Z}} \ \forall k \in \mathbb{I}_{[t,N+t-1]}, \\ \quad z(N+t|t) = z_s, \\ \quad z(t|t) = z(t), \\ \text{where } V^{\text{int}}(z(t), \boldsymbol{v}(t)) = \sum_{k=t}^{N+t-1} \ell^{\text{int}}(z(k|t), v(k|t)). \end{cases}$$

Algorithm 1 Robust Economic MPC Algorithm

given: initial state $x(0)$
  for $t = 0, 1, 2, \ldots$ **do**
    solve (P_{REMPC})
    apply $u(t) = \varphi(v^*(t|t), x(t), z(t))$ to (1)
    apply $v(t) = v^*(t|t)$ to (3)
  end for

Applying Algorithm 1 leads to the nominal closed-loop system

$$z(t+1) = f(z(t), v^*(t|t), 0) \tag{8}$$

and real (disturbed) closed-loop system

$$x(t+1) = f(x(t), \varphi(v^*(t|t), x(t), z(t)), w(t)). \tag{9}$$

Here, $\boldsymbol{v}^* = \{v^*(t|t), \ldots, v^*(N+t-1|t)\}$ denotes the optimal input sequence determined by (P_{REMPC}).

Remark 3. In contrast to the robust MPC approach in Mayne et al. [2005], we will not optimize over the nominal initial condition after each iteration, but determine the initial nominal state $z(t+1)$ at next iteration according to the prediction $z^*(t+1|t)$ at time t (see, e.g., Chisci et al. [2001]). By means of this constraint, the sequence of initial nominal states $\{z(t), z(t+1), \dots\}$ is an actual trajectory of the nominal system. For the initial nominal state $z(0)$ at time $t = 0$, one can think of different possibilities. This initial nominal state only needs to satisfy $x(0) \in z(0) \oplus \Omega$. One could, e.g., take $z(0) = x(0)$ or one could allow to optimize over $z(0)$.

The presented optimization problem (P_{REMPC}) applied within Algorithm 1 has two main features. First, the cost function consists of the sum of the stage cost functions *integrated* over the RCI set. Thus, we consider all possible states which are within the RCI set around the predicted nominal state and robustness is introduced within the economic framework. Second, the "real" input φ, which is actually applied to the system, is taken into consideration within the cost function ℓ^{int}, i.e., instead of only considering the nominal input v, we also take into account the cost of the input needed to keep the error within the RCI set. This is in contrast to the aforementioned set-based robust MPC approaches, where usually only the *nominal* input is considered within the cost functional.

We can now state the following asymptotic average performance result, which is based on a similar result for nominal economic MPC in Angeli et al. [2012].

Theorem 4. Assume that the optimization problem (P_{REMPC}) is feasible at time $t = 0$ for a given initial condition $x(0)$. Then, the MPC Algorithm 1 is recursively feasible and the solution of the resulting closed-loop system (9) has a robust asymptotic average performance which is no worse than that of the robust optimal steady state, i.e.,

$$\ell^{\text{int}}(z_s, v_s) \geq \limsup_{T\to\infty} \frac{\sum_{t=0}^{T-1} \ell^{\text{int}}(z(t), v^*(t|t))}{T}. \tag{10}$$

Remark 5. The statement (10) means that the average integral cost along the closed-loop nominal system (8) is at least as good as the integral cost at the robust optimal steady state. As the real closed-loop system (9) lies in the RCI set around the nominal closed-loop system (8), this can be interpreted as an average performance result for the real closed-loop system (9), averaged over all possible disturbances by integrating over the RCI set.

Proof. The proof for recursive feasibility follows from the appropriate tightening of the sets introduced in Section 2.1, the fact that Ω is an RCI set for the error system (4) and by feasibility of the terminal constraint. Thus, following the standard argumentation in MPC, the input sequence $\{v^*(t+1|t), \dots, v^*(N+t-1|t), v_s\}$, where the steady state input v_s is added, can be used as a new admissible input generating a feasible solution at the next iteration (see, e.g., Chisci et al. [2001]).

Concerning average performance, the proof follows along the lines of Angeli et al. [2012]. We denote the solution of the optimization problem (P_{REMPC}) by $V^{\text{int}^*}(z(t))$. Due to recursive feasibility and because of the terminal constraint, we can derive

$$V^{\text{int}^*}(z(t+1)) - V^{\text{int}^*}(z(t)) \leq \ell^{\text{int}}(z_s, v_s) - \ell^{\text{int}}(z(t), v^*(t|t))$$

Taking the average at both sides leads to

$$\begin{aligned} &\liminf_{T\to\infty} \frac{\sum_{t=0}^{T-1} V^{\text{int}^*}(z(t+1)) - V^{\text{int}^*}(z(t))}{T} \\ &\leq \liminf_{T\to\infty} \frac{\sum_{t=0}^{T-1} \ell^{\text{int}}(z_s, v_s) - \ell^{\text{int}}(z(t), v^*(t|t))}{T} \end{aligned} \tag{11}$$

The left hand side is a telescoping series, i.e.,

$$\begin{aligned} &\liminf_{T\to\infty} \frac{\sum_{t=0}^{T-1} V^{\text{int}^*}(z(t+1)) - V^{\text{int}^*}(z(t))}{T} \\ &= \liminf_{T\to\infty} \frac{V^{\text{int}^*}(z(T)) - V^{\text{int}^*}(z(0))}{T} = 0 \end{aligned} \tag{12}$$

where the last equality holds due to compactness of $\mathbb{Z}$ and continuity of ℓ. For the right hand side of (11), we can derive

$$\begin{aligned} &\liminf_{T\to\infty} \frac{\sum_{t=0}^{T-1} \ell^{\text{int}}(z_s, v_s) - \ell^{\text{int}}(z(t), v^*(t|t))}{T} \\ &= \ell^{\text{int}}(z_s, v_s) - \limsup_{T\to\infty} \frac{\sum_{t=0}^{T-1} \ell^{\text{int}}(z(t), v^*(t|t))}{T}. \end{aligned} \tag{13}$$

Combining (12) and (13), we can see that

$$\ell^{\text{int}}(z_s, v_s) \geq \limsup_{T\to\infty} \frac{\sum_{t=0}^{T-1} \ell^{\text{int}}(z(t), v^*(t|t))}{T}.$$

Hence, the asymptotic average performance is no worse than that of the robust optimal steady state. □

Up to now, for ease of presentation, we limited the problem setup to consider terminal equality constraints. However, it is can be shown that a setting with a terminal cost and terminal set can be used as well, analogous to the setting developed by Amrit et al. [2011] for the nominal case.

Remark 6. In the proposed robust economic MPC formulation ((P_{REMPC}) + Algorithm 1), all errors in the RCI set Ω are weighted equally within the integral. Of course, if there is more information about the distribution of the errors available, it would be beneficial to use this information within the optimization. One possible way would be to optimize over the expected value of the cost, such that the distribution is taken into account. This would lead to a formulation similar to the idea in *stochastic MPC* (see, e.g., Cannon et al. [2007]). Of interest in this framework would be to determine the distribution over the RCI set with a known distribution of the disturbance. However, these approaches are beyond the scope of this paper and are content of ongoing research.

3. STABILITY ANALYSIS

In the following, we want to investigate stability (and convergence) of the proposed Robust Economic MPC algorithm.

Since the cost does not need to satisfy any definiteness assumptions, we cannot rely on the stability proof in standard Robust MPC, where the (possibly quadratic) cost is used as a Lyapunov candidate function, similar to the approach in stabilizing MPC (Mayne et al. [2000]). In economic MPC, due to the general structure of the cost, there may exist some closed-loop sequences of the nominal inputs and states providing a better asymptotic average performance for the disturbed system than operation at the robust optimal steady state. On the other hand, for nominal economic MPC, it was shown in Angeli et al. [2012] that convergence of the closed-loop system can be established under a certain dissipativity assumption. We show now that a similar result can be established in our setting with disturbances.

We introduce the admissible set $\overline{\mathbb{Z}}_N$ as the set of all $(z, \tilde{\boldsymbol{v}})$ pairs, with $\tilde{\boldsymbol{v}} = \{\tilde{v}(0), \tilde{v}(1), \dots, \tilde{v}(N-1)\}$, which satisfy all constraints, i.e.,

$$\overline{\mathbb{Z}}_N = \{(z, \tilde{\boldsymbol{v}}) \mid \exists\, z(1), \dots, z(N) : \\ z(k+1) = f(z(k), \tilde{v}(k), 0), (z(k), \tilde{v}(k)) \in \overline{\mathbb{Z}}, \\ \forall k \in \mathbb{I}_{[0,\dots,N-1]}, z(N) = z_s, z(0) = z\}.$$

The set of admissible states $\mathcal{Z}_N$ is given by the projection of $\overline{\mathbb{Z}}_N$ onto $\mathbb{R}^n$, i.e.,

$$\mathcal{Z}_N = \{z \in \mathbb{R}^n | \exists\, \tilde{\boldsymbol{v}} \text{ such that } (z, \tilde{\boldsymbol{v}}) \in \overline{\mathbb{Z}}_N\}.$$

We assume the following form of weak controllability (see Diehl et al. [2011])

Assumption 7. (Weak controllability). There exists a function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ of class $\mathcal{K}_\infty$such that for each $z \in \mathcal{Z}_N$, there exists a $\tilde{\boldsymbol{v}}$ such that $(z, \tilde{\boldsymbol{v}}) \in \overline{\mathbb{Z}}_N$ and

$$\sum_{k=0}^{N-1} |\tilde{v}(k) - v_s| \leq \alpha(|z - z_s|).$$

We use the following definition, based on the definition in Angeli et al. [2012]. To this end, denote by $\overline{\mathbb{X}}$ the projection of $\overline{\mathbb{Z}}$ on $\mathbb{R}^n$.

Definition 8. The nominal system $z(k+1) = f(z(k), v(k), 0)$ is dissipative with respect to the supply rate $s : \overline{\mathbb{Z}} \to \mathbb{R}$ if there exists a storage function $\lambda : \overline{\mathbb{X}} \to \mathbb{R}$ such that

$$\lambda(f(z, v, 0)) - \lambda(z) \leq s(z, v)$$

for all $(z, v) \in \overline{\mathbb{Z}}$. If in addition $\rho : \overline{\mathbb{X}} \to \mathbb{R}_{\geq 0}$ continuous and positive definite [1] exists such that

$$\lambda(f(z, v, 0)) - \lambda(z) \leq s(z, v) - \rho(z)$$

then the nominal system is said to be strictly dissipative.

By means of this definition, we can state the following theorem which is based on the result in Angeli et al. [2012] without disturbances.

Theorem 9. Let system (1) be given together with an RCI set Ω for the associated error dynamics (4), and suppose that Assumption 7 is satisfied. If the nominal system (3) is strictly dissipative with respect to the supply rate

$$s(z, v) = \ell^{\text{int}}(z, v) - \ell^{\text{int}}(z_s, v_s), \tag{14}$$

then, under application of Algorithm 1, $\mathcal{A} := \{z_s\} \times \{z_s\} \oplus \Omega$ is asymptotically stable for the closed-loop composite system (8) and (9) with region of attraction $\mathcal{Z}_N \times (\mathcal{Z}_N \oplus \Omega)$.

Proof. The proof for the asymptotic stability of the nominal system follows along the lines of the proof for the nominal case in Angeli et al. [2012]. We introduce the rotated stage cost

$$L^{\text{int}}(z(k|t), v(k|t)) = \ell^{\text{int}}(z(k|t), v(k|t)) \\ + \lambda(z(k|t)) - \lambda(f(z(k|t), v(k|t), 0)).$$

With this, let us consider the auxiliary objective

$$\tilde{V}^{\text{int}}(z(t), \boldsymbol{v}(t)) = \sum_{k=t}^{N+t-1} L^{\text{int}}(z(k|t), v(k|t)) \\ = V^{\text{int}}(z(t), \boldsymbol{v}(\cdot)) + \lambda(z(t)) - \lambda(z_s).$$

Moreover, we introduce the optimization problem

$$\tilde{V}^{\text{int}^*}(z(t)) = \min_{\boldsymbol{v}(\cdot)} \tilde{V}^{\text{int}}(z(t), \boldsymbol{v}(\cdot))$$

subject to the same constraints as given in (P_{REMPC}). Comparing this optimization problem with (P_{REMPC}) one can see that these two problems are exactly the same except for two constant terms $\lambda(z(t))$ and $\lambda(z_s)$. Hence, solving this optimization problem provides the same solution as solving (P_{REMPC}). The idea within this proof is to use $\tilde{V}^{\text{int}^*}$ as a Lyapunov function. Thus, we first have to show that $\tilde{V}^{\text{int}^*}$ is positive definite with respect the robust optimal steady state (z_s, v_s). Using the fact that the nominal system is strictly dissipative with supply rate given by (14), we can see that

$$L^{\text{int}}(z_s, v_s) = \ell^{\text{int}}(z_s, v_s) \\ \leq \ell^{\text{int}}(z, v) - \lambda(f(z, v, 0)) + \lambda(z) - \rho(z) = L^{\text{int}}(z, v) - \rho(z)$$

for all $(z, v) \in \overline{\mathbb{Z}}$. Without loss of generality, we can assume $\ell^{\text{int}}(z_s, v_s)$ being zero, and thus, as $\rho(\cdot)$ is positive definite with respect to z_s, there exists a class $\mathcal{K}_\infty$ function $\alpha_1(\cdot)$ such that $\tilde{V}^{\text{int}^*}(z(t)) \geq L^{\text{int}}(z(t), v^*(t|t)) \geq \rho(z(t)) \geq \alpha_1(|z(t) - z_s|)$. By means of Assumption 7, one can show that $\tilde{V}^{\text{int}^*}(z(t)) \leq \alpha_2(|z(t) - z_s|)$, where $\alpha_2(\cdot)$ is a class $\mathcal{K}_\infty$ function (see Diehl et al. [2011] for a detailed discussion). Using the dissipativity in Definition 8, we can state

$$\tilde{V}^{\text{int}^*}(z(t+1)) \leq \tilde{V}^{\text{int}^*}(z(t)) + L^{\text{int}}(z_s, v_s) - L^{\text{int}}(z(t), v(t)) \\ \leq \tilde{V}^{\text{int}^*}(z(t)) - \rho(z(t)).$$

From here, it follows that the robust optimal steady state is an asymptotically stable equilibrium of the closed-loop nominal system (8).

Concerning the stability of the composite system (8) and (9), we can follow the proof in [Rawlings and Mayne, 2009, Proposition 3.15]. As $x(t) = z(t) + e(t)$ and $e(t) \in \Omega$ for all $t \in \mathbb{I}_{\geq 0}$, it follows that $|x(t)|_{\{z_s \oplus \Omega\}} \leq |z(t) - z_s| \leq \beta(|z(0) - z_s|, t)$, where β is a class $\mathcal{KL}$ function. Thus,

$$|(z(t), x(t))|_{\mathcal{A}} \leq 2\beta(|z(0) - z_s|, t) \leq 2\beta(|(z(0), x(0))|_{\mathcal{A}}, t),$$

and hence, $\mathcal{A}$ is asymptotically stable for the composite system with region of attraction $\mathcal{Z}_N \times (\mathcal{Z}_N \oplus \Omega)$. $\square$

Remark 10. In order to show that the optimization problems (P_{REMPC}) and the one using L^{int} instead of ℓ^{int} are equivalent, we made use of the fact that $z(t|t)$, and hence also $\lambda(z(t|t))$ is a constant. If $z(t|t)$ was an optimization variable, as is the case in various robust MPC approaches (compare the discussion in Remark 3), the two optimization problems would not be equivalent anymore and hence the result would not follow.

4. MOTIVATING EXAMPLE REVISITED

In this section, we revisit the motivating example,

$$x(k+1) = 0.9x(k) + u(k) + w(k),$$

and assume the same constraints, namely $\mathbb{Z} = \{(x, u) \in \mathbb{R}^2 |\, |x| \leq 10, |u| \leq 1\}$ and $\mathbb{W} = \{w \in \mathbb{R} | |w| \leq 0.1\}$, and we use $K = -0.9$ within the linear control law, such that the minimal RPI set for the associated error system is given by $\Omega = \mathbb{W}$. The prediction horizon is chosen to be $N = 20$. In the following, we consider a different cost function:

$$\ell(x, u) = \begin{cases} 80x^2 & \text{for } x < 0 \\ 0.5(-2x+3)x^2 & \text{for } 0 \leq x < 1 \\ 0.499(-2x+5)(x-1)^2 + 0.5 & \text{for } 1 \leq x < 2 \\ 0.001 & \text{for } 2 \leq x < 3 \\ (x-3)^2 + 0.001 & \text{for } 3 \leq x \end{cases}$$

The cost is independent of the input u. In Figure 2, the cost

[1] A function $\rho(z)$ is said to be positive definite with respect to some z^*, if $\rho(z^*) = 0$ and $\rho(z) > 0$ for all $z \neq z^*$. In the following, when speaking of strict dissipativity, we assume that ρ is positive definite with respect to $z^* = z_s$.

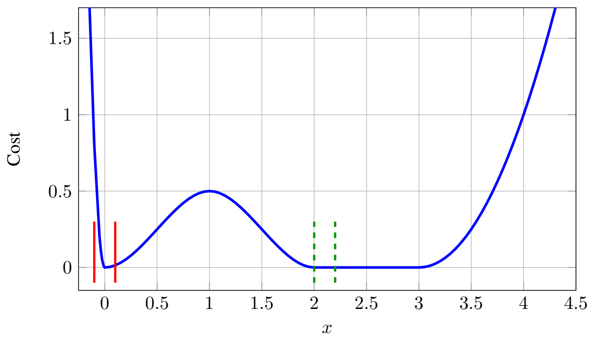

Fig. 2. Cost function of the example in Section 4 with the RPI set Ω centered at the optimal steady state of the nominal system $\bar{z}_s = 0$ (red, solid) and at the robust optimal steady state $z_s = 2.1$ (green, dashed).

function is plotted over x. As in the motivating example, this cost function is positive definite with respect to the origin, i.e., the optimal steady state of the nominal system is $\bar{z}_s = 0$. This example does not have one single robust optimal steady state, but in fact all steady states of the nominal system with $2.1 \leq z_s \leq 2.9$ are robust optimal steady states, with the same associated cost $\ell^{\text{int}}(z_s, v_s)$. The cost $\ell^{\text{int}}(z, v)$ is positive definite in z with respect to the set of robust optimal steady states z_s and constant in v. This means that the nominal system is dissipative with supply rate (14) and $\lambda = 0$ but not strictly dissipative. We choose as a terminal constraint $z(T) = 2.1$ and consider the asymptotic average performance averaged over 10 simulations. This leads to the following asymptotic average performances:

MPC with (P_{class})	MPC with (P_{REMPC})
0.1354	0.001

These numbers provide that when considering the disturbances within the optimization it is on average "cheaper" not to operate the disturbed system at the origin, but rather within an RPI set around a robust optimal steady-state, similar to the motivating example in Section 2.2.

5. CONCLUSION AND OPEN QUESTIONS

We have presented an economic MPC algorithm for disturbance affected systems. Using the key idea of integrating the economic stage cost over the RCI set, robustness against the disturbances is taken into account within the economic MPC optimization. We have shown that - as in the nominal case - the stability result of the closed loop is connected to the concept of dissipativity. We have also presented examples, where the average performance can be improved.

Within this framework, there are several open questions. First, it would be interesting to consider optimal steady-state operations as presented in Angeli et al. [2012] for the nominal case. Second, referring to the concept in Mayne et al. [2005], the initial state of the nominal trajectory could be used as an additional optimization variable. However, as discussed in Remark 10 this gives rise to several difficulties in proving stability. Third, as mentioned in Remark 6, it might be interesting to consider the case when some a priori knowledge on the distribution of the disturbance is available. Finally, using the proposed algorithm in application related examples is also content of our current research.

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