Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

A Simultaneous Stabilization Problem of Linear Systems and their Discretized Models

Zhuo Wang * **and Tongwen Chen** **

* *Department of Electrical and Computer Engineering, University of Alberta, Edmonton, Alberta T6G 2V4, Canada (e-mail: wangzhuo12300@gmail.com).*
** *Department of Electrical and Computer Engineering, University of Alberta, Edmonton, Alberta T6G 2V4, Canada (e-mail: tchen@ualberta.ca).*

Abstract: We present a method to design a feedback controller which stabilizes a continuous-time linear system and its discretized models simultaneously. These discretized models include the Euler approximate discrete-time model and the exact discrete-time model. We present some conditions on solvability of the simultaneous stabilization problem, and a procedure for designing such a controller. We also give three examples to illustrate our research results.

Keywords: Euler approximate discrete-time models; Exact discrete-time models; Simultaneous stabilization; Stabilizing controllers.

1. INTRODUCTION

Digital control, see, e.g., Fadali and Visioli [2009], is an important branch in control theory that uses digital computers to implement controllers. In a digital control system, a continuous-time plant is typically controlled by a discrete-time feedback algorithm. Samplers and hold devices (such as ZOH) provide the interface between the continuous-time and discrete-time parts of the closed-loop system. Since a digital computer has finite precision, causing quantization distortion, see, e.g., Wu *et al.* [2010], Xu *et al.* [2013], in analog-to-digital conversion, extra care is needed to ensure errors in coefficients, analog-to-digital conversion, and digital-to-analog conversion are not producing undesired effects.

Although an analog controller is stabilizing when implemented as an analog controller, it may not provide closed-loop stability when implemented as a digital controller because of pathological sampling or sampling too slow (Babaali and Egerstedt [2004], Middleton and Freudenberg [1995]). One typical way to address this problem is to use a sufficiently small sampling period. However, the hardware used to sample and hold the plant measurements or implement the feedback control algorithm may make it impossible to reduce the sampling period to a level that guarantees acceptable closed-loop performance. In this case, it becomes meaningful to investigate the existence conditions of such stabilizing feedback controllers that can stabilize the original continuous-time system and its discretized models simultaneously, as well as the requirements on the sampling period (Nešić *et al.* [1999], Nešić and Teel [2004, 2006]).

We would like to study this feedback stabilization problem through the following continuous-time linear system:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (t \geq 0), \tag{1}$$

and its Euler approximate discrete-time model

$$x(k+1) = (TA + I)x(k) + TBu(k) \quad (k \geq 0), \tag{2}$$

as well as the exact discrete-time model

$$x(k+1) = e^{AT}x(k) + \int_0^T e^{A\tau}d\tau Bu(k) \quad (k \geq 0), \tag{3}$$

where $x \in R^n$ is the state, $u \in R^m$ is the input, $T > 0$ is the sampling period, and $I \in R^{n\times n}$ is the identity matrix; $A \in R^{n\times n}$ and $B \in R^{n\times m}$ are time-invariant matrices.

Besides, when n is large and A does not have many zero elements, regular methods (such as the undetermined coefficients method, the Jordan canonical form method, and the series summation method) have difficulties to precisely calculate e^{AT} and $\int_0^T e^{A\tau}d\tau$ in the exact discrete-time model (3). Thus in practice, engineers usually use the Euler approximate discrete-time model in (2) instead of the exact discrete-time model in (3) for convenience. However, Nešić and Teel [2004] introduced an example, where they found a feedback gain matrix $K = [\frac{1}{T^3}, \frac{3}{T^2}, \frac{3}{T}]$ that can stabilize the continuous-time linear system (whose closed-loop eigenvalues are $\xi_1 = \xi_2 = \xi_3 = -\frac{1}{T}$) and the Euler approximate discrete-time model (the closed-loop eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = 0$), but cannot stabilize the exact discrete-time model no matter how small the sampling period T is. This example demonstrates that the stabilizing controller for systems (1) and (2) does not naturally stabilize system (3). In this circumstance, we would like to study all the above three systems together for the simultaneous stabilization problem.

In summary, the problems which we will study are listed as follows:

1. Whether there exists a feedback gain matrix $K \in R^{m\times n}$, which can stabilize system (1) and its discrete-time models (2) and (3) simultaneously, or can stabilize two of them at least, e.g., (1) and (2), or (1) and (3)?
2. What are the existence conditions of K? Are there any special requirements on the sampling period T?
3. If K does exist, can we find a method to design such a K?

So far, few research results have been published on how to stabilize these three systems simultaneously. Therefore, it is meaningful to conduct theoretical analysis on the simultaneous stabilization conditions for system (1), and its discrete-time models (2) and (3). Moreover, since controllability and stabilizability are closely related, we will also conduct some analysis on controllability of these three systems.

2. THEORETICAL ANALYSIS ON THE SIMULTANEOUS STABILIZATION CONDITIONS

In this section, we will conduct some theoretical analysis on the simultaneous stabilization conditions.

Theorem 1. If there exists a feedback gain matrix $K \in R^{m\times n}$ that stabilizes system (2), then this K also stabilizes system (1).

Proof. For system (2), let $u(k) = -Kx(k)$, then

$$x(k+1) = [T(A-BK)+I]\,x(k) \quad (k \geq 0). \tag{4}$$

Assume that K can make the eigenvalues of system (4) within the unit circle, such that the corresponding characteristic polynomial is

$$\det[\lambda I - I - T(A-BK)] = (\lambda-\alpha_1)(\lambda-\alpha_2)\cdots(\lambda-\alpha_n),$$

where $|\alpha_i| < 1$ $(1 \leq i \leq n)$ and $|\cdot|$ denotes the absolute value in this paper. Thus, the eigenvalues are $\lambda_i = \alpha_i$ $(1 \leq i \leq n)$, and $|\lambda_i| < 1$. We can use this $K \in R^{m\times n}$ to stabilize system (2).

Let $V_i \in R^n$ denote the eigenvector of $[T(A-BK)+I]$ corresponding to λ_i $(1 \leq i \leq n)$. Then, we have

$$\begin{aligned} &[T(A-BK)+I]\,V_i = \lambda_i V_i, \\ &T(A-BK)V_i = (\lambda_i - 1)V_i, \\ &(A-BK)V_i = \left(\frac{\lambda_i-1}{T}\right)V_i. \end{aligned} \tag{5}$$

When we use the same feedback gain matrix K to control system (1), i.e. $u(t) = -Kx(t)$,

$$\dot{x}(t) = (A-BK)\,x(t) \quad (t \geq 0). \tag{6}$$

By (5), V_i is the eigenvector of $(A-BK)$ corresponding to the eigenvalue $\left(\frac{\lambda_i-1}{T}\right)$. Since $|\lambda_i| < 1$, $\mathrm{Re}\left(\frac{\lambda_i-1}{T}\right) < 0$ $(1 \leq i \leq n)$. Then, the equilibrium point $x = 0$ of system (1) is asymptotically stable, and the system is stabilized by the same feedback gain matrix K. □

Theorem 2. If system (1) is stabilized by a feedback gain matrix $K \in R^{m\times n}$, and if the closed-loop eigenvalues $\xi_i = \sigma_i + j\omega_i$ $(1 \leq i \leq n)$ satisfy

$$\sigma_i < 0, \quad T(\sigma_i^2+\omega_i^2) < -2\sigma_i \quad (1 \leq i \leq n),$$

then this K also stabilizes system (2).

Proof. Let $f_1(\xi)$ and $f_2(\lambda)$ represent the characteristic polynomials of feedback controlled systems (1) and (2), respectively. Then with $u = -Kx$, we have

$$\begin{aligned} f_1(\xi) &= \det[\xi I - (A-BK)], \\ f_2(\lambda) &= \det[\lambda I - I - T(A-BK)] \\ &= T\det\left[\left(\frac{\lambda-1}{T}\right)I - (A-BK)\right]. \end{aligned}$$

Compare $f_1(\xi)$ and $f_2(\lambda)$, if we let $\xi = \left(\frac{\lambda-1}{T}\right)$, then the closed-loop eigenvalues λ_i $(1 \leq i \leq n)$ of feedback controlled system (2) are

$$\lambda_i = 1 + T\xi_i = (1+T\sigma_i) + jT\omega_i \quad (1 \leq i \leq n). \tag{7}$$

With $\sigma_i < 0$ and $T(\sigma_i^2+\omega_i^2) < -2\sigma_i$, we have

$$|\lambda_i| = \sqrt{1 + T\left[T(\sigma_i^2+\omega_i^2)+2\sigma_i\right]} < 1 \quad (1 \leq i \leq n). \tag{8}$$

This implies that all the closed-loop eigenvalues of feedback controlled system (2) are within the unit circle. Then, this K also stabilizes system (2). □

Corollary 1. If systems (1) and (2) are both stabilizable, then they are stabilized by the same feedback gain matrix $K \in R^{m\times n}$ if and only if K makes all the closed-loop eigenvalues $\xi_i = \sigma_i + j\omega_i$ $(1 \leq i \leq n)$ of feedback controlled system (1) satisfy $\sigma_i < 0$, $T(\sigma_i^2+\omega_i^2) < -2\sigma_i$.

Proof. First, we discuss the necessity. Since K stabilizes system (1), then all its closed-loop eigenvalues $\xi_i = \sigma_i + j\omega_i$ $(1 \leq i \leq n)$ should have negative real parts, i.e. $\sigma_i < 0$. Since K also stabilizes system (2), all its closed-loop eigenvalues should stay within the unit circle, i.e. $|\lambda_i[T(A-BK)+I]| < 1$ $(1 \leq i \leq n)$. By (7) and (8), this means $T(\sigma_i^2+\omega_i^2) < -2\sigma_i$ $(1 \leq i \leq n)$. Therefore, the necessity is established.

The sufficiency is implied by Theorem 2. □

Theorem 3. The controllability of system (1) and the controllability of system (2) are equivalent.

Proof. System (1) is completely controllable if and only if $\mathrm{Rank}\left\{\left[B, AB, \ldots, A^{n-1}B\right]\right\} = n$. System (2) is completely controllable if and only if

$$\mathrm{Rank}\left\{\left[TB, (TA+I)TB, \ldots, (TA+I)^{n-1}TB\right]\right\} = n.$$

Since $T > 0$ is a scalar,

$$\begin{aligned} &\mathrm{Rank}\left\{\left[TB, (TA+I)TB, \ldots, (TA+I)^{n-1}TB\right]\right\} \\ &= \mathrm{Rank}\left\{\left[B, (TA+I)B, \ldots, (TA+I)^{n-1}B\right]\right\} \\ &= \mathrm{Rank}\left\{\left[B, TAB, \ldots, (TA)^{n-1}B\right]\right\} \\ &= \mathrm{Rank}\left\{\left[B, \ldots, A^{n-1}B\right]\begin{bmatrix} I_m & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & T^{n-1}I_m \end{bmatrix}\right\}, \end{aligned} \tag{9}$$

where $I_m \in R^{m\times m}$ is the identity matrix. Besides, since $\begin{bmatrix} I_m & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & T^{n-1}I_m \end{bmatrix} \in R^{mn\times mn}$ has full rank,

$$\begin{aligned}&\text{Rank}\left\{\left[TB,(TA+I)TB,\ldots,(TA+I)^{n-1}TB\right]\right\}\\&=\text{Rank}\left\{\left[B,AB,\ldots,A^{n-1}B\right]\right\}.\end{aligned}$$

Therefore, the controllability of system (1) and the controllability of system (2) are equivalent. □

Corollary 2. If either system (1) or system (2) is completely controllable, then there exists a feedback gain matrix $K \in R^{m\times n}$ which stabilizes both systems (1) and (2).

Proof. By Theorem 3, if either system (1) or system (2) is completely controllable, then they are both completely controllable, and hence are both stabilizable.

Thus, we can arbitrarily set the closed-loop eigenvalues of these two systems. If we design a feedback gain matrix $K \in R^{m\times n}$ which can make all the closed-loop eigenvalues $\xi_i = \sigma_i + j\omega_i$ $(1 \le i \le n)$ of feedback controlled system (1) satisfy $\sigma_i < 0$ and $T(\sigma_i^2 + \omega_i^2) < -2\sigma_i$ $(1 \le i \le n)$, then by Corollary 1, K stabilizes both systems (1) and (2). □

Theorem 4. Suppose $A^T = A$. If there exists a feedback gain matrix $K \in R^{m\times n}$ that stabilizes system (3) and if $\left[\int_0^T e^{A\tau}d\tau(A-BK) + (A-BK)^T \int_0^T e^{A\tau}d\tau\right]$ is a negative definite matrix, then K also stabilizes system (1).

Proof. Let $f(A) = \int_0^T e^{A\tau}d\tau$. With $u = -Kx$, feedback controlled system (1) is

$$\dot{x}(t) = (A-BK)x(t), \tag{10}$$

and feedback controlled system (3) is

$$\begin{aligned}x(k+1) &= e^{AT}x(k) - f(A)BKx(k)\\&= [f(A)A+I]\,x(k) - f(A)BKx(k)\\&= [I+f(A)(A-BK)]\,x(k).\end{aligned} \tag{11}$$

Let $D = I + f(A)(A-BK)$. Then, the eigenvalues of D are $\lambda_i(D) = 1 + \lambda_i\left[f(A)(A-BK)\right]$ $(1 \le i \le n)$. Since K stabilizes system (3), $|\lambda_i(D)| < 1$ $(1 \le i \le n)$. Such that $\text{Re}\{\lambda_i[f(A)(A-BK)]\} < 0$ $(1 \le i \le n)$.

Since $A^T = A$, then $f(A) = \int_0^T e^{A\tau}d\tau$ is a symmetric positive definite matrix. For system (10), let $V(x(t)) = x^T(t)f(A)x(t)$. $V(x(t)) \ge 0$ with equality if and only if $x(t) = 0$. Since $\left[f(A)(A-BK) + (A-BK)^T f(A)\right]$ is a negative definite matrix, then

$$\begin{aligned}\dot{V}(x(t)) &= x^T(t)\left[f(A)(A-BK) + (A-BK)^T f(A)\right]x(t)\\&\le 0\end{aligned}$$

with equality if and only if $x(t) = 0$.

Based on the Lyapunov stability theory, the equilibrium point $x = 0$ of (10) is asymptotically stable. So, the same feedback gain matrix K also stabilizes system (1). □

In the above theorems, we provided the existence conditions of the feedback gain matrix $K \in R^{m\times n}$ that can stabilize either systems (1) and (2), or systems (1) and (3). However, the existence conditions of K that stabilizes all these three systems simultaneously are not clear. Besides, we do not know how to design such a K in a general way. In the next theorem, we will show that in some special cases, one can use an explicit method to design K which stabilizes all these three systems simultaneously.

Theorem 5. For systems (1), (2), and (3), if

1) $m \ge n$ and $\text{Rank}[B] = n$;
2) no eigenvalue of A equals $j\dfrac{2l\pi}{T}$, where $l \ne 0$ is an integer;
3) there is a real number $0 < \alpha < \left\|e^{AT}\right\|^{-1}$ (where $\|\cdot\|$ represents the spectral norm of a matrix), such that the eigenvalues of $\bar{A} = A - (1-\alpha)\left(\int_0^T e^{-A\tau}d\tau\right)^{-1}$, $\lambda_i(\bar{A}) = \sigma_i + j\omega_i$ $(1 \le i \le n)$ satisfy $\sigma_i < 0$ and $T\left(\sigma_i^2 + \omega_i^2\right) < -2\sigma_i$ $(1 \le i \le n)$;

then, the feedback gain matrix

$$K(T) = (1-\alpha)B^T\left[BB^T\right]^{-1}\left(\int_0^T e^{-A\tau}d\tau\right)^{-1}$$

stabilizes systems (1), (2), and (3) simultaneously.

Proof. Since $\text{Rank}[B] = n$ and

$$n \ge \text{Rank}\{\left[B,AB,\ldots,A^{n-1}B\right]\} \ge \text{Rank}[B],$$

system (1) is completely controllable. By Theorem 3, system (2) is also completely controllable.

Let $f(A) = \int_0^T e^{A\tau}d\tau$, $g(A) = \int_0^T e^{-A\tau}d\tau$. By Spectral Mapping Theorem, $\sigma[f(A)] = \{f(\lambda):\ \lambda \in \sigma(A)\}$, where $f(\lambda) = \int_0^T e^{\lambda\tau}d\tau = \dfrac{1}{\lambda}\left(e^{\lambda T} - 1\right)$. Then, we can obtain

$$\begin{aligned}\{\text{zeros of } f(\lambda)\} &= \{\lambda:\ e^{\lambda T} = 1,\ \lambda \ne 0\}\\&= \left\{\lambda:\ \lambda = j\frac{2l\pi}{T},\ l \ne 0\right\}.\end{aligned}$$

From Condition 2), we can infer that

$$\{\text{zeros of } f(\lambda)\} \cap \sigma(A) = \emptyset, \quad 0 \notin \sigma[f(A)].$$

Therefore, $f(A)$ is invertible. Similarly, $g(A)$ is invertible. As a consequence, $\text{Rank}\,[f(A)B] = n$ and

$$\text{Rank}\left\{\left[f(A)B, e^{AT}f(A)B, \ldots, \left(e^{AT}\right)^{n-1}f(A)B\right]\right\} = n.$$

Hence, system (3) is completely controllable. Thus, these three systems are all stabilizable. In the following, we will show that the feedback gain matrix $K(T)$ can stabilize all these three systems simultaneously.

For system (1), let $u(t) = -K(T)x(t)$. Then,

$$\dot{x}(t) = [A - BK(T)]\,x(t) \quad (t \ge 0). \tag{12}$$

By Conditions 1) and 3),

$$\begin{aligned}A - BK(T) &= A - B(1-\alpha)B^T\left[BB^T\right]^{-1}g(A)^{-1}\\&= A - (1-\alpha)g(A)^{-1}\\&= \bar{A}.\end{aligned}$$

By Condition 3), the eigenvalues of $[A - BK(T)]$ satisfy

$$\text{Re}\{\lambda_i[A - BK(T)]\} = \text{Re}\{\lambda_i(\bar{A})\} = \sigma_i < 0 \quad (1 \le i \le n).$$

Then, the origin $x = 0$ is asymptotically stable, and system (1) is stabilized by $K(T)$.

For system (3), with $u(k) = -K(T)x(k)$,

$$x(k+1) = \left[e^{AT} - f(A)BK(T)\right]x(k) \quad (k \geq 0). \tag{13}$$

With given conditions, we can obtain

$$\begin{aligned} & e^{AT} - f(A)BK(T) \\ & = e^{AT} - (1-\alpha)f(A)g(A)^{-1} \\ & = e^{AT} - (1-\alpha)f(A)\left[e^{-AT}e^{AT}g(A)\right]^{-1} \\ & = e^{AT} - (1-\alpha)f(A)\left(\int_0^T e^{A(T-\tau)}d\tau\right)^{-1} e^{AT} \\ & = e^{AT} - (1-\alpha)f(A)f(A)^{-1}e^{AT} \\ & = e^{AT} - (1-\alpha)e^{AT} \\ & = \alpha e^{AT}. \end{aligned}$$

From Condition 3), we shall have

$$\begin{aligned} \left\|e^{AT} - f(A)BK(T)\right\| & = \alpha\left\|e^{AT}\right\| \\ & < \left\|e^{AT}\right\|^{-1}\left\|e^{AT}\right\| \\ & = 1. \end{aligned} \tag{14}$$

Then for system (13), for all $k \geq 0$,

$$\left\|x(k+1)\right\| \leq \left\|e^{AT} - f(A)BK(T)\right\|\left\|x(k)\right\| < \left\|x(k)\right\|.$$

So that $\lim_{k\to\infty}\left\|x(k)\right\| = 0$ and $\lim_{k\to\infty} x(k) = 0$. Then, the origin $x = 0$ is asymptotically stable, and system (3) is stabilized by $K(T)$.

On the other hand, with $u(k) = -K(T)x(k)$, feedback controlled system (2) is

$$x(k+1) = \left[I + T\big(A - BK(T)\big)\right]x(k) \quad (k \geq 0). \tag{15}$$

Since $\bar{A} = A - BK(T)$,

$$\lambda_i\left[I + T\big(A - BK(T)\big)\right] = \lambda_i\big(I + T\bar{A}\big) = 1 + T(\sigma_i + j\omega_i).$$

Because $T\left(\sigma_i^2 + \omega_i^2\right) < -2\sigma_i$ and $\sigma_i < 0$,

$$\begin{aligned} \left|\lambda_i\left[I + T\big(A - BK(T)\big)\right]\right| & = \sqrt{(1 + T\sigma_i)^2 + (T\omega_i)^2} \\ & = \sqrt{1 + T\left[T\left(\sigma_i^2 + \omega_i^2\right) + 2\sigma_i\right]} \\ & < 1 \quad (1 \leq i \leq n). \end{aligned}$$

Then, the origin $x = 0$ is asymptotically stable, and system (2) is stabilized by the same $K(T)$. □

Note that, the conditions in Theorem 5 are sufficient for $K(T)$ to exist, but they are not necessary. In addition, since $\alpha \in \left(0, \left\|e^{AT}\right\|^{-1}\right)$, we may choose different values of α, and thus there might be many $K(T)$'s designed using the method introduced in Theorem 5, which can stabilize systems (1), (2), and (3) simultaneously.

3. NUMERICAL EXAMPLES

In this section, we will give three examples to illustrate our theorems. First, for Theorems 1, 2, and 3, as well as Corollaries 1 and 2, we consider the following continuous-time linear system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad (t \geq 0). \tag{16}$$

In this example, we set $T = 1$s. Then, the Euler approximate discrete-time model is

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k) \quad (k \geq 0), \tag{17}$$

and the exact discrete-time model is

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k) \quad (k \geq 0). \tag{18}$$

Then, $\text{Rank}\big\{[B, AB]\big\} = \text{Rank}\big\{[TB, (TA + I)TB]\big\} = 2$. Therefore, systems (16) and (17) are both completely controllable, as stated in Theorem 3. The condition that either system (16) or system (17) is completely controllable in Corollary 2 is satisfied, and the condition that the two systems are both stabilizable in Corollary 1 is also satisfied.

Design $K = [0.5, \ 1]$, and let $u = -Kx$. The closed-loop eigenvalues of feedback controlled system (17) are $\lambda_1 = 0.5 + 0.5j$ and $\lambda_2 = 0.5 - 0.5j$. Then, $|\lambda_i| = 0.7071 < 1$ $(i = 1, 2)$, and system (17) is stabilized by K. The closed-loop eigenvalues of feedback controlled system (16) are $\xi_1 = -0.5 + 0.5j$ and $\xi_2 = -0.5 - 0.5j$. Thus, $\text{Re}(\xi_1) = \text{Re}(\xi_2) = -0.5 < 0$, and system (16) is also stabilized by K, as stated in Theorem 1. Besides, since $T\left[(-0.5)^2 + (\pm 0.5)^2\right] = 0.5 < -2(-0.5) = 1$, the condition that $\sigma_i < 0$ and $T\left(\sigma_i^2 + \omega_i^2\right) < -2\sigma_i$ $(1 \leq i \leq n)$ in Theorem 2 and Corollary 1 is satisfied.

Next, to illustrate Theorem 4, we consider the following continuous-time linear system:

$$\dot{x}(t) = \begin{bmatrix} 0.5 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \quad (t \geq 0). \tag{19}$$

Here, we choose a diagonal matrix A for the sake of simple calculation, since there always exists a similarity transformation to transform a symmetric matrix that is not diagonal into a diagonal one. Therefore, the assumption that A is symmetric in Theorem 4 is satisfied.

Set $T = 1$s. Then, $e^{AT} = \begin{bmatrix} e^{0.5} & 0 \\ 0 & e^{-1} \end{bmatrix}$ and $\int_0^T e^{A\tau}d\tau = \begin{bmatrix} 2\left(e^{0.5} - 1\right) & 0 \\ 0 & \left(1 - e^{-1}\right) \end{bmatrix}$. Such that, the exact discrete-time model is

$$\begin{aligned} & x(k+1) \\ & = \begin{bmatrix} e^{0.5} & 0 \\ 0 & e^{-1} \end{bmatrix} x(k) + \begin{bmatrix} 2\left(e^{0.5} - 1\right) \\ 1 - e^{-1} \end{bmatrix} u(k) \quad (k \geq 0). \end{aligned} \tag{20}$$

We design $K = [0.9, \ 0.4]$, and let $u = -Kx$. Then,

$$\begin{aligned} & \left[\int_0^T e^{A\tau}d\tau(A - BK) + (A - BK)^T \int_0^T e^{A\tau}d\tau\right] \\ & = \begin{bmatrix} -1.038 & -1.0879 \\ -1.0879 & -1.7699 \end{bmatrix}, \end{aligned}$$

which has eigenvalues -2.5517 and -0.2561. The assumption that $\left[\int_0^T e^{A\tau}d\tau(A - BK) + (A - BK)^T \int_0^T e^{A\tau}d\tau\right]$ is a symmetric negative definite matrix, is then satisfied.

The closed-loop eigenvalues of feedback controlled system (20) are $\lambda_1 = 0.8714$ and $\lambda_2 = -0.2753$, which are both within the unit circle; so system (20) is stabilized by K. The closed-loop eigenvalues of feedback controlled system in (19) are $\xi_1 = -0.119 < 0$ and $\xi_2 = -1.681 < 0$; so system (19) is stabilized by the same K, as stated in

Theorem 4.

Finally, to illustrate Theorem 5, we consider the following continuous-time linear system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix} u(t) \quad (t \geq 0). \tag{21}$$

Here, $m = n = 2$, Rank$[B] = 2$, and $\lambda_1(A) = \lambda_2(A) = 0$. Thus, Conditions 1) and 2) of Theorem 5 are satisfied.

We still set $T = 1$s. Then, the Euler approximate discrete-time model is

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix} u(k) \quad (k \geq 0), \tag{22}$$

and the exact discrete-time model is

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.2 & 0.25 \\ 0 & 0.5 \end{bmatrix} u(k) \quad (k \geq 0). \tag{23}$$

Then, $\left\| e^{AT} \right\|^{-1} = 0.618$, and we choose $\alpha = 0.5$. Such that $\bar{A} = A - (1-\alpha)\left(\int_0^T e^{-A\tau} d\tau\right)^{-1} = \begin{bmatrix} -0.5 & 0.75 \\ 0 & -0.5 \end{bmatrix}$, whose eigenvalues are $\lambda_1(\bar{A}) = \lambda_2(\bar{A}) = -0.5$. Since $-0.5 < 0$ and $T\left[(-0.5)^2 + 0\right] = 0.25 < -2(-0.5) = 1$, Condition 3) of Theorem 5 is satisfied.

Using the method introduced in Theorem 5, we shall have

$$\begin{aligned} K(T) &= (1-\alpha)B^T \left[BB^T\right]^{-1} \left(\int_0^T e^{-A\tau} d\tau \right)^{-1} \\ &= (1-0.5) \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2.5 & 1.25 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

The closed-loop eigenvalues of feedback controlled system (21) are $\xi_1 = \xi_2 = -0.5 < 0$. Therefore, the system is stabilized by $K(T)$. $\left|\lambda_i\left[I + T\big(A - BK(T)\big)\right]\right| = 0.5 < 1$ and $\left|\lambda_i\left[e^{AT} - \int_0^T e^{A\tau} d\tau BK(T)\right]\right| = 0.5 < 1$ $(i = 1, 2)$. Therefore, systems (22) and (23) are both stabilized by the same $K(T)$, as stated in Theorem 5.

4. CONCLUSION

In this paper, we studied a simultaneous stabilization problem for continuous-time linear systems and their discretized models. These discretized models include the Euler approximate discrete-time models and the exact discrete-time models. We discussed whether there exists a feedback gain matrix $K \in R^{m \times n}$ that can stabilize all these three systems simultaneously, and studied the related existence conditions. To design K, we presented an explicit method, which, however, has some specific conditions. We then gave three numerical examples to illustrate our research results.

More specifically, in our work, Theorems 1 and 2, and Corollaries 1 and 2 concern the simultaneous stabilization conditions for continuous-time linear systems and their Euler approximate discrete-time models. Theorem 4 studies the simultaneous stabilization conditions for continuous-time linear systems and their exact discrete-time models. In Theorem 5, we propose a method to design K for some special systems. In addition, Theorem 3 shows that continuous-time linear systems and their Euler approximate discrete-time models have the same state controllability.

The contribution of our research is that we provide the general conditions for this simultaneous stabilization problem, which is rarely mentioned in the previous work. However, there is still a disadvantage, that is, we have not found a general method explicit enough to design this K. In the future, we hope to study this problem and obtain some results not so restrictive as in Theorems 4 and 5. Also, we would like to expand the research scope to linear time-varying systems as well as nonlinear systems.

REFERENCES

M. Babaali and M. Egerstedt. Pathwise observability through arithmetic progressions and non-pathological sampling. *Proceedings of the 2004 American Control Conference*, 6: 5824–5828, Boston, MA, USA, June 30–July 2 2004.

M.S. Fadali and A. Visioli. *Digital Control Engineering: Analysis and Design.* Academic Press, Waltham, 2009.

R. Middleton and J. Freudenberg. Non-pathological sampling for generalized sampled-data hold functions. *Automatica*, 31(2): 315–319, Feb. 1995.

D. Nešić, A.R. Teel, and P.V. Kokotović. Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations. *Sys. Contr. Lett.*, 38(4): 259–270, Dec. 1999.

D. Nešić and A.R. Teel. A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models. *IEEE Trans. Autom. Control*, 49(7): 1103–1122, July 2004.

D. Nešić and A.R. Teel. Stabilization of sampled-data nonlinear systems via backstepping on their Euler approximate model. *Automatica*, 42(10): 1801–1808, Oct. 2006.

Y.-D. Wu, C.-M. Lai, C.-C. Lee, and P.-C. Huang. A quantization error minimization method using DDS-DAC for wideband fractional-N frequency synthesizer. *IEEE J. Solid-State Circuits*, 45(11): 2283–2291, Nov. 2010.

X. Xu, Z. Xiong, X. Sheng, J. Wu, and X. Zhu. A new time synchronization method for reducing quantization error accumulation over real-time networks: theory and experiments. *IEEE Trans. Ind. Informat.*, 9(3): 1659–1669, Aug. 2013.