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Structured optimisation dynamics for robust triple mode predictive control

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Abstract:
This paper makes contribution in the area of model based predictive control (MPC) and in particular examines to what extent structured optimisation dynamics can improve regions of attraction, performance and computational burden in the robust case. This paper focuses on the use of structured prediction dynamics within a robust triple mode MPC context and demonstrates that these can simplify the design significantly while retaining good robust feasibility regions and performance. The improvements, with respect to existing algorithms are demonstrated by numerical examples using statistical analysis.

Keywords: Triple mode MPC, robust MPC, region of attraction, generalised function dynamics, demanding optimisation.

1. INTRODUCTION

Model Based Predictive Control (MPC) (Rossiter, 2003), is the general name of different computer control algorithms that use predictions as a basis of forming a control law. MPC has now reached a high level of maturity in academia and is widely used in industry. Nevertheless, there are still many outstanding research issues. There is substantial interest in how to develop algorithms to deal with nonlinearity or uncertainty, in particular because these issues can lead to substantial computation and/or complexity. The main aim of this paper is to contribute to research which reduces complexity while still tackling the robust case, perhaps at some small loss to optimality but critically retaining large regions of attraction. Specifically, the focus is on the potential of structured optimisation dynamics (i.e. Laguerre, Kautz and higher order orthonormal function) which have been shown by (Khan and Rossiter, 2011) to enable enlargement of the region of attraction in the nominal case without (in general) detriment to performance and the online computational burden, that is they enable the use of low numbers of degrees of freedom (d.o.f.) which in turn reduces the complexity of the optimisation.

Several authors have looked at this issue, some well known ones being focussed on multiparametric solutions (Bemporad et al., 2002), fast optimisations (Wang and Boyd, 2010), time varying control laws (Limon et al., 2005), interpolation between two different control strategies, (Rossiter and Ding, 2010) and blocking (Cagienard et al., 2007; Gondhalekar and Imura, 2010); the latter two methods form foundation concepts for parametric methods proposed in (Rossiter et al., 2010) where the key development is that the effective horizon range of the d.o.f (for constraint handling) is far greater than the number of d.o.f. (this is not the case for conventional algorithms such as Generalised Predictive Control (GPC)).

Some of the earlier work for the robust case (Kouvaritakis et al., 2000; Cannon and Kouvaritakis, 2005) is based on ellipsoidal invariance and thus is ill-suited to polyhedral and/or asymmetric constraints. However, it is useful to note that the ellipsoidal approaches indicate (Cannon and Kouvaritakis, 2005) there is no further gain in the ellipsoidal region of attraction when the order of the parameterisation dynamic used to capture the input predictions exceeds the system dimension. These techniques are used in triple mode control (Rossiter et al., 2000, 2001; Imsland et al., 2008) to introduce an additional mode within dual mode MPC; the additional mode is unstructured in the sense that it arises from an optimisation of an ellipsoidal region of attraction, but therefore is optimal (Cannon and Kouvaritakis, 2005) only in the context of symmetric constraints and ellipsoidal invariance. In the case of asymmetric constraints and/or polyhedral constraints it is logical to focus more on pragmatic (computationally tractable) approaches such as arise in the use of structured optimisation dynamics. In the context of dual mode MPC algorithms parameterisations based on Laguerre polynomials and higher order orthonormal functions have been shown to be effective and give good numerical conditioning (Khan and Rossiter, 2013). Specifically, it was shown that in many cases changing the parameterisation allowed substantial improvements in the volume of the region of attraction with little or no detriment and in fact in many cases an improvement to performance. In summary, worthwhile question to ask is: Does the use of structured parameterisation dynamics within robust triple mode MPC yield benefits? A statistical analysis, or comparison of the structured and unstructured parameterisation dynamics within robust triple mode approaches (Cannon and Kouvaritakis, 2005) is presented.

Section 2 will gives a brief outline of dual mode and triple mode MPC structures. Section 3 shows how the structured parameterisations can be used in a robust dual mode approach; this is a straightforward extension to the nominal case already in the literature and forms the basis of the extension to robust triple mode structures in the following section. Numerical examples are in section 5. Finally conclusions and future work are in section 6.

2. BACKGROUND

This section will summarise the background information related to robust MPC and robust triple mode MPC.

2.1 Problem formulation for Robust MPC

Assume discrete time linear parameter varying (LPV) state space models of the form (Pluymers et al., 2005b)

$$x_{k+1} = A_k x_k + B_k u_k \tag{1}$$

with constraints on state $x_k \in R^{n_x}$ and $u_k \in R^{n_u}$

$$L_x x_k + L_u u_k \leq l. \tag{2}$$

The value of (A_k, B_k) are unknown and belong to a polytopic uncertainty class Ω i.e.

$$(A_k, B_k) \in \Omega = Co\{(A_1, B_1), ..., (A_m, B_m)\} \tag{3}$$

where $(A_j, B_j),\ j = 1, ..., m$ are known constant matrices.

Suppose the infinite horizon linear quadratic performance index is given as

$$J_k = \sum_{i=0}^{\infty} x_{k+i|k}^T Q x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k} \tag{4}$$

where $Q = Q^T \succeq 0$ and $R = R^T \succ 0$ are performance matrices. Then the worst case performance index to be minimised is

$$\tilde{J}_k = \max_{(A_{k+i}, B_{k+i}) \in \Omega} J_k \tag{5}$$

subject to system dynamics for prediction

$$x_{k+i+1|k} = A_{k+i|k} x_{k+i|k} + B_{k+i|k} u(k+i|k) \tag{6}$$

In order to guarantee closed loop stability, constraints (2) must be satisfied along predicted trajectories for all possible future model uncertainty; details of how to form robust constraint sets are available in (Pluymers et al., 2005a) so omitted here. The system will be pre-stabilised with a state feedback controller K as was done by (Kouvaritakis et al., 2000; Imsland et al., 2005; Cannon and Kouvaritakis, 2005).

2.2 Robust dual mode MPC structures and ERPC

Consider the autonomous state space model (Cannon and Kouvaritakis, 2005)

$$z_{k+i+1|k} = \psi_{k+i|k} z_{k+i|k}, \quad z_{k|k} = \begin{bmatrix} x_k \\ \underrightarrow{c}_k \end{bmatrix},$$
$$\psi_{k+i|k} \in Co\{\psi_j,\ j = 0, 1 \ldots, m\}, \quad \psi_j = \begin{bmatrix} \Phi_j & B_j D \\ 0 & G_c \end{bmatrix},$$
$$\Phi_j = A_j - B_j K, \quad G_c \in \{G_{c,j},\ j = 0, \ldots, m\},$$
$$x_{k+i|k} = [I\ 0]\, z_{k+i|k}, \quad u_{k+i|k} = [K\ D]\, z_{k+i|k} \tag{7}$$

where $z \in \mathbb{R}^{n_x + n_u n_c}$, $\underrightarrow{c}_k^T = [c_k^T, c_{k+1}^T, \ldots, c_{k+n_c-1}^T]$, D and G_c are variables that are used to optimise the size and the shape of the associated region of attraction. In (Kouvaritakis et al., 2000), $D = E$ and $G_c = I_L$ are known as Efficient Robust Predictive Control (ERPC), where $E = [I_{n_u}\ 0\ \ldots]$ and I_L is a shift matrix.

This approach is improved in (Imsland et al., 2005) by varying parameters in the dynamic feedback law and known as generalised ERPC (GERPC). In (Cannon and Kouvaritakis, 2005), a convex formulation of GERPC is proposed to enlarge the region of attraction using as highly tuned a terminal control as is possible in combination with any other stabilising law. It provides a tuning parameter γ for the size of the region of attraction and online cost trade-off for GERPC.

2.3 Robust triple mode MPC

GERPC can be embedded within the middle and terminal modes of triple mode MPC to enlarge the region of attraction. The prediction dynamics are defined as (Imsland and Rossiter, 2005; Imsland et al., 2008)

$$X_{k+1} = \Psi_k X_k, \quad X_k = \begin{bmatrix} x_k \\ \underrightarrow{f}_k \\ \underrightarrow{c}_k \end{bmatrix}, \tag{8}$$
$$\Psi_k \in Co\{\Psi_j,\ j = 0, 1 \ldots, m\}.$$

where both $\underrightarrow{f}_k$, $\underrightarrow{c}_k$ can be d.o.f. in the predictions and the uncertain description of augmented dynamics is given by (Imsland and Rossiter, 2005; Imsland et al., 2008)

$$\Psi_j = \begin{bmatrix} A_j - B_j K & B_j D & B_j E \\ 0 & G_c & 0 \\ 0 & 0 & I_L \end{bmatrix}. \tag{9}$$

Robust constraint satisfaction by the uncertain predictions can be ensured with inequalities of the form:

$$[L_x - L_u K\ \ L_u D\ \ L_u E]\, X_k \leq l, \quad \forall k \tag{10}$$

3. STRUCTURED OPTIMISATION DYNAMICS FOR ROBUST DUAL MODE MPC

The earlier papers focused on symmetric constraints and thus could use ellipsoids and LMI optimisation methods (Cannon and Kouvaritakis, 2005) to identify the terminal mode in robust dual mode MPC. This paper considers how the design could be extended to asymmetric constraints where those optimisation results are no longer valid. The proposal is to use generalised function parameterisation dynamics as earlier work (Khan and Rossiter, 2013) has indicated that one can select these pragmatically and effectively with a minimum of computation. Specifically, this section explores a more intuitive technique based on predefined structured dynamics to define G_c and D in (7). This section further shows how structured function dynamics are analogous to the additional mode of GERPC based robust triple mode MPC.

The generalised structured function prediction for $3rd$ order dynamics are defined as (Khan and Rossiter, 2011)

$$G_{k+1} = \underbrace{\begin{bmatrix} a_2 & 0 & 0 & 0 & \dots \\ a_2 & a_3 & 0 & 0 & \dots \\ -a_1 a_2 & 1-a_1 a_3 & a_1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{A_G} G_k,$$

$$G_0 = \sqrt{(1-a_1^2)(1-a_2^2)(1-a_3^2)} \begin{bmatrix} 1 \\ 1 \\ -a \\ \vdots \end{bmatrix}. \quad (11)$$

Input predictions are defined using (7) and:

$$c_k = G_k^T \eta, \quad G_{k+1} = A_G G_k, \quad \forall k. \quad (12)$$

An autonomous prediction formulation of (9), using the generalised function dynamic from (12), is defined as: $G_c = A_G^T$ and $D = G_0^T$, where $[A_G, G_0] \in Co\{[A_{G,j}, G_{0,j}],\ j = 0, \dots, m\}$. For the robust case, the predicted cost can be calculated using (12) as:

$$J_G = \max_{(A_{k+i}, B_{k+i}) \in \Omega} [x_k \ \underset{\rightarrow}{c}_k]^T P [x_k \ \underset{\rightarrow}{c}_k] \quad (13)$$

where $P > 0$ satisfies

$$P - \Psi_j{}^T P \Psi_j \geq [I\ 0]^T Q [I\ 0]$$
$$+ [-K\ G_0^T]^T R [-K\ G_0^T], \quad j = 1, ..., m. \quad (14)$$

The matrix P can be efficiently calculated by the SDP

$$\min_P \ tr(P) \quad s.t. \quad (14). \quad (15)$$

Consequently, a robust dual mode algorithm using structured predictions can be summarised as:

Algorithm 3.1. GRMPC

$$\min_{\underset{\rightarrow}{\eta}_k, \underset{\rightarrow}{c}_k} J_G$$
$$s.t. \quad A_S X_k \leq l.$$

Implement $u_k = -K x_k + G_0^T \underset{\rightarrow}{\eta}_k$ to the plant. $A_S X_k \leq l$ is a suitable polyhedral robust control invariant set.

4. ROBUST TRIPLE MODE MPC USING STRUCTURED FUNCTION DYNAMICS

The triple mode approach is under-explored in the literature. It is recognised that a large number of free control moves may be required to get close to the global optimal region of attraction whereas one may not desire this. In triple mode structures (9) an additional mode is introduced into the predicted class to increase the region of attraction; originally this choice was based on the analysis of invariant ellipsoidal sets. However, to form an efficient algorithm, that is with few optimisation variables, it is necessary to make implicit assumptions for the terminal mode and the middle mode (that is $\underset{\rightarrow}{f}_k$) while selecting the initial mode $\underset{\rightarrow}{c}_k$ explicitly using polytopic constraints.

In (Imsland et al., 2008), the triple mode prediction setup is modified in conjunction with GERPC to formulate a robust triple mode MPC algorithm. The proposed algorithm allows a tractable QP-based MPC algorithm for the robust case, it allows a large region of attraction with just a small number of online optimisation variables. However, that work assumed symmetric constraints and hence this paper makes a proposal for possible modifications to deal with non-symmetric constraints. This paper proposes structured function parameterisation to handle non-symmetric constraints and significantly, proposes a method which requires only elementary offline computations; the original work for symmetric constraints required a relatively demanding LMI/BMI optimisation.

It is noted that the middle mode (or signal $\underset{\rightarrow}{f}_k$) of a triple mode law can be defined either implicitly or explicitly; obviously the latter is more computationally efficient and that is the route used in this paper and explained shortly. However that issue is not central to this paper. Rather, the novel proposal here is to use structured parameterisations in both the middle and terminal modes.

Specifically the contribution here is to:

- Use a structured parameterisation in the middle mode and thus remove the need for a complex optimisation to determine G_c, D; moreover those optimisations are not valid for asymmetric constraints anyway.
- Use a structured function parameterisation for the terminal mode. This is logical based on evidence in the literature (Khan and Rossiter, 2013), but has not been tried for triple mode.

4.1 Structured function parameterisations for the middle mode of robust triple mode MPC

The offline problems of ERPC and GERPC can be used to **implicitly** specify the additional mode control moves for LTI and robust triple mode MPC, (Rossiter et al., 2001; Imsland and Rossiter, 2005; Imsland et al., 2008). It is temping to use the first control move of the middle mode control $f_{k+i} = H x_{n_c}$, where $H = -P_{22}^{-1} P_{21}$. The P_{22} and P_{21} are corresponding to an augmented invariant ellipsoid $\mathcal{E}_z = \left(z : z^T Q_z^{-1} z \leq 1\right)$, with size to some degree decided by choice of γ. The $\mathcal{E}_z$ in $z = [x^T \ \ c^T]^T$ can be written as (Rossiter et al., 2001; Imsland et al., 2008):

$$z^T \underbrace{\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}}_{Q_z^{-1}} z \leq 1. \quad (16)$$

This fixed control law ensures robust invariance (Cannon and Kouvaritakis, 2005; Imsland et al., 2008) and could be used as first control move in a terminal mode in lieu of the optimised feedback $u_k = -K x_k$. Hence, define middle mode varying terminal state feedback control law as

$$u_k = (-K + DH) x_k, \quad (17)$$

which has the same (robustly) region of attraction as (G)ERPC, a dual mode robust MPC algorithm can be constructed by the method (Pluymers et al., 2005a) using this state feedback as a terminal control (defining a terminal set and cost).

In this paper, the augmented dynamics are based on a generalised function parameterisation and are defined as:

$$\tilde{\Psi}_j = \begin{bmatrix} A_j - B_j K + DH & B_j E \\ 0 & A_G^T \end{bmatrix} \quad (18)$$

where $c_k = G_0^T \underset{\rightarrow}{\eta}_k$ and $\underset{\rightarrow}{\eta}_{k+1} = A_G^T \underset{\rightarrow}{\eta}_k$. These dynamics should fulfil the constraints,

$$\left[L x_k - LuK + LuDH \ \ LuG_0^T\right] z_k \leq l. \quad (19)$$

The associated worst case performance/constraints can be represented as:

$$\min_{\underrightarrow{\eta}_k}[x_k \ \underrightarrow{\eta}_k]^T \tilde{P}[x_k \ \underrightarrow{\eta}_k]$$
$$s.t. \quad A_S X_k \leq l. \tag{20}$$

While this choice of middle model is simple (minimal off-line computation) and gives good feasible regions, because the choice is fixed (see eqn.17), performance is limited. Consequently, it is logical to add an initial mode which is totally free and is used to improve the performance feasibility trade-off while utilising just a few d.o.f. and thus retaining low on line computational loads.

4.2 Robust triple mode MPC using structured function parameterisation for the terminal mode

The model structure for a triple mode prediction is given in eqn.(9). The previous section discussed the definition of matrices G_c, D used to form vector $\underrightarrow{f}_k$. This section looks at the initial mode, or signal $\underrightarrow{c}_k$ which originally was governed by a shift matrix I_L. Here, this will be replaced by a structured function parameterisation (i.e. Laguerre, Kautz or higher order dynamic function) in order to enlarge the robust region of attraction.

The input perturbations c_k parameterised using structured functions and the predicted cost can be represented in terms of the perturbations $\tilde{\rho}_k$, hence

$$\tilde{J}_G = [x_k \ \underrightarrow{f}_k \ \underrightarrow{\eta}_k]^T P [x_k \ \underrightarrow{f}_k \ \underrightarrow{\eta}_k], \tag{21}$$

with $f_{k+i} = G_i^T \underrightarrow{\eta}_k$, $c_{k+i} = \tilde{G}_i^T \underrightarrow{\tilde{\eta}}_k$, $G_i = A_G G_{i-1}$ and $\tilde{G}_i = \tilde{A}_G \tilde{G}_{i-1}$. It is noted that different structured function dynamics can be used for middle/initial modes.

From (9), the triple mode augmented dynamics can be modified by replacing I_L by $\tilde{A_G}$ where $[\tilde{A_G}, \tilde{G}_0] \in Co\{[\tilde{G}, j, \tilde{G}_{0,j}], \ j = 0, \ldots, m\}$. Robust satisfaction of constraints can be tested with linear inequalities such as:

$$\left[L_x - L_u K \ L_u G_0^T \ L_u \tilde{G}_0^T\right] X_k \leq l. \tag{22}$$

In summary, a triple mode algorithm deploying structured prediction parameterisations is given by:

Algorithm 4.1. GRTMPC

$$\min_{\underrightarrow{f}_k \ \underrightarrow{\eta}_k} \tilde{J}_G$$
$$s.t. \quad A_S X_k \leq l.$$

Remark 1. Recursive feasibility and robust asymptotic stability of GRMPC, GR(E)TMPC and GRTMPC can be proved using conventional Lypanunov methods as in (Imsland et al., 2008).

4.3 Selection of structured function dynamics

The structured function dynamic offers a systematic tool for creating a flexible prediction structure that works well within MPC, the remaining question is how does one best deploy this flexibility to achieve the desired benefits. From equation (11), in structured function parameterisation dynamics there are two main choices within the future input predictions. Firstly, select the order of prediction dynamics that is the number of poles a_i in A_G and secondly, select specific values for the poles a_i.

Higher order structured function dynamics have more flexibility in choosing dynamic parameters to overcome the trade off between the region of attraction and closed loop performance loss. However, there is a commonsense observation that $n_c \geq m$, that is to fully utilise the flexibility in having m poles of generalised function dynamics, one should use at least m d.o.f.. Consequently, where one knows that a given value of n_c is sufficient, it is not recommended to use a higher number of poles. It was shown by Cannon and Kouvaritakis (2005) that in terms of size of ellipsoidal regions of attraction, there is no advantage in choosing $m > n_x$.

The optimal selection of pole locations can be determined using a pareto surface with multiobjective optimisation(Khan and Rossiter, 2013). Nevertheless multi-objective optimisations can been computationally demanding, albeit offline, so although these offer good insight into the trade offs, it is not simplistic; there are easier routes to an effective, albeit slightly suboptimal, design. One can use system information to judge the likely outcome without extensive optimisation, or to infer an 'ideal' prediction structure for given system. This selection is based on the underlying convex hull of the closed loop stable system. The pole location(s) of parameterised dynamics can be selected to be equal to or in vicinity of pole(s) of the optimal closed loop system; often these provide a good starting point. The authors make no claim that this pragmatic selection can be proven in any objective sense, but it is based on observation results from numerous tests using the multi-objective approach.

5. NUMERICAL EXAMPLE

This section will illustrate the efficacy of the structured algorithms using numerical examples and statistical analysis. The comparison will be done using volumes of the regions of attraction and number of inequalities required to describe the robust invariant set.

5.1 Statistical Analysis

The pragmatic selection is analysed in this section using a statistical analysis for structured function parameterisation algorithms (i.e. $1st$ (LOMPC), $2nd$ (KOMPC), $3rd$ (GOMPC $3rd, 4th$ order)). The prime interest is to compare the MCAS (Maximum Controllable Admissible Set) volume of structured dynamic algorithms with GERPC (Cannon and Kouvaritakis, 2005) and ERPC (Kouvaritakis et al., 2000) algorithms. The comparisons are based on 500 random systems with $x \in \mathbb{R}^2$, $x \in \mathbb{R}^3$, and $x \in \mathbb{R}^4$ (total of 1500 systems) using $n_c = n_x$.

Consider single input single output random systems subject to input and state constraints i.e. $-1 \leq u_k \leq 1$ and $[4 \ \ 1.5]^T \leq |x_k|$ (for $x_k \in \mathbb{R}^2$), $[4 \ \ 1.5 \ \ 2]^T \leq |x_k|$ (for $x_k \in \mathbb{R}^3$), $[4 \ 1.5 \ 2 \ 1]^T \leq |x_k|$ (for $x_k \in \mathbb{R}^4$). The tuning parameters are $Q = C^T C$, $R = 1$, $n_c = n_x$ and $\gamma = \infty$. The region of attractions are calculated using the Multi-Parametric Toolbox (MPT) (Kvasnica et al., 2004).

The MCAS volume is analysed using two sample hypothesis tests. The volume of invariant set representing a non-

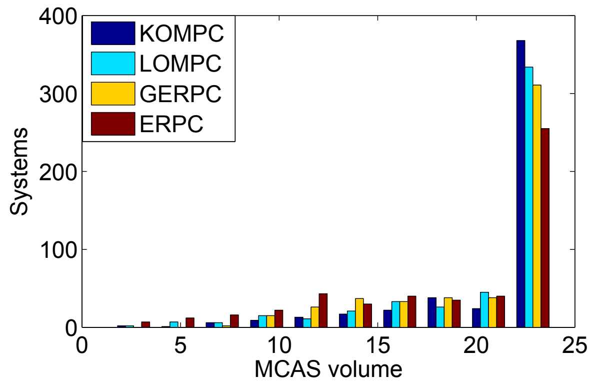

Fig. 1. Histogram comparison of MCAS volume for 2 dimensional Example

normally distributed data is shown in Figure 1. Therefore a non-parametric test can be used to test the null hypothesis (i.e. have the same median) between structured algorithms (i.e. $1st$, $2nd$ and $3rd$ order) and GERPC or alternatively, whether the region of attraction of structured algorithms tend to be larger than GERPC. The Mann-Whitney U test (also called the Wilcoxon rank-sum test) (Mann and Whitney, 1947) is used and results are shown in Table 1.

Table 1 shows the statistical significance level using a P-value; this is the probability that the null hypothesis is true. The null hypothesis is rejected when significance level is less than 5%. The rejection of hypothesis is represented using h-value. If h = 0, it indicates the failure to reject the null hypothesis, whereas h = 1 indicates that the result would be highly unlikely under the null hypothesis.

It is shown that for 2 dimensional random systems, $1st$ and $2nd$ order structured dynamics enlarge the region of attraction as compared to GERPC as for both algorithms the P-value is less than 5% and h = 1. In 3 dimensional examples, the significance level test shows that both 3rd order and 2nd order structured dynamics enlarge the region of attraction compared to GERPC as the P-values are less than 5% and h = 1. However, $1st$ order fail to reject the null hypothesis as significance level is 15% and h = 0. In 4 dimensional systems, the significance level test indicates that for both 4th order and 3rd order fail to reject the null hypothesis test with 72% and 99% significance level. However, $1st$ and $2nd$ order reject the null hypothesis test as GERPC has a larger MCAS volume.

The statistical analysis shown in Table 1 and Figure 1 demonstrate that in many cases structured function dynamics using pragmatic selection enlarge the region of attraction. Table 1 suggests that for some randomly selected systems GERPC may be a better choice than structured function dynamics (e.g. $1st$ order) algorithms using pragmatic selection; this is expected as the pragmatic choice is a suboptimal choice which provides a good starting point to tune the parameterisation dynamics.

5.2 Comparison of robust dual mode and triple mode MPC using structured function dynamics

This section will further illustrate the efficacy of the parameterisation within robust triple mode MPC algorithms by numerical example. The aim is to compare two aspects: (i) the MCAS volume; (ii) the number of inequalities required to describe the robust MCAS. The robust and nominal cases are simulated using both symmetric and non-symmetric constraints. The nominal dynamics are given by $A = 0.5(A_1 + A_2)$ and $B = 0.5(B_1 + B_2)$.

Consider a linear uncertain system representing a double integrator with an uncertainty polytope defined by:

$$A = \begin{bmatrix} 1 & \zeta_1 \\ 0 & 1 \end{bmatrix}; \; B = \begin{bmatrix} 0 \\ \zeta_2 \end{bmatrix}; \zeta_1 = (0.1, 0.2),$$
$$\zeta_2 = (1, 1.5)\; Q = I, R = 0.1, m_c = 2, \gamma = 10^{10}. \quad (23)$$

The system is subject to symmetric constraints

$$-1 \le u_k \le 1; \qquad -10 \le x_{i_k} \le 10;\; i = 1, 2; \quad (24)$$

and non-symmetric constraints

$$-1.5 \le u_k \le 1; \qquad -15 \le x_{i_k} \le 10;\; i = 1, 2; \quad (25)$$

The structured function dynamics i.e. Laguerre ($1st$ order) and Kautz ($2nd$ order) dynamics are selected in the vicinity of closed loop stable pole(s). These dynamics are selected as a combination to define both mode 2 and 3 respectively. Laguerre dynamics are $a = (0.6, 0.7)$ and Kautz dynamics are $(a, b) = ((0.7, 0.1), (0.6, 0.1))$. The middle mode within the robust triple mode MPC is introduced using GERPC (with $\gamma = \infty$), Laguerre or Kautz function dynamics. The selection of structured function dynamics is made with $n_c \le n_x$. The results using both symmetric and non-symmetric constraints are presented in Table 2 and 3. The d.o.f. shown in Table 2 and 3 is the sum of both modes. The regions of attraction are compared using GERPC, Kautz and Laguerre function dynamics for $n_c = 2$ and $n_c = 4$ for uncertain and nominal cases. For both symmetric and non-symmetric constraints, the region of attraction for both Laguerre and Kautz dynamics are larger than GERPC, utilising the different number of inequalities as shown in Table 2 and 3.

Table 1. Mann-Whitney U test of MCAS volume between structured algorithms and GERPC for random systems.

Example	$x_k \in \mathbb{R}^2$		$x_k \in \mathbb{R}^3$		$x_k \in \mathbb{R}^4$	
Algorithms with GERPC	P	h	P	h	P	h
$1st$ order	0.02	1	0.15	0	$2{\times}10^{-8}$	1
$2nd$ order	8×10^{-5}	1	0.03	1	3×10^{-2}	1
$3rd$ order	-	-	2×10^{-8}	1	0.99	0
$4th$ order	-	-	-	-	0.72	0

Table 2. Comparison of MCAS volume and number of inequalities for robust triple mode MPC

	Symmetric Const.			Non-symmetric Const.		
Algorithms	vol.	Ineq.	d.o.f.	vol.	Ineq.	d.o.f.
RTMPC	344.94	227	4	505.44	140	4
($\gamma = \infty$)	307.39	36	2	436.68	68	2
LRTMPC	380.00	136	4	592.50	122	4
($1st$ order)	338.24	88	2	456.22	89	2
KRTMPC	380.00	130	4	592.50	113	4
($2nd$ order)	338.24	24	2	456.22	37	2
R(E)TMPC ($\gamma = \infty$)	152.21	24	2	237.54	23	2
R(L)TMPC ($1st$ order)	359.08	50	2	557.61	50	2
R(K)TMPC ($2nd$ order)	366.48	40	2	565.82	40	2

For completeness, it is important to compare the number of inequalities required to describe the robust MCAS as the complexity of these set descriptions has an impact on the online computational burden, the more inequalities the higher the computational burden in solving the associated QP optimisation (this paper does not discuss issues linked to the exploitation of structure and efficient QP optimisers). The number of inequalities to define the MCAS is compared with the number of d.o.f. in Table 2 and 3.

The structured function dynamic algorithms (i.e. using Laguerre and Kautz function dynamics) enlarge the MCAS volume at the price of an increase in the number of constraints in the online problem using an implicit or an explicit choice of triple mode MPC. The higher order function dynamics seems to reduce the number of inequalities, although this cannot be proved generally.

6. CONCLUSION

The main contribution of this paper is to present the potential advantages of structured functions to **robust** triple mode MPC: (i) Structured functions are embedded into the middle mode of both nominal and robust triple mode MPC. This provides a pragmatic choice to enlarge the region of attraction which simplifies the offline design by removing the need for a demanding optimisation and (ii) The structured function are also used to parameterise the degrees of freedom within the initial mode of triple mode MPC. The examples demonstrate that such parameterisations can enlarge the robust region of attraction but sometimes with an increase in number of inequalities required to describe the corresponding robust MCAS compared to a more conventional robust approach. The structured function dynamics were analysed using statistical analysis and this demonstrated significant volume improvements in general. Consequently, the use of the structured function dynamics within a robust triple mode MPC seems to be beneficial in many cases. Nevertheless, an obvious outstanding issue is to determine a systematic and rigorous selection algorithm for the structured dynamics for robust MPC, but critically using a tractable optimisation scheme.

Table 3. Comparison of MCAS volume and number of inequalities for nominal triple mode MPC

	Symmetric Constraints			Non-symmetric Constraints		
Algorithms	vol.	Ineq.	d.o.f.	vol.	Ineq.	d.o.f.
TMPC ($\gamma = \infty$)	351.34	62	4	593.48	72	4
LTMPC (1*st* order)	380.99	110	4	600.62	32	4
KTMPC (2*nd* order)	385.00	32	4	600.62	28	4
(E)TMPC ($\gamma = \infty$)	152.21	24	2	237.54	23	2
L(E)TMPC (1*st* order)	359.08	50	2	557.61	50	2
K(E)TMPC (2*nd* order)	366.48	40	2	565.82	40	2

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