Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Controllability and Observability Preservation for Networked Systems with Time Varying Topologies

Lorenzo Sabattini * **Cristian Secchi** * **Cesare Fantuzzi** *

* *Department of Sciences and Methods for Engineering (DISMI), University of Modena and Reggio Emilia, Italy*
{lorenzo.sabattini, cristian.secchi, cesare.fantuzzi}@unimore.it

Abstract: When dealing with networked systems, the concept of controllability refers to the possibility of controlling the overall state of a group of agents by providing an external input only to a subset of them, that are referred to as leaders. On the same lines, the concept of observability is related to the possibility of observing the whole state of the networked system from only a subset of the agents. In this paper we introduce a decentralized interaction rule that ensures preservation of the controllability and observability properties of a networked system. The proposed strategy is based on the fact that these properties are strongly related to the topology of the underlying communication graph. We will show that connectivity maintenance and a random choice of the edge–weights are key concepts to ensure controllability and observability preservation for a networked system.

Keywords: Control of networks; Cooperative systems; Multi-agent systems; Networks of sensors and actuators; Networked robotic systems; Distributed control and estimation.

1. INTRODUCTION

This paper introduces a control strategy to ensure the preservation of the controllability and observability properties for a networked system, in a completely decentralized manner.

The idea of interacting with a networked system has been increasingly investigated in the last few years [Liu et al., 2011, Cocetti et al., 2013, Sabattini et al., 2014]. The main idea is that of controlling the overall state of a multi–agent system by directly interacting with only a subset of the agents, which are commonly referred to as the *leaders*. In order to obtain this objective, a communication and interaction network among the agents is exploited: clearly, the topology of the network strongly influences the possibility of obtaining this objective [Egerstedt et al., 2012].

To the best of the authors' knowledge, one of the first attempts to apply the classical concepts of controllability and observability to networked systems can be found in [Tanner, 2004]. In this work, the multi–agent system is partitioned between leaders and followers, and the overall networked system is represented as a standard LTI system: this representation lets the author apply the classical concept of controllability.

Based on these ideas, several strategies for the analysis of the controllability property of networked systems have recently appeared in the literature. Sufficient conditions are derived in [Ji and Egersted, 2007, Rahmani et al., 2009] from a graph–theoretic perspective, that guarantee controllability of a networked system: given the topology of the communication graph, it is possible to understand if the corresponding networked system is controllable from a given set of leaders. On the same lines, a strategy to verify the controllability is presented in [Franceschelli et al., 2010], and is based on decentralized estimation of the spectrum of the Laplacian matrix.

Considering a linear system defined by a generic structure, and a set of variable parameters, the concept of *structural controllability* [Lin, 1974, Shields and Pearson, 1976] refers to LTI systems for which controllability is verified for a particular choice of the entries of the matrices that define the systems themselves. This concept has been recently applied to networked systems [Liu et al., 2009, Sundaram and Hadjicostis, 2013, Sorrentino, 2007, Tan et al., 2010, Zamani and Lin, 2009] interconnected by means of an edge–weighted graph. Specifically, structural controllability of a weighted graph identifies networked systems that can be made controllable with an opportune choice of the edge–weights.

Hence, while the analysis of the controllability property for networked systems has been deeply investigated in the literature, the *synthesis* problem has not been thoroughly addressed yet. For this reason, in this paper we propose a methodology that, in a completely decentralized manner, leads a group of networked agents to create a weighted communication graph in such a way that the overall system is both controllable and observable. The proposed methodology is based on a decentralized strategy first presented in [Sabattini et al., 2013a,b] for the maintenance of the connectivity of the communication graph in a networked system, that will be here exploited for guaranteeing structural controllability. Subsequently we will show that, given a structurally controllable networked system, a random choice of the edge–weights ensures controllability almost surely. Duality principle is invoked to guarantee that, if controllability is ensured, than observability is ensured as well.

The structure of the paper is as follows. Section 2 summarizes background notions on graph theory, that will be exploited throughout the paper. The model of the networked system is described as well, and the concept of structural controllability is introduced. A decentralized strategy for the preservation of the structural controllability property is introduced in Section 3. Then, Section 4 introduces the concept of almost sure controllability, and describes a decentralized methodology to ensure this property. Simulation results are presented in Section 5, for validation purposes. Concluding remarks are then given in Section 6.

2. PRELIMINARIES

2.1 Background on graph theory

In this section we summarize some of the main notions on graph theory used in the paper. Further details can be found for instance in [Godsil and Royle, 2001].

Let $\mathscr{G}$ indicate a generic undirected graph: throughout the paper we will always refer to undirected graphs, unless otherwise specified. Let $\mathsf{V}(\mathscr{G})$ and $\mathsf{E}(\mathscr{G})$ be the vertex set and the edge set of the graph $\mathscr{G}$, respectively. Moreover, let N be the cardinality of $\mathsf{V}(\mathscr{G})$ (i.e. the number of vertices, or nodes, of the graph), and let M be the cardinality of $\mathsf{E}(\mathscr{G})$ (i.e. the number of edges, or link, of the graph). Clearly, $\mathsf{E} \subseteq \mathsf{V} \times \mathsf{V}$.

Let $v_i, v_j \in \mathsf{V}(\mathscr{G})$ be the i–th and the j–th vertices of the graph, respectively. Then, v_i and v_j are neighbors if $(v_i, v_j) \in \mathsf{E}(\mathscr{G})$. Given an undirected graph, $(v_i, v_j) \in \mathsf{E}(\mathscr{G})$ if and only if $(v_j, v_i) \in \mathsf{E}(\mathscr{G})$.

Define then an indexing of the edges of the graph, namely:

$$\mathsf{E} = \{e_1, \ldots, e_M\} \tag{1}$$

Defining then an arbitrary orientation of each edge, the incidence matrix $\mathcal{I}(\mathscr{G}) \in \mathbb{R}^{N\times M}$ can be defined as a matrix whose (i,k)–th element ι_{ik} is [Ji and Egersted, 2007]:

$$\iota_{ik} = \begin{cases} -1 \text{ if } v_i \text{ is the head of } e_k \\ 1 \text{ if } v_i \text{ is the tail of } e_k \\ 0 \text{ otherwise} \end{cases} \tag{2}$$

The (unweighted) Laplacian matrix $\mathcal{L}(\mathscr{G})$ is then defined as follows:

$$\mathcal{L}(\mathscr{G}) = \mathcal{I}(\mathscr{G})\,\mathcal{I}^T(\mathscr{G}) \tag{3}$$

In an edge–weighted graph, a positive number (the weight) is associated to each edge of the graph. Let $w_k > 0$ be the weight associated with the k–th edge, and let $w = [w_1, \ldots, w_M] \in \mathbb{R}^M$. Then, the weight matrix $\mathcal{W}(\mathscr{G}) \in \mathbb{R}^{M\times M}$ is defined as $\mathcal{W}(\mathscr{G}) = \mathrm{diag}(w)$.

The (weighted) Laplacian matrix $\mathcal{L}_{\mathcal{W}}(\mathscr{G})$ of the graph $\mathscr{G}$, associated with the weight matrix $\mathcal{W}(\mathscr{G})$ can then be defined as follows:

$$\mathcal{L}_{\mathcal{W}}(\mathscr{G}) = \mathcal{I}(\mathscr{G})\,\mathcal{W}(\mathscr{G})\,\mathcal{I}^T(\mathscr{G}) \tag{4}$$

The (weighted or unweighted) Laplacian matrix exhibits some remarkable properties:

(1) Let $\mathbf{1}$ be the column vector of all ones. Then, $\mathcal{L}(\mathscr{G})\,\mathbf{1} = \mathbf{0}$.
(2) Let $\lambda_i(\mathcal{L}(\mathscr{G}))$, $i = 1, \ldots, N$ be the eigenvalues of the Laplacian matrix.
    - The eigenvalues can be ordered such that
    $$0 = \lambda_1(\mathcal{L}(\mathscr{G})) \leq \lambda_2(\mathcal{L}(\mathscr{G})) \leq \ldots \leq \lambda_N(\mathcal{L}(\mathscr{G})) \tag{5}$$
    - $\lambda_2(\mathcal{L}(\mathscr{G})) > 0$ if and only if the graph is connected: then, $\lambda_2(\mathcal{L}(\mathscr{G}))$ is defined as the algebraic connectivity of the graph.

2.2 Model of the system

Consider a group of N agents, namely mobile robots, sensors or other entities, interconnected by means of a graph $\mathscr{G}$. Let $x_i \in \mathbb{R}^m$ be the state of the i–th agent: without loss of generality, we will hereafter consider the case where the state corresponds to each agent's position. Then, let the agents be interconnected according to the well known (weighted) consensus protocol [Olfati–Saber et al., 2007]:

$$\dot{x}_i = -\sum_{j\in \mathsf{N}_i} w_{ij}(x_i - x_j) \tag{6}$$

where $w_{ij} > 0$ is the edge weight, and $\mathsf{N}_i \subseteq \mathsf{V}(\mathscr{G})$ is the neighborhood of the i–th agent, defined as the set of the agents that are interconnected to the i–th one, namely:

$$\mathsf{N}_i = \{j \in \mathsf{V}(\mathscr{G}) \text{ such that } (v_i, v_j) \in \mathsf{E}(\mathscr{G})\} \tag{7}$$

Without loss of generality, we will hereafter refer to the scalar case, namely $x_i \in \mathbb{R}$. It is however possible to extend all the results to the multi–dimensional case, considering each component independently.

Hence, let $x = [x_1, \ldots, x_N]^T \in \mathbb{R}^N$ be the state of the multi–agent system. As is well known [Olfati–Saber et al., 2007], the interaction rule defined in Eq. (6) can be rewritten as follows:

$$\dot{x} = -\mathcal{L}_{\mathcal{W}}(\mathscr{G})\,x \tag{8}$$

Under the consensus protocol, the states of the agents converge to a common value. Assume now that the goal is to control the states of the networked agents: for this purpose, define a few leader agents, to whom it is possible to inject a control action. The state of the other agents, referred to as the followers, evolves according to the consensus protocol.

More specifically, let $\mathsf{V}_L(\mathscr{G}) \subset \mathsf{V}(\mathscr{G})$ be the set of the leader agents, and let $\mathsf{V}_F(\mathscr{G}) = \mathsf{V}(\mathscr{G}) - \mathsf{V}_L(\mathscr{G})$ be the set of the follower agents. Then, as shown in [Egerstedt et al., 2012] for unweighted graphs, the interaction rule introduced in Eq. (6) is modified as follows:

$$\begin{cases} \dot{x}_i = -\sum_{j\in \mathsf{N}_i} w_{ij}(x_i - x_j) & \text{if } v_i \in \mathsf{V}_F(\mathscr{G}) \\ x_i = u_i & \text{if } v_i \in \mathsf{V}_L(\mathscr{G}) \end{cases} \tag{9}$$

where $u_i = u_i(t) \in \mathbb{R}$ is a control input. We suppose that both u_i and $\dot{u}_i$ are bounded, namely

$$\exists u_L \in \mathbb{R} \text{ such that } \|u_i\| \leq u_L,\ \|\dot{u}_i\| \leq u_L \ \ \forall i = 1, \ldots, N \tag{10}$$

Let N_L be the number of leaders. It is always possible to index the agents such that the last N_L agents are the leaders, and the first $N_F = N - N_L$ are the followers. Then, as shown in [Egerstedt et al., 2012], it is possible to decompose the Laplacian matrix $\mathcal{L}_{\mathcal{W}}(\mathscr{G})$ as follows:

$$\mathcal{L}_{\mathcal{W}}(\mathscr{G}) = -\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{B}^T & \mathcal{E} \end{array}\right] \tag{11}$$

where $\mathcal{A} = \mathcal{A}^T \in \mathbb{R}^{N_F \times N_F}$ represents the interconnection among the followers, $\mathcal{B} \in \mathbb{R}^{N_F \times N_L}$ represents the interconnection among leaders and followers, and $\mathcal{E} = \mathcal{E}^T \in \mathbb{R}^{N_L \times N_L}$ represents the interconnections among the leaders.

Define now $x_F \in \mathbb{R}^{N_F}$ as the state vector of the followers, namely $x_F = [x_1, \ldots, x_{N_F}]^T$. Define also $u \in \mathbb{R}^{N_L}$ as the input vector, namely $u = [u_{N_F+1}, \ldots, u_N]^T$. Moreover, let $y \in \mathbb{R}^{N_L}$ be the output vector, that is the vector containing the state variables that are measurable by the leaders: it is reasonable to assume that each leader is able to measure the state of its neighbors.

We assume that the leader nodes are able to directly exchange information among each other. Namely, the following assumption is made:

Assumption 1. A complete communication graph exists among the leader nodes.

Under this assumption, each leader can broadcast its measured output to all the other leaders: therefore, it is possible to assume that each leader has access to the entire output vector y. Hence, the dynamics of the networked system can be rewritten as a standard LTI system as follows:

$$\begin{cases} \dot{x}_F &= \mathcal{A}x_F + \mathcal{B}u \\ y &= \mathcal{B}^T x_F \end{cases} \tag{12}$$

Hence, the classical notion of controllability can be applied to solve the problem of controlling the state of the multi–agent system through the leaders. The controllability property is related to the matrices $\mathcal{A}$ and $\mathcal{B}$. Namely, the controllability matrix $\mathcal{R}$ is defined as follows:

$$\mathcal{R} = \left[\mathcal{B} \,|\, \mathcal{A}\mathcal{B} \,|\, \mathcal{A}^2\mathcal{B} \,|\, \ldots \,|\, \mathcal{A}^{N_F-1}\mathcal{B}\right] \tag{13}$$

As is well known from the classical control theory, a LTI system is controllable if and only if its controllability matrix has full rank.

Duality principle can be invoked to show that a networked system is controllable if and only if it is observable. In fact, according to Eq. (12), the observability matrix $\mathcal{O}$ is given by:

$$\mathcal{O} = \mathcal{R}^T \tag{14}$$

A LTI system is controllable if and only if its controllability matrix has full rank, and it is observable if and only if its observability matrix has full rank. According to the definition of the LTI representation of the networked system given in Eq. (12), and as shown in Eq. (14), a networked system is controllable if and only if it is observable.

Therefore, without loss of generality, we will hereafter refer only to the controllability property. However, all the proposed results can be exploited to ensure observability as well.

As stated in the Introduction, several works have recently appeared in the literature to study the influence of the network topology on the controllability property [Egerstedt et al., 2012]. In this paper we will show how to exploit edge weights to ensure the controllability in a decentralized manner. For this purpose, we exploit the concept of *structural controllability*, that has been introduced in [Lin, 1974, Shields and Pearson, 1976, Zamani and Lin, 2009, Lou and Hong, 2012], and that is referred to as *weight controllability* in [Goldin and Raisch, 2013]. The structural controllability property is defined as follows:

Definition 1. (*Structural controllability*).
A networked system is said to be *structurally controllable* if and only if it is controllable for almost any choice of the edge weights.

In [Zamani and Lin, 2009, Lou and Hong, 2012, Goldin and Raisch, 2013] it was shown that structural controllability is strongly related to the connectivity of the underlying graph. Specifically, the following Property was derived:

Property 1. A networked system is structurally controllable if and only if the underlying graph is connected.

3. STRUCTURAL CONTROLLABILITY PRESERVATION

In this section we introduce a decentralized strategy to ensure the preservation of the structural controllability property for the networked system.

The proposed strategy relies on Property 1: specifically, we will exploit a connectivity maintenance control strategy, in order to ensure structural controllability preservation.

Let $\mathcal{C}(\mathscr{G}) \in \mathbb{R}^{M \times M}$ be the communication edge weight matrix. This matrix is defined as $\mathcal{C}(\mathscr{G}) = \text{diag}(c)$, where $c \in \mathbb{R}^M$ will be defined subsequently as a vector of weights related to the communication constraints in the networked system.

Consider then the weighed Laplacian matrix $\mathcal{L}_{\mathcal{C}}(\mathscr{G})$. In [Sabattini et al., 2013a,b], the following kinematic model was considered:

$$\dot{x}_i = u_i^{\mathcal{C}} \tag{15}$$

with the control law $u_i^{\mathcal{C}}$ defined as follows:

$$u_i^{\mathcal{C}} = -\frac{\partial V\left(\lambda_2\left(\mathcal{L}_{\mathcal{C}}(\mathscr{G})\right)\right)}{\partial x_i} \tag{16}$$

where the energy function $V\left(\lambda_2\left(\mathcal{L}_{\mathcal{C}}(\mathscr{G})\right)\right)$ will be defined hereafter. The aim of this control law is to ensure that, given a desired threshold $\epsilon > 0$, then the value of the algebraic connectivity does never go below this threshold, as the system evolves. For this purpose, the energy function $V\left(\lambda_2\left(\mathcal{L}_{\mathcal{C}}(\mathscr{G})\right)\right)$ is chosen according to the following definition.

Definition 2. (*Energy Function*).
A function $V(\lambda_2(\cdot)) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is defined as an *energy function* if the following properties hold:

(P1) It is continuously differentiable $\forall \lambda_2(\cdot) > \epsilon$.
(P2) It is non–negative.
(P3) It is non–increasing with respect to $\lambda_2(\cdot)$, $\forall \lambda_2(\cdot) \geq \epsilon$.
(P4) It approaches a constant value, as $\lambda_2(\cdot)$ increases.
(P5) $\lim_{\lambda_2(\cdot) \mapsto \epsilon} V(\lambda_2(\cdot)) = \infty$

(P6) $\lim_{\lambda_2(\cdot)\mapsto\epsilon} \left\| \frac{\partial V(\lambda_2(\cdot))}{\partial \lambda_2(\cdot)} \right\| = \infty$

In [Sabattini et al., 2013a], the energy function was defined as follows:

$$V(\lambda_2(\mathcal{L}_\mathcal{C}(\mathscr{G}))) = \coth(\lambda_2(\mathcal{L}_\mathcal{C}(\mathscr{G})) - \epsilon) \qquad (17)$$

Let R be the maximum communication range for each agent, i.e. the j–th agent is inside N_i if $\|x_i - x_j\| \le R$. Then, the vector c is defined as the collection of the edge–weights c_{ij}, defined as follows:

$$c_{ij} = \begin{cases} e^{-\left(\|x_i-x_j\|^2\right)/\left(2\kappa^2\right)} & \text{if } \|x_i - x_j\| \le R \\ 0 & \textit{otherwise} \end{cases} \qquad (18)$$

The scalar parameter κ is chosen to satisfy the threshold condition $e^{-\left(R^2\right)/\left(2\kappa^2\right)} = \Delta$, where Δ is a small predefined threshold.

This definition of the edge–weights is motivated by the fact that $\lambda_2(\cdot)$ is a non–increasing function of each edge–weight [Godsil and Royle, 2001]: hence, as two connected robots increase their distance, the value of $\lambda_2(\cdot)$ decreases, until they disconnect.

It is worth noting that, even though the algebraic connectivity of the communication graph is a global quantity, the connectivity maintenance control action can be implemented exploiting an estimate of $\lambda_2(\cdot)$. Specifically, a bounded–error decentralized estimation procedure was introduced in [Sabattini et al., 2013a] for the estimation of $\lambda_2(\cdot)$.

The dynamics of the networked system introduced in Eq. (9) can then be extended as follows:

$$\begin{cases} \dot{x}_i = -\sum_{j\in\mathsf{N}_i} w_{ij}(x_i - x_j) + u_i^\mathcal{C} & \text{if } v_i \in \mathsf{V}_F(\mathscr{G}) \\ \dot{x}_i = \dot{u}_i + u_i^\mathcal{C} & \text{if } v_i \in \mathsf{V}_L(\mathscr{G}) \end{cases} \qquad (19)$$

with the control law $u_i^\mathcal{C}$ defined according to Eq. (16), $\forall v_i \in \mathsf{V}(\mathscr{G})$, and the control law u_i is defined as in Eq. (9).

Theorem 1. Consider the dynamical system described in Eq. (19), and let the networked system be structurally controllable at time $t = 0$. Then, the control strategy in Eq. (16) ensures the preservation of the structural controllability property.

Proof. According to Property 1, if the system is structurally controllable at time $t = 0$, then the underlying communication graph is connected.

Inspired by [Sabattini et al., 2013a,b], we will then show that the control strategy in Eq. (16) guarantees that, if a graph is initially connected, then it will stay connected as the system evolves.

Consider an energy function $V(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ defined according to Definition 2. We will now show that this function does not decrease, as the system evolves.

The dynamical system described in Eq. (19) can be rewritten as follows:

$$\dot{x}_i = u_i^\mathcal{C} + u_i^\mathcal{E} \quad \forall i = 1, \ldots, N \qquad (20)$$

where the external input $u_i^\mathcal{E}$ is bounded, namely

$$\exists u_M \in \mathbb{R} \text{ such that } \left\|u_i^\mathcal{E}\right\| \le u_M, \ \forall i = 1, \ldots, N \qquad (21)$$

It is worth noting that, according to Eq. (10), this constraint holds for the leader inputs, namely u_i, as well.

The time derivative of the energy function can be computed as follows:

$$\dot{V}(\cdot) = \nabla_x V(\cdot)^T \dot{x} = \sum_{i=1}^N \frac{\partial V(\cdot)}{\partial x_i}^T \dot{x}_i \qquad (22)$$

Considering Eqs. (20) and (16), the time derivative of the energy function can be rewritten as follows:

$$\dot{V}(\cdot) = \sum_{i=1}^N \frac{\partial V(\cdot)}{\partial x_i}^T \left(-\frac{\partial V(\cdot)}{\partial x_i} + u_i^\mathcal{E} \right) \qquad (23)$$

According to Eq. (21), and considering that

$$\frac{\partial V(\cdot)}{\partial x_i} = \frac{\partial V(\cdot)}{\partial \lambda_2(\cdot)} \frac{\partial \lambda_2(\cdot)}{\partial x_i}$$

the following inequality can be computed:

$$\dot{V}(\cdot) \le -\left\| \frac{\partial V(\cdot)}{\partial \lambda_2(\cdot)} \right\|^2 \sum_{i=1}^N \left\| \frac{\partial \lambda_2(\cdot)}{\partial x_i} \right\|^2 + \left\| \frac{\partial V(\cdot)}{\partial \lambda_2(\cdot)} \right\| u_M \sum_{i=1}^N \left\| \frac{\partial \lambda_2(\cdot)}{\partial x_i} \right\| \qquad (24)$$

Then, it is possible to conclude that $\dot{V}(\cdot) \le 0$ if the following inequality holds:

$$\left\| \frac{\partial V(\cdot)}{\partial \lambda_2(\cdot)} \right\| \sum_{i=1}^N \left\| \frac{\partial \lambda_2(\cdot)}{\partial x_i} \right\|^2 \ge u_M \sum_{i=1}^N \left\| \frac{\partial \lambda_2(\cdot)}{\partial x_i} \right\| \qquad (25)$$

Assuming that the following condition holds:

$$\sum_{i=1}^N \left\| \frac{\partial \lambda_2(\cdot)}{\partial x_i} \right\|^2 \ne 0 \qquad (26)$$

then the inequality in Eq. (25) can be rewritten as follows:

$$\left\| \frac{\partial V(\cdot)}{\partial \lambda_2(\cdot)} \right\| \ge u_M \frac{\sum_{i=1}^N \left\| \frac{\partial \lambda_2(\cdot)}{\partial x_i} \right\|}{\sum_{i=1}^N \left\| \frac{\partial \lambda_2(\cdot)}{\partial x_i} \right\|^2} < \infty \qquad (27)$$

According to the Property (P6) in Definition 2, $\exists\bar{\lambda} > \epsilon$ such that, $\forall \lambda_2(\cdot) \le \bar{\lambda}$, the inequality in Eq. (27) holds. This implies that the value of $\lambda_2(\cdot)$ is always bounded away from $\epsilon > 0$.

In case the condition in Eq. (26) does not hold, then $\dot{\lambda}_2(\cdot) = 0$. Hence, $\lambda_2(\cdot)$ does not change its value, and then remains greater than $\epsilon > 0$.

It is then possible to conclude that the control strategy in Eq. (16) guarantees that, if a graph is initially connected, then it will stay connected as the system evolves.

According to Property 1, this guarantees also preservation of the structural controllability property. □

It is worth remarking that, according to the Property (P4) in Definition 2, the control action $u_i^\mathcal{C}$ vanishes at steady state, namely when the algebraic connectivity of the graph is sufficiently big. For this reason, we will hereafter consider the dynamics of the system described in Eqs. (9) and (12), assuming that the control action $u_i^\mathcal{C}$ becomes non–negligible only when, due to the relative displacements of the agents, it becomes necessary to enforce the structural controllability property.

4. DECENTRALIZED CONTROLLABILITY OF THE NETWORK

In Section 3 we defined a control strategy that ensures preservation of the structural controllability property. According to Definition 1, given a structurally controllable graph, a choice of edge weights exists such that the corresponding weighted graph is controllable.

Hence, assuming that the graph is structurally controllable, in this section we will define a completely decentralized strategy to define the edge weights in order to ensure the controllability of the graph, while maintaining the graph undirected.

For this purpose, we introduce the following definition of *almost sure controllability*:

Definition 3. (*Almost sure controllability*).
A LTI system is *almost surely controllable* if it is controllable with probability one.

Almost sure controllability can be ensured modifying the control law in Eq. (19) as follows:

$$\begin{cases} \dot{x}_i = -\sum_{j \in \mathsf{N}_i} (w_i + w_j)(x_i - x_j) + u_i^{\mathcal{C}} & \text{if } v_i \in \mathsf{V}_F(\mathscr{G}) \\ \dot{x}_i = \dot{u}_i + u_i^{\mathcal{C}} & \text{if } v_i \in \mathsf{V}_L(\mathscr{G}) \\ w_i \in \mathsf{G},\ w_i > 0 & \forall v_i \in \mathsf{V} \end{cases} \tag{28}$$

Specifically, each agent i computes the Gaussian random variable $w_i \in \mathsf{G}$. Subsequently, the edge weight w_{ij} is computed as

$$w_{ij} = w_i + w_j \tag{29}$$

which can be computed in a decentralized manner by each agent, exploiting only information from the neighbors.

Since, according to Eq. (29), $w_{ij} = w_{ji}$, $\forall v_i, v_j \in \mathsf{V}_F$, then the weighted graph is guaranteed to be undirected.

According to Theorem 1, the control term $u_i^{\mathcal{C}}$ ensures preservation of the structural controllability property. Therefore, assigning randomly chosen edge weights, almost sure controllability is guaranteed.

5. SIMULATIONS

Repeated Matlab simulations were performed, to validate the proposed strategy to ensure controllability of a networked system.

5.1 Controllability of random graphs

The results described in Sections 3 and 4 were validated on random graphs, defined with the following parameters:

- N: randomly chosen between 3 and 15.
- M: randomly chosen between $N-1$ and $\dfrac{N(N-1)}{2}$.

We randomly generated 3000 connected graphs. Then, for each graph:

- We considered the presence of a single leader and, without loss of generality, we considered node N as the leader node.
- We then computed matrices $\mathcal{A}$ and $\mathcal{B}$ from the Laplacian matrix of the graph $\mathcal{L}(\mathscr{G})$, as in Eq. (11).
- We computed the rank of the corresponding controllability matrix.
- When the controllability matrix did not have full rank (i.e. the system was not controllable), then random edge–weights were computed.
- Matrices $\mathcal{A}$ and $\mathcal{B}$ were then extracted again, from the weighted Laplacian matrix $\mathcal{L}_{\mathcal{W}}(\mathscr{G})$.
- We then computed the rank of the corresponding controllability matrix.

The results of the simulations are summarized in Table 1. As expected, introducing random edge–weights always ensures the controllability of the networked system.

Table 1. Simulation results

Number of nodes N	$3 \rightarrow 15$
Number of edges M	$(N-1) \rightarrow \dfrac{N(N-1)}{2}$
Number of generated connected graphs	3000
Number of uncontrollable graphs (unweighted)	1487
Number of uncontrollable graphs with random weights	0

5.2 Preservation of the controllability

In order to validate the results provided in Theorem 1, several Matlab simulations were performed. Considering point agents moving in a two–dimensional environment, the topology of the communication graph was based on the *R–disk model*, namely:

$$\mathsf{N}_i = \{v_j \in \mathsf{V} \text{ such that } \|x_i - x_j\| \leq R\} \tag{30}$$

where $R > 0$ is the communication radius, and $x_i, x_j \in \mathbb{R}^2$.

Hence, the control law in Eq. (19) was applied to a group of N agents, with N varying between $N = 3$ and $N = 15$, and with edge weights w_{ij} defined as in Eqs. (28) and (29). The communication radius was set to $R = 0.5m$, and the initial position of each agent was randomly selected within a circle of radius $3m$, chosen in order to guarantee an initially connected topology. Without loss of generality, the N–th agent was supposed to be the leader, and different control inputs $u_N(t) \in \mathbb{R}^2$ were implemented.

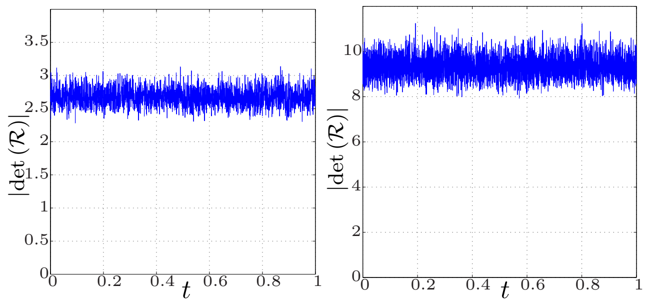
In order to evaluate the performance of the proposed controllability preservation algorithm, the value of the determinant of the controllability matrix $\mathcal{R}$ was monitored. The results of two typical simulation runs are represented in Fig. 1, with the control input $u_N(t) \in \mathbb{R}^2$ defined as follows:

$$u_N(t) = \begin{bmatrix} t \\ \sin(t) \end{bmatrix} \tag{31}$$

As expected, the determinant of the controllablility matrix is always different from zero, which implies controllability of the corresponding networked system.

6. CONCLUSIONS

This paper introduces a decentralized control strategy that ensures preservation of the controllability and observ-



(a) $N = 6$ (b) $N = 10$

Fig. 1. Determinant (absolute value) of the controllability matrix with random weights: simulations implemented with 6 and 10 agents, respectively

ability properties of a networked system. The proposed strategy is based on duality principle, and on the fact that these properties are strongly related to the topology of the underlying communication graph. Specifically, we introduced a decentralized strategy to ensure structural controllability of the networked system. Subsequently, we demonstrated that a random choice of the edge–weights guarantees almost sure controllability. Invoking duality, observability is guaranteed as well.

From an implementation point of view, it is important to remark that this result holds only if it is possible to define the random edge–weights with infinite precision. In fact, as shown in [Sundaram and Hadjicostis, 2013], if the values of the edge–weights are chosen on a finite field F, then the probability of having a controllable graph is related to the size of the field F. A precise characterization of the probability of having a controllable graph will be investigated in future works.

REFERENCES

M. Cocetti, L. Sabattini, C. Secchi, and C. Fantuzzi. Decentralized control strategy for the implementation of cooperative dynamic behaviors in networked systems. In *Proceedings of the IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS)*, pages 5902–5907, Tokyo, Japan, nov 2013.

M. Egerstedt, S. Martini, M. Cao, K. Camlibel, and A. Bicchi. Interacting with networks: How does structure relate to controllability in single-leader, consensus networks? *Control Systems, IEEE*, 32(4):66 –73, aug. 2012.

M. Franceschelli, S. Martini, M. Egerstedt, A. Bicchi, and A. Giua. Observability and controllability verification in multi-agent systems through decentralized laplacian spectrum estimation. In *Decision and Control (CDC), 2010 49th IEEE Conference on*, pages 5775 –5780, dec. 2010.

C. Godsil and G. Royle. *Algebraic Graph Theory*. Springer, 2001.

D. Goldin and J. Raisch. On the weight controllability of consensus algorithms. In *Proceedings of the European Control Conference (ECC)*, pages 233–238, 2013.

M. Ji and M. Egersted. A graph-theoretic characterization of controllability for multi-agent systems. In *American Control Conference, 2007. ACC '07*, pages 4588 –4593, july 2007.

C. T. Lin. Structural controllability. *Automatic Control, IEEE Transactions on*, 19(3):201 – 208, jun 1974.

X. Liu, H. Lin, and B. M. Chen. A graph-theoretic characterization of structural controllability for multi-agent system with switching topology. In *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*, pages 7012 –7017, dec. 2009.

Y. Y. Liu, J. J. Slotine, and A. L. Barabasi. Controllability of complex networks. *Nature*, 473, 2011.

Y. Lou and Y. Hong. Controllability analysis of multi-agent systems with directed and weighted interconnection. *International Journal of Control*, 85(10):1486–1496, 2012.

R. Olfati–Saber, J. A. Fax, and R. M. Murray. Consensus and cooperation in networked multi–agent systems. *Proceedings of the IEEE*, 95(1):215–233, 2007.

A. Rahmani, M. Ji, M. Mesbahi, and M. Egerstedt. Controllability of multi-agent systems from a graph-theoretic perspective. *SIAM Journal on Control and Optimization*, 48(1):162–186, Feb. 2009.

L. Sabattini, N. Chopra, and C. Secchi. Decentralized connectivity maintenance for cooperative control of mobile robotic systems. *The International Journal of Robotics Research (SAGE)*, 32(12):1411–1423, Oct. 2013a.

L. Sabattini, C. Secchi, N. Chopra, and A. Gasparri. Distributed control of multi-robot systems with global connectivity maintenance. *IEEE Transactions on Robotics*, 29(5):1326 – 1332, Oct. 2013b.

L. Sabattini, C. Secchi, M. Cocetti, and C. Fantuzzi. Implementation of arbitrary periodic dynamic behaviors in networked systems. In *Proceedings of the IEEE International Conference on Robotics and Automation (ICRA)*, Hong Kong, China, jun. 2014.

R. Shields and J. Pearson. Structural controllability of multiinput linear systems. *Automatic Control, IEEE Transactions on*, 21(2):203 – 212, apr 1976.

F. Sorrentino. Effects of the network structural properties on its controllability. *Chaos*, 17(3), 2007.

S. Sundaram and C. N. Hadjicostis. Structural controllability and observability of linear systems over finite fields with applications to multi-agent systems. *Automatic Control, IEEE Transactions on*, 58(1):60 –73, jan. 2013.

N. Tan, H. Lin, and Z. Ji. New results on controllability of multi-agent systems. In *Intelligent Control and Automation (WCICA), 2010 8th World Congress on*, pages 574 –579, july 2010.

H. G. Tanner. On the controllability of nearest neighbor interconnections. In *Decision and Control, 2004. CDC. 43rd IEEE Conference on*, volume 3, pages 2467 – 2472 Vol.3, dec. 2004.

M. Zamani and H. Lin. Structural controllability of multi-agent systems. In *American Control Conference, 2009. ACC '09.*, pages 5743 –5748, june 2009.