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# On Linear Estimation Fusion under Unknown Correlations of Estimator Errors

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**Abstract:** The linear fusion of estimators is widely used in decentralised state estimation. Because the maintaining of estimation error cross-correlations between local estimators is not affordable in large-scale problems, approaches dealing with unknown correlations were developed. The Covariance Intersection fusion is a linear fusion of estimators and it provides a fused estimator quality matrix that does not undervalue the mean square error matrix. This paper derives the matrix of the fused estimator quality for arbitrary weights of the linear fusion rule that considers the unknown correlations. It also shows that there can exist better matrices of the fused estimator quality than the ones proposed by the Covariance Intersection fusion rule.

Keywords: Stochastic systems; State estimation; Decentralised fusion; Unknown correlations; Covariance Intersection.

## 1. INTRODUCTION

Estimation deals with inferring the value of a quantity of interest from observed data that are affected by random errors. The elementary requirement of point estimation, Lehmann and Casella [1998], is to provide an estimate that is close to the estimated quantity in some statistical sense. In state estimation of dynamic systems, Simon [2006], Bar-Shalom et al. [2011], the availability of stochastic description of the estimation error is no less important than the error itself.

Using the mean square error criterion for evaluating the mapping from the data to the estimates, that means for evaluating the estimator, the Kalman filter is the optimal linear estimation algorithm. If the measured data are not available in one central processor, decentralised estimation is needed. There are several approaches that fuses local estimators, i.e. that fuses locally processed data. The decentralised Kalman filter, Chong [1979], Hashemipour et al. [1988], provides the same outputs as the hypothetical Kalman filter that fuses all local data directly, but it is the most restrictive one. Besides the local estimators, the filter needs to handle the common prior information and employs an affine fusion of local estimators.

Leaving the requirement to obtain the optimal estimator and contenting with the exact description of the estimator error, the linear fusion can be adopted, Chang et al. [1997], Li et al. [2003], Shin et al. [2007], Yuan et al. [2010]. The optimal linear fusion of local estimators is determined by the correlations of their errors. The mean square error matrix of the fused estimator depends on the correlations as well. However, the maintaining of the knowledge of exact cross-correlations is expensive. It requires an extra communication every time the local estimators are updated by the local measurements. In complex networks of estimators, it also requires the knowledge of the whole network, that cannot be achieved in practice.

Because of that, a further concession is made and the requirement to compute the mean square error matrix exactly is retracted. Instead of the exact value of the mean square error matrix, its upper bound is constructed in the Covariance Intersection fusion, Julier and Uhlmann [1997], Arambel et al. [2001], Chen et al. [2002], Reinhardt et al. [2012] . In order to compute the upper bound, the weights of the linear combination of the local estimators are selected from a set given by a vector parameter. The value of the parameter is chosen according to a predefined criterion. However, it is not the actual mean square error of the fused estimator that is optimised, but it is a measure of the provided upper bound. Fast approaches that approximate the optimal parameter are proposed in Niehsen [2002], Fränken and Hüpper [2005]. Covariance Intersection supposes that the cross-correlations of the local estimator errors are completely unknown. A partial knowledge of the cross-correlation is dealt with in Hanebeck and Briechle [2001], Hanebeck et al. [2001], Julier [2009], a comparison with other approaches is made in Ajgl and Šimandl [2013].

This paper considers the linear combination of the local estimators with the weights that are selected from a wider set than in the Covariance Intersection fusion. The goal of the paper is to cast new light on the fusion under unknown correlations, namely on the choice of the weights and on the optimality of the upper bounds of the mean square error matrices.

In Section 2, the problem of linear fusion under unknown correlations is formulated. Section 3 focuses on fusion of full state estimates, Section 4 dissects partial state estimates. Section 5 summarises the paper. Appendix discusses the parametrisation of matrix upper bounds.

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## 2. PROBLEM SETTINGS

Let $\mathbf{x}$ be the quantity to be estimated and let the quantity be a realisation of an unobservable random variable $\mathcal{X}$ with probability density $p_{\mathcal{X}}$ and support $\Omega_{\mathcal{X}}$. In classical approach, $p_{\mathcal{X}}$ is considered to be an unknown Dirac function, i.e. $\mathbf{x}$ is an unknown deterministic value. In a more general approach, the density $p_{\mathcal{X}}$ is not Dirac and it is considered to be known. Let $\mathcal{Z}$ be an observable random variable with density $p_{\mathcal{Z}|\mathbf{x}}$ for each $\mathbf{x} \in \Omega_{\mathcal{X}}$. The observation $\mathbf{z}$ of $\mathcal{Z}$ is called a measurement.

An estimator $\hat{\mathcal{X}}$ is the random variable given by a function $\hat{x}$ of the random variable $\mathcal{Z}$, $\hat{\mathcal{X}} = \hat{x}(\mathcal{Z})$, its realisation is the estimate $\hat{\mathbf{x}}$, $\hat{\mathbf{x}} = \hat{x}(\mathbf{z})$. An estimator is unbiased, if the expectation of the estimator error is zero, i.e. if $\mathrm{E}\{\mathcal{X} - \hat{x}(\mathcal{Z})\} = 0$ holds. Note that the expectation is unconditional. That means that the estimate needs not be the conditional expectation of $\mathcal{X}$ given by $\mathbf{z}$ and that the conditional expectation of $\hat{x}(\mathcal{Z})$ given by $\mathbf{x}$ need not be equal to $\mathbf{x}$.

The quality of an estimator is often assessed by the mean square error matrix $\mathbb{P}$, $\mathbb{P} = \mathrm{E}\{(\mathcal{X} - \hat{x}(\mathcal{Z}))(\mathcal{X} - \hat{x}(\mathcal{Z}))^{\mathrm{T}}\}$. Again, the expectation is unconditional, i.e. the matrix $\mathbb{P}$ cannot depend on the actual measurement $\mathbf{z}$, and so it has to be possible to compute $\mathbb{P}$ without the knowledge of $\mathbf{z}$.

In fusion, multidimensional random variables $\mathcal{Z}$ are considered and are partitioned into $N$ parts, $\mathcal{Z} = [\mathcal{Z}_1^{\mathrm{T}}, \ldots, \mathcal{Z}_N^{\mathrm{T}}]^{\mathrm{T}}$. Local estimators $\hat{\mathcal{X}}_n$, $n = 1, \ldots, N$, are given by $\hat{\mathcal{X}}_n = \hat{x}_n(\mathcal{Z}_n)$, their quality by the mean square error matrices $\mathbb{P}_n$ and the matrices $\mathbb{P}_{ij}$, $i = 1, \ldots, N$, $j = 1, \ldots, N$, $i \neq j$, are given by $\mathbb{P}_{ij} = \mathrm{E}\{(\mathcal{X} - \hat{x}_i(\mathcal{Z}_i))(\mathcal{X} - \hat{x}_j(\mathcal{Z}_j))^{\mathrm{T}}\}$. The fusion provides an estimator $\hat{\mathcal{X}}_F$, $\hat{\mathcal{X}}_F = \hat{x}_F(\hat{\mathcal{X}}_1, \ldots, \hat{\mathcal{X}}_N)$, its quality is measured by analogically defined matrix $\mathbb{P}_F$.

If the matrices $\mathbb{P}_{ij}$ are not available, it is not possible to compute the matrix $\mathbb{P}_F$. Covariance Intersection, see Julier and Uhlmann [1997], Arambel et al. [2001], Chen et al. [2002], performs a linear fusion of the local estimators $\hat{\mathcal{X}}_n$ and provides an upper bound $\sqcap_F$ of the mean square error matrix $\mathbb{P}_F$, while the upper bound is meant in the sense that the matrix $\sqcap_F - \mathbb{P}_F$ is positive semidefinite. The estimator provided by Covariance Intersection is given by

$$\hat{\mathcal{X}}_F = \sum_{n=1}^{N} \mathbf{W}_n \hat{\mathcal{X}}_n, \tag{1a}$$

where the weights $\mathbf{W}_n$ fulfil $\sum_{n=1}^{N} \mathbf{W}_n = \mathbf{I}$, $\mathbf{I}$ denotes the identity matrix of the corresponding order. For the weights $\mathbf{W}_n$ given by

$$\mathbf{W}_n = (\sum_{m=1}^{N} \omega_m \sqcap_m^{-1})^{-1} \omega_n \sqcap_n^{-1}, \tag{1b}$$

where $\omega_n$, $0 \leq \omega_n \leq 1$, $\sum_{n=1}^{N} \omega_n = 1$, are free parameters and $\sqcap_n$ are known upper bounds of the local mean square error matrices $\mathbb{P}_n$, $\sqcap_n - \mathbb{P}_n \geq 0$, the Covariance Intersection fusion proposes a matrix of the fused estimator quality $\sqcap_F$,

$$\sqcap_F = (\sum_{n=1}^{N} \omega_n \sqcap_n^{-1})^{-1}, \tag{1c}$$

that is an upper bound of $\mathbb{P}_F$ for all admissible values of the free parameters $\omega_n$.

The question posed in this paper is if this upper bound $\sqcap_F$ is optimal for all admissible values of the free parameters $\omega_n$ in (1b). That is if there exist another matrix upper bound which achieves a lower value of some criterion.

## 3. FULL STATE FUSION ANALYSIS

In this section, upper bounds of mean square error matrices in linear fusion are treated. The increase of the upper bounds that is caused by the non-optimal choice of the weights of the linear combination is expressed in a closed form. By the means of an example, it is shown that for the weights given by (1b), the upper bound (1c) needs not be optimal for non-optimal values of the free parameters $\omega_n$.

### *3.1 Expression of the upper bounds*

The analysis starts with the linear fusion of two estimators, $N = 2$. The matrices $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{P}_{12}$ are composed into the matrix $\mathbb{P}_C$ and its upper bound $\sqcap_C$ is composed of the upper bounds $\sqcap_1$ and $\sqcap_2$,

$$\sqcap_C = \begin{bmatrix} \sqcap_1/\omega_1 & \mathbf{0} \\ \mathbf{0} & \sqcap_2/(1-\omega_1) \end{bmatrix}, \quad \mathbb{P}_C = \begin{bmatrix} \mathbb{P}_1 & \mathbb{P}_{12} \\ \mathbb{P}_{12}^{\mathrm{T}} & \mathbb{P}_2 \end{bmatrix}. \tag{2}$$

Note that the parametrisation of the upper bound $\sqcap_C$ by a scalar parameter is discussed in the Appendix.

The requirement $\mathbf{W}_1 + \mathbf{W}_2 = \mathbf{I}$ is made in order to be able to express $\mathbb{P}_F$ by using only $\mathbf{W}_1$, $\mathbf{W}_2$ and $\sqcap_C$. Note that if the weights does not sum up to the identity matrix, then according to its definition, $\sqcap_F$ is dependent on $\mathrm{E}\{\mathcal{X}\mathcal{X}^{\mathrm{T}}\}$ and some cross-terms. However, it cannot be supposed that $\mathrm{E}\{\mathcal{X}\mathcal{X}^{\mathrm{T}}\}$ is available in the decentralised fusion. So, suppose that it holds $\mathbf{W}_2 = \mathbf{I} - \mathbf{W}_1$ and combine the weights into $\mathbf{W}_C$, $\mathbf{W}_C = [\mathbf{W}_1, \mathbf{W}_2]$. Then the mean square error matrix $\mathbb{P}_F$ is obtained by $\mathbb{P}_F = \mathbf{W}_C \mathbb{P}_C \mathbf{W}_C^{\mathrm{T}}$. If the matrix $\sqcap_F$ is constructed as $\sqcap_F = \mathbf{W}_C \sqcap_C \mathbf{W}_C^{\mathrm{T}}$, then it is an upper bound of $\mathbb{P}_F$ if $\sqcap_C$ is an upper bound of $\mathbb{P}_C$, because $\sqcap_F - \mathbb{P}_F = \mathbf{W}_C(\sqcap_C - \mathbb{P}_C)\mathbf{W}_C^{\mathrm{T}}$ holds then, see Hanebeck and Briechle [2001], Hanebeck et al. [2001]. Constructed in such a way, the matrix $\sqcap_F$ is given by

$$\sqcap_F = \mathbf{W}_1 \frac{\sqcap_1}{\omega_1} \mathbf{W}_1^{\mathrm{T}} + (\mathbf{I} - \mathbf{W}_1) \frac{\sqcap_2}{1-\omega_1} (\mathbf{I} - \mathbf{W}_1)^{\mathrm{T}} = \frac{\sqcap_2}{1-\omega_1} + \mathbf{W}_1 (\frac{\sqcap_1}{\omega_1} + \frac{\sqcap_2}{1-\omega_1}) \mathbf{W}_1^{\mathrm{T}} - \mathbf{W}_1 \frac{\sqcap_2}{1-\omega_1} - \frac{\sqcap_2}{1-\omega_1} \mathbf{W}_1^{\mathrm{T}}. \tag{3}$$

*Lemma 1.* The upper bound $\sqcap_F$ can be expressed as

$$\sqcap_F = \mathbf{\Delta} (\frac{\sqcap_1}{\omega_1} + \frac{\sqcap_2}{1-\omega_1}) \mathbf{\Delta}^{\mathrm{T}} + \frac{\sqcap_2}{1-\omega_1} - \frac{\sqcap_2}{1-\omega_1} (\frac{\sqcap_1}{\omega_1} + \frac{\sqcap_2}{1-\omega_1})^{-1} \frac{\sqcap_2}{1-\omega_1}, \tag{4a}$$

$$\mathbf{\Delta} = \mathbf{W}_1 - \frac{\sqcap_2}{1-\omega_1} (\frac{\sqcap_1}{\omega_1} + \frac{\sqcap_2}{1-\omega_1})^{-1}. \tag{4b}$$

*Proof:* Complete (3) to square. □

*Corollary 2.* The upper bound $\sqcap_F$ is also given by

$$\sqcap_F = \mathbf{\Delta} (\frac{\sqcap_1}{\omega_1} + \frac{\sqcap_2}{1-\omega_1}) \mathbf{\Delta}^{\mathrm{T}} + (\omega_1 \sqcap_1^{-1} + (1-\omega_1) \sqcap_2^{-1})^{-1}, \tag{5a}$$

$$\mathbf{\Delta} = \mathbf{W}_1 - (\omega_1 \sqcap_1^{-1} + (1-\omega_1) \sqcap_2^{-1})^{-1} \omega_1 \sqcap_1^{-1}. \tag{5b}$$

*Proof:* Apply the identities (22) and (23) on (4). □

*Remark 3.* The relation between (5) and (1b), (1c) is evident. From (4), (5), it follows that if the parameter $\omega_1$ is fixed, then the weight $\mathbf{W}_1$ that leads to the best upper bound $\mathbb{\Pi}_F$ is given by zeroing $\mathbf{\Delta}$. The increase in the upper bound $\mathbb{\Pi}_F$ if another weight $\mathbf{W}_1$ is chosen is also shown.

*Theorem 4.* If the weight $\mathbf{W}_1$ is given by letting $\mathbf{\Delta}$ to be zero for $\omega_1$ that does not minimise (1c) in a predetermined sense, the best upper bound needs not be given by (1c).

*Proof:* A counterexample is provided in Section 3.2. □

Now, an analysis analogous to the above performed one will be done. Three estimators are considered, $N = 3$, it is supposed that $\mathbf{W}_C = [\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3]$, $\mathbf{W}_C * [\mathbf{I}, \mathbf{I}, \mathbf{I}]^{\mathrm{T}} = \mathbf{I}$. Further, $\mathbb{P}_C$ is constructed analogously to (2) and $\mathbb{\Pi}_C$ has the blocks $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ on its diagonal, where $\mathcal{P}_1 = \mathbb{\Pi}_1/\omega_1$, $\mathcal{P}_2 = \mathbb{\Pi}_2/\omega_2$, $\mathcal{P}_3 = \mathbb{\Pi}_3/\omega_3$ and $\omega_1 + \omega_2 + \omega_3 = 1$. Again, $\mathbb{\Pi}_F$ is constructed as $\mathbb{\Pi}_F = \mathbf{W}_C \mathbb{\Pi}_C \mathbf{W}_C^{\mathrm{T}}$. Then it holds

$$\begin{aligned}\mathbb{\Pi}_F &= \mathbf{W}_1\mathcal{P}_1\mathbf{W}_1^{\mathrm{T}} + \mathbf{W}_2\mathcal{P}_2\mathbf{W}_2^{\mathrm{T}} + \\ &+ (\mathbf{I} - \mathbf{W}_1 - \mathbf{W}_2)\mathcal{P}_3(\mathbf{I} - \mathbf{W}_1 - \mathbf{W}_2)^{\mathrm{T}} = \mathbf{W}_1\mathcal{P}_1\mathbf{W}_1^{\mathrm{T}} + \\ &+ \mathbf{W}_2\mathcal{P}_2\mathbf{W}_2^{\mathrm{T}} + \mathcal{P}_3 + \mathbf{W}_1\mathcal{P}_3\mathbf{W}_1^{\mathrm{T}} + \mathbf{W}_2\mathcal{P}_3\mathbf{W}_2^{\mathrm{T}} - \mathbf{W}_1\mathcal{P}_3 - \\ &- \mathcal{P}_3\mathbf{W}_1^{\mathrm{T}} - \mathbf{W}_2\mathcal{P}_3 - \mathcal{P}_3\mathbf{W}_2^{\mathrm{T}} + \mathbf{W}_1\mathcal{P}_3\mathbf{W}_2^{\mathrm{T}} + \mathbf{W}_2\mathcal{P}_3\mathbf{W}_1^{\mathrm{T}}. \end{aligned} \tag{6}$$

*Lemma 5.* For $N = 3$, the upper bound $\mathbb{\Pi}_F$ is given by

$$\begin{aligned}\mathbb{\Pi}_F = \mathbf{\Delta}\left(\begin{bmatrix}\mathcal{P}_1 & \mathbf{0}\\ \mathbf{0} & \mathcal{P}_2\end{bmatrix} + \begin{bmatrix}\mathbf{I}\\ \mathbf{I}\end{bmatrix}\mathcal{P}_3\,[\mathbf{I}\ \mathbf{I}]\right)\mathbf{\Delta}^{\mathrm{T}} + \mathcal{P}_3 - \\ - \mathcal{P}_3\,[\mathbf{I}\ \mathbf{I}]\left(\begin{bmatrix}\mathcal{P}_1 & \mathbf{0}\\ \mathbf{0} & \mathcal{P}_2\end{bmatrix} + \begin{bmatrix}\mathbf{I}\\ \mathbf{I}\end{bmatrix}\mathcal{P}_3\,[\mathbf{I}\ \mathbf{I}]\right)^{-1}\begin{bmatrix}\mathbf{I}\\ \mathbf{I}\end{bmatrix}\mathcal{P}_3,\end{aligned} \tag{7a}$$

$$\mathbf{\Delta} = [\mathbf{W}_1\ \mathbf{W}_2] - \mathcal{P}_3\,[\mathbf{I}\ \mathbf{I}]\left(\begin{bmatrix}\mathcal{P}_1 & \mathbf{0}\\ \mathbf{0} & \mathcal{P}_2\end{bmatrix} + \begin{bmatrix}\mathbf{I}\\ \mathbf{I}\end{bmatrix}\mathcal{P}_3\,[\mathbf{I}\ \mathbf{I}]\right)^{-1}. \tag{7b}$$

*Proof:* Complete (6) to square. □

*Corollary 6.* For $N = 3$, $\mathbb{\Pi}_F$ can be written as

$$\mathbb{\Pi}_F = \mathbf{\Delta}\begin{bmatrix}\mathcal{P}_1 + \mathcal{P}_3 & \mathcal{P}_3\\ \mathcal{P}_3 & \mathcal{P}_2 + \mathcal{P}_3\end{bmatrix}\mathbf{\Delta}^{\mathrm{T}} + (\mathcal{P}_1^{-1} + \mathcal{P}_2^{-1} + \mathcal{P}_3^{-1})^{-1}, \tag{8a}$$

$$\mathbf{\Delta} = [\mathbf{W}_1\ \mathbf{W}_2] - (\mathcal{P}_1^{-1} + \mathcal{P}_2^{-1} + \mathcal{P}_3^{-1})^{-1}\left[\mathcal{P}_1^{-1}\ \mathcal{P}_2^{-1}\right], \tag{8b}$$

*Proof:* Apply the identities (22) and (23) on (7). □

*Remark 7.* It can be observed that if the parameters $\omega_1$, $\omega_2$ are fixed, then the weights $\mathbf{W}_1$, $\mathbf{W}_2$ that lead to the best upper bound are given by letting $\mathbf{\Delta}$ to be zero. An analysis for $N > 3$ would be analogous and it is not done.

### 3.2 Example – full state estimation

In this example, the dependence of the value of the upper bound $\mathbb{\Pi}_F$ on the values of the weight $\mathbf{W}_1$ and of the parameter $\omega_1$ is inspected numerically.

Suppose that the upper bounds of the mean square error of two local estimators are given as $\mathbb{\Pi}_1 = 1$ and $\mathbb{\Pi}_2 = 2$. Let the linear combination of the estimators (1a) be convex, i.e. $0 \leq \mathbf{W}_1 \leq 1$. The proposed upper bound is given by (3). For the purpose of a comparison, a function of the weight $\mathbf{W}_1$ is considered, namely $\omega_{\mathbf{W}} = \mathbf{W}_1\mathbb{\Pi}_1/(\mathbf{W}_1\mathbb{\Pi}_1 + (1 - \mathbf{W}_1)\mathbb{\Pi}_1)$. Note that the value $\omega_{\mathbf{W}}$ is the value of $\omega_1$ that produces the weight $\mathbf{W}_1$ according to (1b). According to the Covariance Intersection fusion, the proposed upper bound is given by (1c) for the given $\omega_{\mathbf{W}}$.

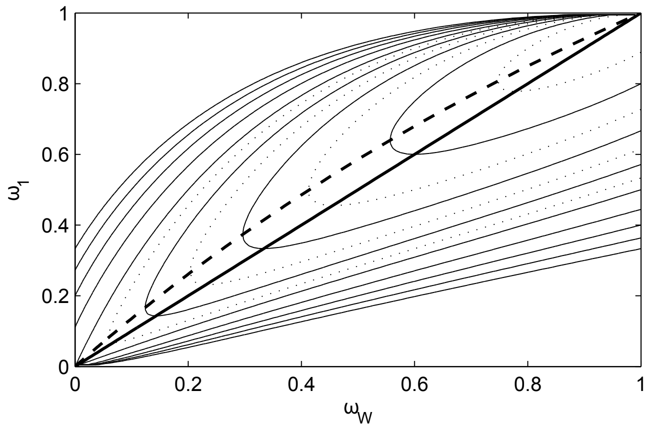


Fig. 1. Contours of $\mathbb{\Pi}_F$ (contours $1.25, 1.5, \ldots, 3$ – solid lines, contours $1.125, 1.375, \ldots, 1.875$ – dotted lines, $\mathbb{\Pi}_F$ achieves its minimum 1 at $\omega_{\mathbf{W}} = 1$, $\omega_1 = 1$). The best choice of $\omega_{\mathbf{W}}$ for given $\omega_1$ (thick solid line), the best choice of $\omega_1$ for given $\omega_{\mathbf{W}}$ (thick dashed line).

This example shows that for fixed weight $\mathbf{W}_1$, i.e. for fixed $\omega_{\mathbf{W}}$, there can exist a better upper bound $\mathbb{\Pi}_F$ than the value obtained by substituting $\omega_1$ by $\omega_{\mathbf{W}}$ in (1c).

Fig. 1 shows the contours of the upper bound (3) for the choices of $\omega_{\mathbf{W}}$ and $\omega_1$. If the parameter $\omega_1$ is fixed first, then the value of $\mathbf{W}_1$ that minimises $\mathbb{\Pi}_F$ is given by (1b), see (5), that means that the value of $\omega_{\mathbf{W}}$ is given by $\omega_{\mathbf{W}} = \omega_1$. However, if the weight $\mathbf{W}_1$ is fixed first, that means if $\omega_{\mathbf{W}}$ is fixed, then the value of $\omega_1$ that minimises $\mathbb{\Pi}_F$ is not given by $\omega_1 = \omega_{\mathbf{W}}$, see the thick dashed line in Fig. 1. The exceptions are the limit cases $\omega_{\mathbf{W}} = 0$, $\omega_{\mathbf{W}} = 1$ and the case when $\omega_{\mathbf{W}}$ minimises (1c). In this example however, (1c) is minimised by $\omega_{\mathbf{W}} = 1$.

Thus, if the Fast Covariance Intersection is used, see Niehsen [2002], Fränken and Hüpper [2005], then $\omega_{\mathbf{W}} = \mathbb{\Pi}_2/(\mathbb{\Pi}_1 + \mathbb{\Pi}_2) = 2/3$ is used in (1). That means that $\mathbf{W}_1 = 0.8$ and (1c) gives an upper bound 1.2. However, the minimal upper bound is approximately equal to 1.17 for $\omega_1$ approximately equal to 0.73. If a consensus is used, i.e. if the value of $\omega_{\mathbf{W}}$ is equal to the inverse of the number of local estimators, $\omega_{\mathbf{W}} = 1/2$, then $\mathbf{W}_1 = 2/3$ and an upper bound 4/3 is suggested. However, the minimal upper bound is approximately equal to 1.23 for $\omega_1 = 2 - \sqrt{2}$.

## 4. PARTIAL STATE FUSION ANALYSIS

In this section, it is supposed that the local estimators do not estimate $\mathcal{X}$, but only linear combinations of its components. That means suppose that $\hat{\mathcal{X}}_n$ estimates $\mathbf{T}_n\mathcal{X}$, where $\mathbf{T}_n$ is a matrix. This settings is usual in large-scale systems, Kim et al. [2007], Khan and Moura [2008], Maestre et al. [2010], where the state is too big to be estimated in one stroke. Further, suppose that the matrix $\mathbf{T}_C$, $\mathbf{T}_C = [\mathbf{T}_1^{\mathrm{T}}, \ldots, \mathbf{T}_N^{\mathrm{T}}]^{\mathrm{T}}$ has full column rank. Roughly speaking, suppose that each component of $\mathcal{X}$ is estimated by at least one estimator. Next, suppose that the local upper bounds $\mathbb{\Pi}_n$ of the mean square error matrices are known and finite. According to Arambel et al. [2001], Julier [2009], the Covariance Intersection fusion now proposes weights $\mathbf{W}_n$ that are given by free parameters $\omega_n$,

$0 \leq \omega_n \leq 1$, $\sum_{n=1}^{N} \omega_n = 1$, and fulfil $\sum_{n=1}^{N} \mathbf{W}_n \mathbf{T}_n = \mathbf{I}$. Namely, the weights are given by

$$\mathbf{W}_n = (\sum_{m=1}^{N} \omega_m \mathbf{T}_m^{\mathrm{T}} \Pi_m^{-1} \mathbf{T}_m)^{-1} \omega_n \mathbf{T}_n^{\mathrm{T}} \Pi_n^{-1}, \tag{9a}$$

and the proposed upper bound $\Pi_F$ is given by

$$\Pi_F = (\sum_{n=1}^{N} \omega_n \mathbf{T}_n^{\mathrm{T}} \Pi_n^{-1} \mathbf{T}_n)^{-1}. \tag{9b}$$

In the following section, the analysis that has been made in Section 3 is extended to partial state fusion. For simplicity, two estimators are considered, $N = 2$.

### 4.1 Expression of the upper bounds

The analysis starts with the assumption $\mathbf{T}_2 = \mathbf{I}$. Now, the off-diagonal blocks of (2) are not square. Similarly to (3), the proposed upper bound $\Pi_F$ is given by

$$\Pi_F = \mathbf{W}_1 \frac{\Pi_1}{\omega_1} \mathbf{W}_1^{\mathrm{T}} + (\mathbf{I} - \mathbf{W}_1 \mathbf{T}_1) \frac{\Pi_2}{1-\omega_1} (\mathbf{I} - \mathbf{W}_1 \mathbf{T}_1)^{\mathrm{T}}. \tag{10}$$

*Lemma 8.* The upper bound $\Pi_F$ can be expressed as

$$\begin{aligned} \Pi_F =& \boldsymbol{\Delta}(\frac{\Pi_1}{\omega_1} + \mathbf{T}_1 \frac{\Pi_2}{1-\omega_1} \mathbf{T}_1^{\mathrm{T}}) \boldsymbol{\Delta}^{\mathrm{T}} + \frac{\Pi_2}{1-\omega_1} - \qquad (11a) \\ & - \frac{\Pi_2}{1-\omega_1} \mathbf{T}_1^{\mathrm{T}} (\frac{\Pi_1}{\omega_1} + \mathbf{T}_1 \frac{\Pi_2}{1-\omega_1} \mathbf{T}_1^{\mathrm{T}})^{-1} \mathbf{T}_1 \frac{\Pi_2}{1-\omega_1}, \\ \boldsymbol{\Delta} =& \mathbf{W}_1 - \frac{\Pi_2}{1-\omega_1} \mathbf{T}_1^{\mathrm{T}} (\frac{\Pi_1}{\omega_1} + \mathbf{T}_1 \frac{\Pi_2}{1-\omega_1} \mathbf{T}_1^{\mathrm{T}})^{-1}. \qquad (11b) \end{aligned}$$

*Proof:* Complete (10) to square. □

*Corollary 9.* The upper bound $\Pi_F$ is also given by

$$\begin{aligned} \Pi_F =& \boldsymbol{\Delta}(\frac{\Pi_1}{\omega_1} + \mathbf{T}_1 \frac{\Pi_2}{1-\omega_1} \mathbf{T}_1^{\mathrm{T}}) \boldsymbol{\Delta}^{\mathrm{T}} + \\ & + (\mathbf{T}_1^{\mathrm{T}} \omega_1 \Pi_1^{-1} \mathbf{T}_1 + (1-\omega_1) \Pi_2^{-1})^{-1}, \qquad (12a) \\ \boldsymbol{\Delta} =& \mathbf{W}_1 - (\mathbf{T}_1^{\mathrm{T}} \omega_1 \Pi_1^{-1} \mathbf{T}_1 + (1-\omega_1) \Pi_2^{-1})^{-1} \mathbf{T}_1^{\mathrm{T}} \omega_1 \Pi_1^{-1}. \qquad (12b) \end{aligned}$$

*Proof:* Apply the identities (22) and (23) on (11). □

*Remark 10.* Again, if $\omega_1$ is fixed, the weight $\mathbf{W}_1$ that leads to the best upper bound $\Pi_F$ is obtained by letting $\boldsymbol{\Delta}$ to be zero. Also, the relation between (12) and (9a), (9b) is evident. The choice of $\omega_1$ for a fixed $\mathbf{W}_1$ is discussed in Section 4.2.

Now, suppose that $\mathbf{T}_2 \neq \mathbf{I}$. For simplicity, suppose also that $\mathbf{T}_1$, $\mathbf{T}_2$ choose components of $\mathcal{X}$ according to

$$\mathbf{T}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \mathbf{T}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{13}$$

where the second block row of $\mathbf{T}_1$ is the same as the first block row of $\mathbf{T}_2$. Next, partition the matrices $\Pi_1/\omega_1$, $\Pi_2/(1-\omega_1)$ accordingly,

$$\frac{\Pi_1}{\omega_1} = \begin{bmatrix} \mathcal{P}_{\mathbf{A}} & \mathcal{P}_{\mathbf{B}} \\ \mathcal{P}_{\mathbf{C}} & \mathcal{P}_{\mathbf{D}} \end{bmatrix}, \quad \frac{\Pi_2}{1-\omega_1} = \begin{bmatrix} \mathcal{P}_{\mathbf{E}} & \mathcal{P}_{\mathbf{F}} \\ \mathcal{P}_{\mathbf{G}} & \mathcal{P}_{\mathbf{H}} \end{bmatrix}. \tag{14}$$

From the definition of $\mathbf{T}_1$ and $\mathbf{T}_2$ and the requirement $\mathbf{W}_1 \mathbf{T}_1 + \mathbf{W}_2 \mathbf{T}_2 = \mathbf{I}$, it follows that $\mathbf{W}_1$, $\mathbf{W}_2$ have the following form,

$$\mathbf{W}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{W}_{\mathbf{A}} \\ \mathbf{0} & \mathbf{W}_{\mathbf{B}} \\ \mathbf{0} & \mathbf{W}_{\mathbf{C}} \end{bmatrix}, \quad \mathbf{W}_2 = \begin{bmatrix} -\mathbf{W}_{\mathbf{A}} & \mathbf{0} \\ \mathbf{I} - \mathbf{W}_{\mathbf{B}} & \mathbf{0} \\ -\mathbf{W}_{\mathbf{C}} & \mathbf{I} \end{bmatrix}, \tag{15}$$

where $\mathbf{W}_{\mathbf{A}}$, $\mathbf{W}_{\mathbf{B}}$, $\mathbf{W}_{\mathbf{C}}$ have corresponding dimensions.

*Proposition 11.* If the upper bound $\Pi_F$ is constructed as before, the completion to square leads to

$$\begin{aligned} \Pi_F =& \boldsymbol{\Delta}(\mathcal{P}_{\mathbf{D}} + \mathcal{P}_{\mathbf{E}}) \boldsymbol{\Delta}^{\mathrm{T}} + \\ & + \begin{bmatrix} \mathcal{P}_{\mathbf{A}} - \mathcal{P}_{\mathbf{B}} \mathbf{Y}_{\mathbf{DE}} \mathcal{P}_{\mathbf{C}} & \mathcal{P}_{\mathbf{B}} \mathbf{Y}_{\mathbf{DE}} \mathcal{P}_{\mathbf{E}} & \mathcal{P}_{\mathbf{B}} \mathbf{Y}_{\mathbf{DE}} \mathcal{P}_{\mathbf{F}} \\ \mathcal{P}_{\mathbf{E}} \mathbf{Y}_{\mathbf{DE}} \mathcal{P}_{\mathbf{C}} & \mathcal{P}_{\mathbf{D}} \mathbf{Y}_{\mathbf{DE}} \mathcal{P}_{\mathbf{E}} & \mathcal{P}_{\mathbf{D}} \mathbf{Y}_{\mathbf{DE}} \mathcal{P}_{\mathbf{F}} \\ \mathcal{P}_{\mathbf{G}} \mathbf{Y}_{\mathbf{DE}} \mathcal{P}_{\mathbf{C}} & \mathcal{P}_{\mathbf{G}} \mathbf{Y}_{\mathbf{DE}} \mathcal{P}_{\mathbf{D}} & \mathcal{P}_{\mathbf{H}} - \mathcal{P}_{\mathbf{G}} \mathbf{Y}_{\mathbf{DE}} \mathcal{P}_{\mathbf{F}} \end{bmatrix}, \end{aligned} \tag{16a}$$

$$\boldsymbol{\Delta} = \begin{bmatrix} \mathbf{W}_{\mathbf{A}} + \mathcal{P}_{\mathbf{B}} (\mathcal{P}_{\mathbf{D}} + \mathcal{P}_{\mathbf{E}})^{-1} \\ \mathbf{W}_{\mathbf{B}} - \mathcal{P}_{\mathbf{E}} (\mathcal{P}_{\mathbf{D}} + \mathcal{P}_{\mathbf{E}})^{-1} \\ \mathbf{W}_{\mathbf{C}} - \mathcal{P}_{\mathbf{G}} (\mathcal{P}_{\mathbf{D}} + \mathcal{P}_{\mathbf{E}})^{-1} \end{bmatrix}, \tag{16b}$$

where the shorthand notation $\mathbf{Y}_{\mathbf{DE}} = (\mathcal{P}_{\mathbf{D}} + \mathcal{P}_{\mathbf{E}})^{-1}$ has been used in the right hand side matrix in (16a).

Now, the identities (22) and (23) have to be used multiple times in order to arrive at (9a), (9b). Using the notation $\mathbf{J} = (\mathcal{P}_{\mathbf{A}} - \mathcal{P}_{\mathbf{B}} \mathcal{P}_{\mathbf{D}}^{-1} \mathcal{P}_{\mathbf{C}})^{-1}$, $\mathbf{L} = (\mathcal{P}_{\mathbf{H}} - \mathcal{P}_{\mathbf{G}} \mathcal{P}_{\mathbf{E}}^{-1} \mathcal{P}_{\mathbf{F}})^{-1}$, $\mathbf{K} = \mathcal{P}_{\mathbf{D}}^{-1} + \mathcal{P}_{\mathbf{D}}^{-1} \mathcal{P}_{\mathbf{C}} \mathbf{J} \mathcal{P}_{\mathbf{B}} \mathcal{P}_{\mathbf{D}}^{-1} + \mathcal{P}_{\mathbf{E}}^{-1} + \mathcal{P}_{\mathbf{E}}^{-1} \mathcal{P}_{\mathbf{F}} \mathbf{L} \mathcal{P}_{\mathbf{G}} \mathcal{P}_{\mathbf{E}}^{-1}$, it is needed to show that for $\boldsymbol{\Delta} = \mathbf{0}$, it holds

$$\Pi_F^{-1} = \begin{bmatrix} \mathbf{J} & -\mathbf{J} \mathcal{P}_{\mathbf{B}} \mathcal{P}_{\mathbf{D}}^{-1} & \mathbf{0} \\ -\mathcal{P}_{\mathbf{D}}^{-1} \mathcal{P}_{\mathbf{C}} \mathbf{J} & \mathbf{K} & -\mathcal{P}_{\mathbf{E}}^{-1} \mathcal{P}_{\mathbf{F}} \mathbf{L} \\ \mathbf{0} & -\mathbf{L} \mathcal{P}_{\mathbf{G}} \mathcal{P}_{\mathbf{E}}^{-1} & \mathbf{L} \end{bmatrix}. \tag{17}$$

The equality (17) can be proved with the use of (21). If $\mathbf{M}$ is given by the right hand side matrix in (17) and $\mathbf{M}$ is given by $\mathbf{A} = \mathbf{J}$, $\mathbf{B} = [-\mathbf{J} \mathcal{P}_{\mathbf{B}} \mathcal{P}_{\mathbf{D}}^{-1}, \mathbf{0}]$, $\mathbf{C} = \mathbf{B}^{\mathrm{T}}$ and $\mathbf{D}$ given by the remaining block, then $\mathbf{E}$ in (21a) can be inverted by (21b). Due to the lack of space, full details are not provided in the paper, nevertheless, (22), (23) and (19) are repeatedly used in the derivation.

*Remark 12.* The weight $\mathbf{W}_{\mathbf{B}}$ given by $\boldsymbol{\Delta} = \mathbf{0}$ and the central block of the matrix in (16a) show that the components of $\mathcal{X}$ that are estimated by both $\hat{\mathcal{X}}_1$ and $\hat{\mathcal{X}}_2$ are combined by the full state Covariance Intersection fusion, see (5), (23) and (19). Further, if $\boldsymbol{\Delta} = \mathbf{0}$, then the components of $\mathcal{X}$ that are estimated by either $\hat{\mathcal{X}}_1$ or $\hat{\mathcal{X}}_2$ are updated only if they are correlated with the common component, i.e. if $\mathcal{P}_{\mathbf{B}} \neq \mathbf{0}$, $\mathcal{P}_{\mathbf{G}} \neq \mathbf{0}$. The blocks on the block diagonal above the main block diagonal of the proposed upper bound $\Pi_F$ are then given by $\mathcal{P}_{\mathbf{B}} \mathbf{W}_{\mathbf{B}}^{\mathrm{T}}$ and $(\mathbf{I} - \mathbf{W}_{\mathbf{B}})^{\mathrm{T}} \mathcal{P}_{\mathbf{F}}$. That means that the original blocks $\mathcal{P}_{\mathbf{B}}$ and $\mathcal{P}_{\mathbf{F}}$ are multiplied by the corresponding fusion weight.

In a general case, the matrices $\mathbf{T}_1$, $\mathbf{T}_2$ need not be given by (13). Instead of components of $\mathcal{X}$, their linear combinations can be chosen by $\mathbf{T}_1$, $\mathbf{T}_2$. Nevertheless, there are linear combinations that are estimated by both $\hat{\mathcal{X}}_1$ and $\hat{\mathcal{X}}_2$ and those estimated by either $\hat{\mathcal{X}}_1$ or $\hat{\mathcal{X}}_2$. So, considering a change of coordinates of $\mathcal{X}$, the above proposed insights do not lack generality.

### 4.2 Example – partial state estimation

In this example, the dependence of the determinant of the upper bound $\Pi_F$ on $\mathbf{W}_1$ and $\omega_1$ is inspected.

Suppose that the estimator $\hat{\mathcal{X}}_1$ estimates the first component of $\mathcal{X}$, that means that $\mathbf{T}_1 = [1, 0]$, and the estimator $\hat{\mathcal{X}}_2$ estimates full state, $\mathbf{T}_2 = \mathbf{I}$. Let the upper bounds of the mean square error of the local estimators be given by

$$\Pi_1 = 0.25, \quad \Pi_2 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}. \tag{18}$$

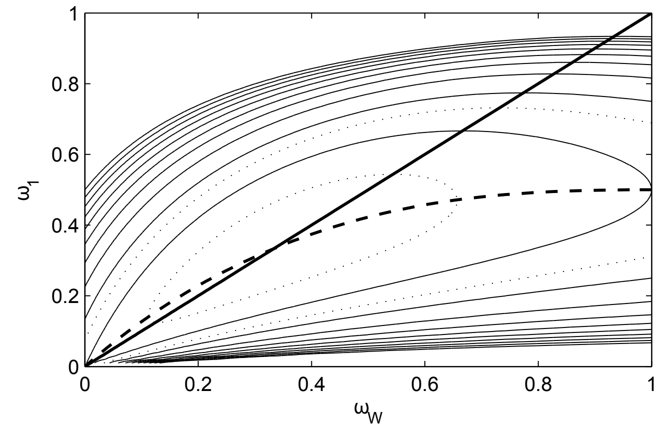


Fig. 2. Contours of the determinant of $\mathbb{P}_F$ (contours $0.75, 1, \ldots, 3$ – solid lines, contours 0.625, 0.875 – dotted lines, the determinant of $\mathbb{P}_F$ achieves its minimum 0.5625, at $\omega_{\mathbf{W}} = \omega_1 = 1/3$). The best choice of $\mathbf{W}_1(\omega_{\mathbf{W}})$ for given $\omega_1$ (thick solid line), the best choice of $\omega_1$ for given $\mathbf{W}_1(\omega_{\mathbf{W}})$ (thick dashed line).

and consider the weight $\mathbf{W}_1$ to be given by $\mathbf{W}_1(\omega_{\mathbf{W}}) = 4\omega_{\mathbf{W}}/(1+3\omega_{\mathbf{W}}) \cdot [1, 0.5]^{\mathrm{T}}$, where $\omega_{\mathbf{W}}$ fulfils $0 \le \omega_{\mathbf{W}} \le 1$. Note that the value $\omega_{\mathbf{W}}$ is the value of $\omega_1$ that produces the weight $\mathbf{W}_1$ according to (9a). The proposed upper bound is given by (10). According to the Covariance Intersection fusion, the proposed upper bound is given by (9b) for the given $\omega_{\mathbf{W}}$.

This example shows that for fixed weight $\mathbf{W}_1$, i.e. for fixed $\omega_{\mathbf{W}}$, there can exist a better upper bound $\mathbb{P}_F$ than the value obtained by substituting $\omega_1$ by $\omega_{\mathbf{W}}$ in (9b).

Fig. 2 shows the contours of the upper bound (10) for the choices of $\omega_{\mathbf{W}}$ and $\omega_1$. If the parameter $\omega_1$ is fixed first, then the value of $\mathbf{W}_1(\omega_{\mathbf{W}})$ that minimises $\mathbb{P}_F$ is given by (9a), see (12), that means that the value of $\omega_{\mathbf{W}}$ is given by $\omega_{\mathbf{W}} = \omega_1$. However, if the weight $\mathbf{W}_1(\omega_{\mathbf{W}})$ is fixed first, that means is $\omega_{\mathbf{W}}$ is fixed, then the value of $\omega_1$ that minimises the determinant of $\mathbb{P}_F$ is not given by $\omega_1 = \omega_{\mathbf{W}}$, see the thick dashed line in Fig. 2. The exceptions are the limit case $\omega_{\mathbf{W}} = 0$ and the case when $\omega_{\mathbf{W}}$ minimises the determinant of $\mathbb{P}_F$ in (9b), i.e. when $\omega_{\mathbf{W}} = 1/3$. Note that in the limit case $\omega_{\mathbf{W}} = 1$, the Covariance Intersection fusion proposes $\mathbb{P}_F$ with infinite determinant, while the optimal upper bound has determinant equal to 0.75 in such a case.

## 5. SUMMARY

The paper has dealt with unknown cross-correlations of the estimation errors. It has been shown that it is sufficient to parametrise the upper bounds of general matrices with two blocks on the diagonal by a scalar parameter. The upper bound of the mean square error matrix of the estimator given by the linear fusion has been expressed by using the completion to square. The fusion has been analysed for full state as well as for partial state estimators. Also, it has been shown that the upper bounds provided by the Covariance Intersection fusion need not be optimal for all admissible values of the free parameters of the fusion rule. The proposition of better upper bounds without numerical optimisation is a possible direction of the future research.

## APPENDIX

In this section, some useful lemmas are given and the parametrisation of upper bounds is discussed.

First, consider two invertible matrices $\mathbf{A}$ and $\mathbf{B}$. It holds,

$$\mathbf{A}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{B} = (\mathbf{A}^{-1}+\mathbf{B}^{-1})^{-1} = \mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}. \quad (19)$$

Second, consider an invertible block matrix $\mathbf{M}$ with blocks $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$, where $\mathbf{A}$ and $\mathbf{D}$ are invertible. It is possible to perform the following two matrix decompositions,

$$\mathbf{M} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{C}\mathbf{A}^{-1} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{A}^{-1}\mathbf{B}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}, \quad (20a)$$

$$\mathbf{M} = \begin{bmatrix}\mathbf{I} & \mathbf{B}\mathbf{D}^{-1}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C} & \mathbf{0}\\ \mathbf{0} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{D}^{-1}\mathbf{C} & \mathbf{I}\end{bmatrix}. \quad (20b)$$

It is easy to observe that $\mathbf{M}^{-1}$ can be written in the following two equivalent forms,

$$\mathbf{M}^{-1} = \begin{bmatrix}\{\mathbf{A}^{-1}+\mathbf{A}^{-1}\mathbf{B}\mathbf{E}\mathbf{C}\mathbf{A}^{-1}\} & -\mathbf{A}^{-1}\mathbf{B}\mathbf{E}\\ -\mathbf{E}\mathbf{C}\mathbf{A}^{-1} & \mathbf{E}\end{bmatrix}, \quad (21a)$$

$$\mathbf{M}^{-1} = \begin{bmatrix}\mathbf{F} & -\mathbf{F}\mathbf{B}\mathbf{D}^{-1}\\ -\mathbf{D}^{-1}\mathbf{C}\mathbf{F} & \{\mathbf{D}^{-1}+\mathbf{D}^{-1}\mathbf{C}\mathbf{F}\mathbf{B}\mathbf{D}^{-1}\}\end{bmatrix}, \quad (21b)$$

where $\mathbf{E} = (\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}$ and $\mathbf{F} = (\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}$ is used to shorthand the notation. Comparing the corresponding blocks, the following identities are obtained,

$$(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} = \mathbf{A}^{-1}+\mathbf{A}^{-1}\mathbf{B}(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1}, \quad (22)$$

$$(\mathbf{A}-\mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} = \mathbf{A}^{-1}\mathbf{B}(\mathbf{D}-\mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}. \quad (23)$$

Last, let the matrix $\mathbf{M}$ be symmetric, $\mathbf{C} = \mathbf{B}^{\mathrm{T}}$, and the blocks $\mathbf{A}$, $\mathbf{D}$ be positive semidefinite. Decompose $\mathbf{A}$ and $\mathbf{D}$ as follows, $\mathbf{A} = \mathbf{S}_A\mathbf{S}_A^{\mathrm{T}}$, $\mathbf{D} = \mathbf{S}_D\mathbf{S}_D^{\mathrm{T}}$. Then, it holds

$$\mathbf{M} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{B}^{\mathrm{T}} & \mathbf{D}\end{bmatrix} = \begin{bmatrix}\mathbf{S}_A & \mathbf{0}\\ \mathbf{0} & \mathbf{S}_D\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{\Omega}\\ \mathbf{\Omega}^{\mathrm{T}} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{S}_A & \mathbf{0}\\ \mathbf{0} & \mathbf{S}_D\end{bmatrix}^{\mathrm{T}}, \quad (24)$$

where $\mathbf{\Omega} = \mathbf{S}_A^{-1}\mathbf{B}(\mathbf{S}_B^{\mathrm{T}})^{-1}$. Using (20a), the middle matrix in (24) can be decomposed as

$$\begin{bmatrix}\mathbf{I} & \mathbf{\Omega}\\ \mathbf{\Omega}^{\mathrm{T}} & \mathbf{I}\end{bmatrix} = \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{\Omega}^{\mathrm{T}} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}-\mathbf{\Omega}^{\mathrm{T}}\mathbf{\Omega}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{\Omega}\\ \mathbf{0} & \mathbf{I}\end{bmatrix} \quad (25)$$

and it is evident that the matrix $\mathbf{M}$ is positive semidefinite if and only if the eigenvalues of $\mathbf{\Omega}^{\mathrm{T}}\mathbf{\Omega}$ are not greater than one. That means that $\mathbf{\Omega}$ has to be a contraction matrix, i.e. its singular values must not be greater that one.

In Covariance Intersection, the matrix $\mathbf{M}$ with unknown off-diagonal block $\mathbf{B}$ is replaced by an upper bound $\mathbb{M}$ that uses only the diagonal blocks of $\mathbf{M}$. The upper bounds use the parameter $\omega$, $0 \le \omega \le 1$, and scales the diagonal blocks. So, the matrix $\mathbb{M}$ can be decomposed as follows,

$$\mathbb{M} = \begin{bmatrix}\mathbf{S}_A & \mathbf{0}\\ \mathbf{0} & \mathbf{S}_D\end{bmatrix}\begin{bmatrix}1/\omega\,\mathbf{I} & \mathbf{0}\\ \mathbf{0} & 1/(1-\omega)\,\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{S}_A & \mathbf{0}\\ \mathbf{0} & \mathbf{S}_D\end{bmatrix}^{\mathrm{T}}. \quad (26)$$

If a reparametrisation is used, namely $\gamma = (1-\omega)/\omega$, then $0 \le \gamma$ and the diagonal blocks of the middle matrix on the right hand side of (26) becomes $(1+\gamma)\mathbf{I}$ and $(1+1/\gamma)\mathbf{I}$. Subtracting $\mathbf{M}$ from $\mathbb{M}$ and using the decompositions (26), (24), the following decomposition is obtained.

$$\mathbb{M} - \mathbf{M} = \begin{bmatrix}\mathbf{S}_A & \mathbf{0}\\ \mathbf{0} & \mathbf{S}_D\end{bmatrix}\begin{bmatrix}\gamma\mathbf{I} & -\mathbf{\Omega}\\ -\mathbf{\Omega}^{\mathrm{T}} & 1/\gamma\,\mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{S}_A & \mathbf{0}\\ \mathbf{0} & \mathbf{S}_D\end{bmatrix}^{\mathrm{T}}. \quad (27)$$

Applying (20a) on the middle matrix on the right hand side of (27), it is evident that the difference $\mathbb{M} - \mathbf{M}$ is a positive semidefinite matrix.

There arises the question if another diagonal blocks of the middle matrix on the right hand side of (27) can lead to a better upper bound $\mathbb{M}$. So, the positive semidefiniteness of matrix $\mathbf{N}$,

$$\mathbf{N} = \begin{bmatrix} \mathbf{E} & -\mathbf{\Omega} \\ -\mathbf{\Omega}^{\mathrm{T}} & \mathbf{F} \end{bmatrix}, \tag{28}$$

is inspected for arbitrary symmetric positive definite matrices $\mathbf{E}$, $\mathbf{F}$ and all contraction matrices $\mathbf{\Omega}$.

Using (20a) again, $\mathbf{N}$ is positive semidefinite if and only if $\mathbf{F} - \mathbf{\Omega}^{\mathrm{T}}\mathbf{E}^{-1}\mathbf{\Omega}$ is positive semidefinite. Since the semidefiniteness is required for all contraction matrices $\mathbf{\Omega}$, the matrix $\mathbf{F}$ has to be an upper bound of the identity matrix scaled by the maximum eigenvalue of $\mathbf{E}^{-1}$, i.e. by the inverse of the minimum eigenvalue of $\mathbf{E}$. However, if the minimum upper bound $\mathbf{F}$ is chosen, than in order to provide as small $\mathbf{N}$ as possible, the matrix $\mathbf{E}$ should be replaced by the identity matrix scaled by the minimum eigenvalue of $\mathbf{E}$. So, the answer is negative, there is no better upper bound $\mathbb{M}$ than the one given by (26).

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