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Dynamic Programming for Fractional Discrete-time Systems.

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Abstract: Dynamic programming problem for fractional discrete-time systems with quadratic performance index has been formulated and solved. A new method for numerical computation of optimal dynamic programming problem has been presented. The efficiency of the method has been demonstrated on numerical example and illustrated by graphs. Graphs also show the differences between the fractional and integer-order systems theory. Results for other cases of the coefficient α, and not illustrated with numerical examples, have been obtained through a computer algorithm written for this purpose

Keywords: optimal control, LQ control, dynamic programming, fractional order, discrete-time systems.

1. INTRODUCTION

Dynamic optimization problems for integer (not fractional) order systems have been widely considered in literature (see e.g. Bellman (1957); Kaczorek (1981); Lewis and Syrmos (1995); Naidu (2002)). Mathematical fundamentals of the fractional calculus are given in the monographes Ostalczyk (2008); Podlubny (1999); Samko et al. (1993) and the fractional differential equations and their applications have been addressed in e.g. Kilbas et al. (2006); F.Liu et al. (2010). The numerical simulation of the fractional order control systems has been investigated in Cai and F.Liu (2007). One of the fractional discretization method has been presented in Meerschaerti and Tadjeran (2004). Some optimal control problems for fractional order systems have been investigated in Frederico and Torres (2008); Jelic and Petrovacki (2008); Agrawal (2008, 2007, 2006, 2004, 2002); Sierociuk and Vinagre (2010). Fractional Kalman filter and its application have been addressed in Sierociuk et al. (2011); Sierociuk and Dzielinski (2006). Some recent interesting results in fractional systems theory and its applications for standard and positive systems can be found in Kaczorek (2011). In this paper dynamic programming problem for fractional discrete-time systems with quadratic performance index will be formulated and solved. A new method for numerical computation of optimal dynamic programming problem will be presented. The efficiency of the method will be demonstrated on numerical example and illustrated by graphs. Graphs also show the differences between the fractional and classical (standard) systems theory. Results for other cases of the coefficient α and not illustrated with numerical examples will be obtained through a computer algorithm written for this purpose. The paper is organized as follows. In section II some preliminaries are recalled and the problem will be formulated. The solutions of the problem are presented in section III. In section IV a procedure for computation of the optimal control is proposed and illustrated by numerical example. A relarion with the integer-order systems theory is demonstrated in section V. Conclusions of the paper are given in section VI. The following notation will be used: $\mathbb{R}$ - the set of real numbers, $\mathbb{R}^{n\times n}$ - the set of $n \times n$ real matrices (in particular $\mathbb{R}^n$ is the set of real vectors), I_n - the $n \times n$ identity matrix, $\mathbb{O}$ - the null matrix of appropriate dimensions, W_a^b, V_a^b are $n \times m$ or $n \times n$ matrices and a is the lower right index and b is an upper right index. Power index is not used.

2. PROBLEM FORMULATION

Consider a fractional discrete-time system, obtained by use of Grunwald-Letnikov's (shifted) approximation, described by equations

$$x_{k+1} = \sum_{j=0}^{k} d_j x_{k-j} + Bu_k \ , \qquad k \in \mathbb{Z}_+ \ , \tag{1a}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ are respectively the state and control vectors, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and

$$d_0 = A_\alpha = A + \alpha I_n \ , \qquad 0 < \alpha < 1 \ , \tag{1b}$$

$$d_j = (-1)^j \binom{\alpha}{j+1} I_n \ , \qquad j = 1, \ldots, k \ . \tag{1c}$$

Consider a performance index of the form

$$\begin{aligned} J_i(u) &= G(x_N) + \sum_{k=i}^{N-1} F_k(x_k, u_k) \\ &= x_N^T S x_N + \sum_{k=i}^{N-1} \left(x_k^T Q x_k + u_k^T R u_k \right) , \end{aligned} \tag{2}$$

where $R \in \mathbb{R}^{m\times m}$, $Q \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{n\times n}$ and $S \geq 0$, $Q \geq 0$ and $R > 0$.

Optimal trajectory starting at the point x_0 and ending at the point x_k has been divided into N elementary time intervals $[0, N]$. It is desired to find optimal control sequence $u_0, u_1, \ldots, u_{N-1}$, $u \in \mathbb{U}$, $\mathbb{U}$-set of admissible inputs, which minimizes the performance index (2) and satisfies the differential equation (1). The solution of this task by searching for a conditional minimum of the performance index (2) requires the solution of N equations with N unknown variables of the form

$$\frac{\partial J(u)}{\partial u_k} = 0 \ , \quad (k = 0, \ldots, N-1) \ ,$$

where $J(u)$ is the performance index (2) after substituting (1) for $k = 1, 2, \ldots, N-1$.

3. PROBLEM SOLUTION

We shall show that the task of determining the $u_0, u_1, \ldots, u_{N-1}$ can be reduced to N tasks minimizing functions of one variable.

For $i = N$ the performance index has the form

$$J_N(u) = G(x_N) = x_N^T S x_N \ ,$$

which in the general case is a function of final state.

Consider the last N-th section of the optimal trajectory. The corresponding performance index of that section has the form

$$\begin{aligned} J_{N-1}(u) &= J_N(u) + F_{N-1}(x_{N-1}, u_{N-1}) \\ &= x_N^T S x_N + x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1} \ , \end{aligned} \tag{3}$$

Denoting $S_{N-1}(\Sigma x_{N-1}) = S_{N-1}(\sum_{j=0}^{N-1} x_{N-1-j})$, as a minimum of the performance index $J_{N-1}(u)$ we can write

$$S_{N-1}(\Sigma x_{N-1}) = \min_{u_{N-1} \in \mathbb{U}} \left\{ J_{N-1}(u) \right\} \ . \tag{4}$$

Substituting (1) for $k = N-1$ to (4) equation above takes the form

$$\begin{aligned} S_{N-1}(\Sigma x_{N-1}) = \min_{u_{N-1} \in \mathbb{U}} \Bigg\{ & x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1} \\ & + \left(\sum_{j=0}^{N-1} d_j x_{N-1-j} + B u_{N-1} \right)^T S \left(\sum_{j=0}^{N-1} d_j x_{N-1-j} + B u_{N-1} \right) \Bigg\} \end{aligned} \tag{5}$$

Calculating the first derivative of the equation (5) and comparing it to zero we obtain

$$\begin{aligned} 0 = \frac{\partial J_{N-1}(u)}{\partial u_{N-1}} &= \left(R + R^T \right) u_{N-1} \\ &+ B^T \left(S + S^T \right) \left(\sum_{j=0}^{N-1} d_j x_{N-1-j} + B u_{N-1} \right) . \end{aligned}$$

We determine u_{N-1} as a function of $x_{N-1}, \ldots, x_0$, i.e.

$$u_{N-1} = \sum_{j=0}^{N-1} W_{N-1}^1 d_j x_{N-1-j} \ , \tag{6}$$

where

$$W_{N-1}^1 = - \left[R + R^T + B^T \left(S + S^T \right) B \right]^{-1} B^T \left(S + S^T \right) .$$

Substituting (6) to (5) we obtain

$$\begin{aligned} S_{N-1}(\Sigma x_{N-1}) &= x_{N-1}^T Q x_{N-1} \\ &+ \left[\sum_{j=0}^{N-1} V_{N-1}^{R_{01}} d_j x_{N-1-j} \right]^T R \left[\sum_{j=0}^{N-1} V_{N-1}^{R_{01}} d_j x_{N-1-j} \right] \\ &+ \left[\sum_{j=0}^{N-1} V_{N-1}^{S_1} d_j x_{N-1-j} \right]^T S \left[\sum_{j=0}^{N-1} V_{N-1}^{S_1} d_j x_{N-1-j} \right] . \end{aligned} \tag{7}$$

where

$$V_{N-1}^{R_{01}} = W_{N-1}^1, \quad V_{N-1}^{S_1} = \left(I_n + B W_{N-1}^1 \right) .$$

Consider the N-th and $N-1$-th sections of the optimal trajectory. The corresponding performance index for those sections has the form

$$\begin{aligned} J_{N-2}(u) &= J_{N-1}(u) + F_{N-2}(x_{N-2}, u_{N-2}) \\ &= J_{N-1}(u) + x_{N-2}^T Q x_{N-2} + u_{N-2}^T R u_{N-2} \ . \end{aligned} \tag{8}$$

Denoting $S_{N-2}(\Sigma x_{N-2}) = S_{N-2}(\sum_{j=0}^{N-2} x_{N-2-j})$ as minimum of the performance index $J_{N-2}(u)$ we can write

$$\begin{aligned} S_{N-2}(\Sigma x_{N-2}) &= \min_{\substack{u_{N-1} \in \mathbb{U} \\ u_{N-2} \in \mathbb{U}}} \left\{ J_{N-2}(u_{N-2}) \right\} \\ &= \min_{u_{N-2} \in \mathbb{U}} \left\{ \min_{u_{N-1} \in \mathbb{U}} J_{N-1}(u_{N-1}) + F_{N-2}(x_{N-2}, u_{N-2}) \right\} \\ &= \min_{u_{N-2} \in \mathbb{U}} \left\{ S_{N-1}(\Sigma x_{N-1}) + F_{N-2}(x_{N-2}, u_{N-2}) \right\} . \end{aligned} \tag{9}$$

Substituting (1) for $k = N-2$ to (9) and calculating the first derivative of the equation and comparing it to zero we obtain

$$\begin{aligned} 0 = \frac{\partial J_{N-2}(u)}{\partial u_{N-2}} &= \left(R + R^T \right) u_{N-2} + B^T \left[Q + Q^T \right] \\ & \left[\sum_{j=0}^{N-2} d_j x_{N-2-j} + B u_{N-2} \right] + \left[V_{N-1}^{R_{01}} d_0 B \right]^T \left[R + R^T \right] \\ & \left[V_{N-1}^{R_{01}} d_0 \left(\sum_{j=0}^{N-2} d_j x_{N-2-j} + B u_{N-2} \right) + \sum_{j=0}^{N-2} V_{N-1}^{R_{01}} d_{j+1} x_{N-2-j} \right] \\ & + \left[V_{N-1}^{S_1} d_0 B \right]^T \left[S + S^T \right] \left[V_{N-1}^{S_1} d_0 \left(\sum_{j=0}^{N-2} d_j x_{N-2-j} + B u_{N-2} \right) \right. \\ & \left. + \sum_{j=0}^{N-2} V_{N-1}^{S_1} d_{j+1} x_{N-2-j} \right] . \end{aligned}$$

We determine u_{N-2} as a function of $x_{N-2}, \ldots, x_0$, i.e.

$$u_{N-2} = \sum_{j=0}^{N-2} \left[W_{N-2}^1 d_j + W_{N-2}^2 d_{j+1} \right] x_{N-2-j} \ , \tag{10}$$

Substituting (1) for $k = N-2$ and (10) to (9) we obtain

$$S_{N-2}(\Sigma x_{N-2}) = x_{N-2}^T Q x_{N-2} + \left[\sum_{j=0}^{N-2}\left(V_{N-2}^{Q_{01}} d_j + V_{N-2}^{Q_{02}} d_{j+1}\right) x_{N-2-j}\right]^T \times Q \left[\sum_{j=0}^{N-2}\left(V_{N-2}^{Q_{01}} d_j + V_{N-2}^{Q_{02}} d_{j+1}\right) x_{N-2-j}\right] + \left[\sum_{j=0}^{N-2}\left(V_{N-2}^{R_{01}} d_j + V_{N-2}^{R_{02}} d_{j+1}\right) x_{N-2-j}\right]^T \times R \left[\sum_{j=0}^{N-2}\left(V_{N-2}^{R_{01}} d_j + V_{N-2}^{R_{02}} d_{j+1}\right) x_{N-2-j}\right] + \left[\sum_{j=0}^{N-2}\left(V_{N-2}^{R_{11}} d_j + V_{N-2}^{R_{12}} d_{j+1}\right) x_{N-2-j}\right]^T \times R \left[\sum_{j=0}^{N-2}\left(V_{N-2}^{R_{11}} d_j + V_{N-2}^{R_{12}} d_{j+1}\right) x_{N-2-j}\right] + \left[\sum_{j=0}^{N-2}\left(V_{N-2}^{S_1} d_j + V_{N-2}^{S_2} d_{j+1}\right) x_{N-2-j}\right]^T \times S \left[\sum_{j=0}^{N-2}\left(V_{N-2}^{S_1} d_j + V_{N-2}^{S_2} d_{j+1}\right) x_{N-2-j}\right], \tag{11}$$

where

$$V_{N-2}^{Q_{01}} = I_n + BW_{N-2}^1 , \quad V_{N-2}^{Q_{02}} = BW_{N-2}^2 ,$$
$$V_{N-2}^{R_{01}} = W_{N-2}^1 , \quad V_{N-2}^{R_{02}} = W_{N-2}^2 ,$$
$$V_{N-2}^{R_{11}} = V_{N-1}^{R_{01}} d_0 V_{N-2}^{Q_{01}} , \quad V_{N-2}^{R_{12}} = V_{N-1}^{R_{01}} d_0 V_{N-2}^{Q_{02}} + V_{N-1}^{R_{01}} ,$$
$$V_{N-2}^{S_1} = V_{N-1}^{S_1} d_0 V_{N-2}^{Q_{01}} , \quad V_{N-2}^{S_2} = V_{N-1}^{S_1} d_0 V_{N-2}^{Q_{02}} + V_{N-1}^{S_1} .$$

In the same way for the last three sections of the optimal trajectory we obtain the corresponding performance index for those sections in the form

$$J_{N-3}(u) = J_{N-2}(u) + F_{N-3}(x_{N-3}, u_{N-3}) = J_{N-2}(u) + x_{N-3}^T Q x_{N-3} + u_{N-3}^T R u_{N-3} . \tag{12}$$

Denoting $S_{N-3}(\Sigma x_{N-3}) = S_{N-3}(\sum_{j=0}^{N-3} x_{N-3-j})$ as minimum of the performance index $J_{N-3}(u)$ we can write

$$S_{N-3}(\Sigma x_{N-3}) = \min_{\substack{u_{N-1}\in\mathbb{U}\\ u_{N-2}\in\mathbb{U}\\ u_{N-3}\in\mathbb{U}}} \left\{J_{N-3}(u)\right\} = \min_{u_{N-3}\in\mathbb{U}} \left\{ \min_{\substack{u_{N-1}\in\mathbb{U}\\ u_{N-2}\in\mathbb{U}}} J_{N-2}(u) + F_{N-3}(x_{N-3}, u_{N-3})\right\} = \min_{u_{N-3}\in\mathbb{U}} \left\{S_{N-2}(\Sigma x_{N-2}) + F_{N-3}(x_{N-3}, u_{N-3})\right\} . \tag{13}$$

Substituting (1) for $k = N-3$ to (13) and calculating the first derivative of the equation and comparing it to zero we obtain

$$0 = \frac{\partial J_{N-3}(u)}{\partial u_{N-3}} = \left(R + R^T\right) u_{N-3} + B^T \left(Q + Q^T\right) \times \left(\sum_{j=0}^{N-3} d_j x_{N-3-j} + B u_{N-3}\right) + \left[\left(V_{N-2}^{Q_{01}} d_0 + V_{N-2}^{Q_{02}} d_1\right) B\right]^T \times \left(Q + Q^T\right) \left[\left(V_{N-2}^{Q_{01}} d_0 + V_{N-2}^{Q_{02}} d_1\right)\left(\sum_{j=0}^{N-3} d_j x_{N-3-j} + B u_{N-3}\right) + \sum_{j=0}^{N-3}\left(V_{N-2}^{Q_{01}} d_{j+1} + V_{N-2}^{Q_{02}} d_{j+2}\right) x_{N-3-j}\right] + \left[\left(V_{N-2}^{R_{01}} d_0 + V_{N-2}^{R_{02}} d_1\right) B\right]^T \left(R + R^T\right) \times \left[\left(V_{N-2}^{R_{01}} d_0 + V_{N-2}^{R_{02}} d_1\right)\left(\sum_{j=0}^{N-3} d_j x_{N-3-j} + B u_{N-3}\right) + \sum_{j=0}^{N-3}\left(V_{N-2}^{R_{01}} d_{j+1} + V_{N-2}^{R_{02}} d_{j+2}\right) x_{N-3-j}\right] + \left[\left(V_{N-2}^{R_{11}} d_0 + V_{N-2}^{R_{12}} d_1\right) B\right]^T \left(R + R^T\right) \times \left[\left(V_{N-2}^{R_{11}} d_0 + V_{N-2}^{R_{12}} d_1\right)\left(\sum_{j=0}^{N-3} d_j x_{N-3-j} + B u_{N-3}\right) + \sum_{j=0}^{N-3}\left(V_{N-2}^{R_{11}} d_{j+1} + V_{N-2}^{R_{12}} d_{j+2}\right) x_{N-3-j}\right] + \left[\left(V_{N-2}^{S_1} d_0 + V_{N-2}^{S_2} d_1\right) B\right]^T \left(S + S^T\right) \times \left[\left(V_{N-2}^{S_1} d_0 + V_{N-2}^{S_2} d_1\right)\left(\sum_{j=0}^{N-3} d_j x_{N-3-j} + B u_{N-3}\right) + \sum_{j=0}^{N-3}\left(V_{N-2}^{S_1} d_{j+1} + V_{N-2}^{S_2} d_{j+2}\right) x_{N-3-j}\right].$$

We determine u_{N-3} as a function of $x_{N-3}, \ldots, x_0$, i.e.

$$u_{N-3} = \sum_{j=0}^{N-3} \left[W_{N-3}^1 d_j + W_{N-3}^2 d_{j+1} + W_{N-3}^3 d_{j+2}\right] x_{N-3-j} \tag{14}$$

Substituting (1) for $k = N-3$ and (14) to (13) we obtain

$$S_{N-3}(\Sigma x_{N-3}) = x_{N-3}^T Q x_{N-3} + \left[\sum_{j=0}^{N-3}\left(V_{N-3}^{Q_{01}} d_j + V_{N-3}^{Q_{02}} d_{j+1} + V_{N-3}^{Q_{03}} d_{j+2}\right) x_{N-3-j}\right]^T Q \times \left[\sum_{j=0}^{N-3}\left(V_{N-3}^{Q_{01}} d_j + V_{N-3}^{Q_{02}} d_{j+1} + V_{N-3}^{Q_{03}} d_{j+2}\right) x_{N-3-j}\right] + \left[\sum_{j=0}^{N-3}\left(V_{N-3}^{Q_{11}} d_j + V_{N-3}^{Q_{12}} d_{j+1} + V_{N-3}^{Q_{13}} d_{j+2}\right) x_{N-3-j}\right]^T Q \times \left[\sum_{j=0}^{N-3}\left(V_{N-3}^{Q_{11}} d_j + V_{N-3}^{Q_{12}} d_{j+1} + V_{N-3}^{Q_{13}} d_{j+2}\right) x_{N-3-j}\right] + \left[\sum_{j=0}^{N-3}\left(V_{N-3}^{R_{01}} d_j + V_{N-3}^{R_{02}} d_{j+1} + V_{N-3}^{R_{03}} d_{j+2}\right) x_{N-3-j}\right]^T R \times \left[\sum_{j=0}^{N-3}\left(V_{N-3}^{R_{01}} d_j + V_{N-3}^{R_{02}} d_{j+1} + V_{N-3}^{R_{03}} d_{j+2}\right) x_{N-3-j}\right] + \left[\sum_{j=0}^{N-3}\left(V_{N-3}^{R_{11}} d_j + V_{N-3}^{R_{12}} d_{j+1} + V_{N-3}^{R_{13}} d_{j+2}\right) x_{N-3-j}\right]^T R \times \left[\sum_{j=0}^{N-3}\left(V_{N-3}^{R_{11}} d_j + V_{N-3}^{R_{12}} d_{j+1} + V_{N-3}^{R_{13}} d_{j+2}\right) x_{N-3-j}\right]$$

$$+\left[\sum_{j=0}^{N-3}\left(V_{N-3}^{R_{21}}d_j+V_{N-3}^{R_{22}}d_{j+1}+V_{N-3}^{R_{23}}d_{j+2}\right)x_{N-3-j}\right]^T R$$
$$\times\left[\sum_{j=0}^{N-3}\left(V_{N-3}^{R_{21}}d_j+V_{N-3}^{R_{22}}d_{j+1}+V_{N-3}^{R_{23}}d_{j+2}\right)x_{N-3-j}\right]$$
$$+\left[\sum_{j=0}^{N-3}\left(V_{N-3}^{S_1}d_j+V_{N-3}^{S_2}d_{j+1}+V_{N-3}^{S_3}d_{j+2}\right)x_{N-3-j}\right]^T S$$
$$\times\left[\sum_{j=0}^{N-3}\left(V_{N-3}^{S_1}d_j+V_{N-3}^{S_2}d_{j+1}+V_{N-3}^{S_3}d_{j+2}\right)x_{N-3-j}\right]. \quad (15)$$

In the general case for q last sections of the optimal trajectory the value which minimize performance index (2) with constraints (1) is given by the relation

$$\begin{aligned}S_{N-q}(\Sigma x_{N-q}) &= x_{N-q}^T Q x_{N-q}\\
&+\sum_{l=0}^{q-2}\left\{\left[\sum_{j=0}^{N-q}\left(\sum_{p=0}^{q-1}V_{N-q}^{Q_{l,p+1}}d_{j+p}\right)x_{N-q-j}\right]^T Q\right.\\
&\times\left.\left[\sum_{j=0}^{N-q}\left(\sum_{p=0}^{q-1}V_{N-q}^{Q_{l,p+1}}d_{j+p}\right)x_{N-q-j}\right]\right\}\\
&+\sum_{r=0}^{q-1}\left\{\left[\sum_{j=0}^{N-q}\left(\sum_{p=0}^{q-1}V_{N-q}^{R_{r,p+1}}d_{j+p}\right)x_{N-q-j}\right]^T R\right.\\
&\times\left.\left[\sum_{j=0}^{N-q}\left(\sum_{p=0}^{q-1}V_{N-q}^{R_{r,p+1}}d_{j+p}\right)x_{N-q-j}\right]\right\}\\
&+\left[\sum_{j=0}^{N-q}\left(\sum_{p=0}^{q-1}V_{N-q}^{S_{p+1}}d_{j+p}\right)x_{N-q-j}\right]^T S\\
&\times\left[\sum_{j=0}^{N-q}\left(\sum_{p=0}^{q-1}V_{N-q}^{S_{p+1}}d_{j+p}\right)x_{N-q-j}\right].\end{aligned} \quad (16)$$

Control u_{N-q}, which minimizes the performance index $J_{N-q}(u)$, in the general case is given by the relation

$$u_{N-q}=\sum_{j=0}^{N-q}\left(\sum_{p=0}^{q-1}W_{N-q}^{p+1}d_{j+p}\right)x_{N-q-j}\ , \quad (17)$$

4. PROCEDURE AND EXAMPLES

From the above considerations, the following procedure followed for solving the dynamic optimization problem:

Procedure:
Step 1. Knowing the matrices A and B of the system (1) and the coefficient α and the number of elementary sections N of the optimal trajectory, we determine the matrix A_α and coefficients d_j for $j = 0, 1, \dots, N$.
Step 2. Knowing the matrices R, Q, S of the performance index (2) and the coefficients d_j for $j = 0, 1, \dots, N$ and using known methods of minimization, we determine the value of the control (6) which minimizes the performance index (3), and its minimum value (7) for $q = 1$. Knowing (7) we determine the value of control (10) which minimizes the performance index (8) and its minimum value (11) for $q = 2$. Continuing the procedure we determine the equations (16) and (17) for $q = 3, 4, \dots, N$.
Step 3. Using the formula (17) for $q = N$ we determine u_0, the control value in a discrete time $k = 0$ depending on the initial conditions x_0. Using (16) we determine minimum of the performance index $S_0(\Sigma x_0)$. Knowing u_0 and x_0 from the relation (1) for $k = 0$ we determine x_1. Using the formula (17) for $q = N-1$ we determine u_1 as a function of x_1, x_0. Using (16) we determine the minimum value of the performance index $S_1(\Sigma x_1)$. Knowing u_1 and x_1, x_0 from the relation (1) for $k = 1$ we can find the x_2. Using the formula (17) for $q = N - 2$ we determine u_2 as a function of x_2, x_1, x_0 and using (16) we can determine $S_2(\Sigma x_2)$. Continuing this procedure we can determine the discrete values of control sequence $u_0, u_1, \dots, u_{N-1} \in \mathbb{U}$, which minimizes the performance index (2) and satisfies the differential equation (1) for given initial conditions x_0 and the subsequent minimum value $S_0(\Sigma x_0), \dots, S_N(\Sigma x_N)$ of the performance index (2).

Example 1. Consider the fractional discrete-time system (1) with matrices

$$A=\begin{bmatrix}1&2\\3&4\end{bmatrix},\quad B=\begin{bmatrix}1\\2\end{bmatrix},\quad x_0=\begin{bmatrix}0.5\\0.7\end{bmatrix}, \quad (18)$$

and the performance index (2) with matrices

$$S=\begin{bmatrix}4&1\\1&4\end{bmatrix},\quad Q=\begin{bmatrix}3&2\\2&3\end{bmatrix},\quad R=[\,1\,]. \quad (19)$$

Using the above Procedure we obtain.
Step 1. Assuming $\alpha = 0.5$ and $N = 3$, the matrix A_α has the form

$$A_\alpha = A+\alpha I_n=\begin{bmatrix}1.5&2\\3&4.5\end{bmatrix}, \quad (20)$$

and the coefficients d_j for $j = 0, 1, \dots, N$ are as follows

$$\begin{aligned}d_0&=\begin{bmatrix}1.5&2\\3&4.5\end{bmatrix}, & d_2&=\begin{bmatrix}0.063&0\\0&0.063\end{bmatrix},\\ d_1&=\begin{bmatrix}0.125&0\\0&0.125\end{bmatrix}, & d_3&=\begin{bmatrix}0.039&0\\0&0.039\end{bmatrix}.\end{aligned} \quad (21)$$

Step 2. Taking into account the matrices (19) and the coefficients (21) and for $q = 1$ we determine a matrix

$$W_{N-1}^1=[\,-0.2400\ -0.3600\,], \quad (22a)$$

and matrices

$$\begin{aligned}V_{N-1}^{R_{01}}&=[\,-0.2400\ -0.3600\,],\\ V_{N-1}^{S_1}&=\begin{bmatrix}0.7600&-0.3600\\-0.4800&0.2800\end{bmatrix}.\end{aligned} \quad (22b)$$

for $q = 2$ we determine matrices

$$\begin{aligned}W_{N-2}^1&=[\,-0.2677\ -0.3575\,],\\ W_{N-2}^2&=[\,-0.0028\ -0.0474\,],\end{aligned} \quad (23a)$$

and matrices

$$\begin{aligned}V_{N-2}^{Q_{01}}&=\begin{bmatrix}0.7323&-0.3575\\-0.5353&0.2850\end{bmatrix},\\ V_{N-2}^{Q_{02}}&=\begin{bmatrix}-0.0028&-0.0474\\-0.0055&-0.0949\end{bmatrix},\end{aligned} \quad (23b)$$

$$\begin{aligned}V_{N-2}^{R_{01}}&=[\,-0.2677\ -0.3575\,],\\ V_{N-2}^{R_{02}}&=[\,-0.0028\ -0.0474\,],\\ V_{N-2}^{R_{11}}&=[\,0.0696\ -0.0836\,],\\ V_{N-2}^{R_{12}}&=[\,-0.2244\ -0.0925\,],\end{aligned} \quad (23c)$$

$$V_{N-2}^{S_1} = \begin{bmatrix} 0.0975 & -0.0499 \\ -0.0727 & 0.0426 \end{bmatrix}, \quad V_{N-2}^{S_2} = \begin{bmatrix} 0.7604 & -0.3534 \\ -0.4820 & 0.2458 \end{bmatrix}. \tag{23d}$$

Continuing the procedure for $q = 3$ we obtain

$$W_{N-3}^{1} = [\,-0.2689 \;\; -0.3566\,], \quad W_{N-3}^{2} = [\,-0.0187 \;\; -0.0403\,], \quad W_{N-3}^{3} = [\,0.0019 \;\; -0.0068\,], \tag{24a}$$

$$V_{N-3}^{Q_{01}} = \begin{bmatrix} 0.7311 & -0.3569 \\ -0.5378 & 0.2861 \end{bmatrix}, \quad V_{N-3}^{Q_{02}} = \begin{bmatrix} -0.0187 & -0.0403 \\ -0.0373 & -0.0806 \end{bmatrix}, \quad V_{N-3}^{Q_{03}} = \begin{bmatrix} 0.0019 & -0.0068 \\ 0.0039 & -0.0135 \end{bmatrix},$$
$$V_{N-3}^{Q_{11}} = \begin{bmatrix} 0.0994 & -0.0521 \\ -0.0701 & 0.0389 \end{bmatrix}, \quad V_{N-3}^{Q_{12}} = \begin{bmatrix} 0.7375 & -0.3465 \\ -0.5437 & 0.2668 \end{bmatrix}, \quad V_{N-3}^{Q_{13}} = \begin{bmatrix} -0.0033 & -0.0456 \\ -0.0047 & -0.0979 \end{bmatrix}, \tag{24b}$$

$$\begin{aligned} V_{N-3}^{R_{01}} &= [\,-0.2689 \;\; -0.3569\,], \\ V_{N-3}^{R_{02}} &= [\,-0.0187 \;\; -0.0403\,], \\ V_{N-3}^{R_{03}} &= [\,0.0019 \;\; -0.0067\,], \\ V_{N-3}^{R_{11}} &= [\,0.0785 \;\; -0.0889\,], \\ V_{N-3}^{R_{12}} &= [\,-0.1599 \;\; -0.1249\,], \\ V_{N-3}^{R_{13}} &= [\,-0.0141 \;\; -0.0085\,], \\ V_{N-3}^{R_{21}} &= [\,0.0062 \;\; -0.0089\,], \\ V_{N-3}^{R_{22}} &= [\,0.0821 \;\; -0.0566\,], \\ V_{N-3}^{R_{23}} &= [\,-0.2257 \;\; -0.0879\,], \end{aligned} \tag{24c}$$

$$V_{N-3}^{S_1} = \begin{bmatrix} 0.1066 & -0.0538 \\ -0.0718 & 0.0369 \end{bmatrix}, \quad V_{N-3}^{S_2} = \begin{bmatrix} 0.0985 & -0.0477 \\ -0.0748 & 0.0381 \end{bmatrix}, \quad V_{N-3}^{S_3} = \begin{bmatrix} 0.7603 & -0.3529 \\ -0.4818 & 0.2451 \end{bmatrix}. \tag{24d}$$

Step 3. Using (17) for $q = N = 3$ and (21), (24a) we determine

$$u_0 = -2.2429 \,. \tag{25}$$

Knowing u_0 and x_0 from (1) for $k = 0$ we determine

$$x_1 = \begin{bmatrix} -0.0929 \\ 0.1642 \end{bmatrix}. \tag{26}$$

Using (16) and (21), (24b)-(24d) we determine the minimum value of the performance index as

$$J_0(\Sigma x_0) = 8.7699 \,. \tag{27}$$

Continuing this procedure we can determine subsequent discrete values of control as

$$u_1 = -0.2662 \,, \quad u_2 = -0.0386 \,. \tag{28}$$

The values of state vector are given as

$$x_2 = \begin{bmatrix} -0.0147 \\ 0.0152 \end{bmatrix}, \quad x_3 = \begin{bmatrix} -0.0106 \\ 0.0114 \end{bmatrix}. \tag{29}$$

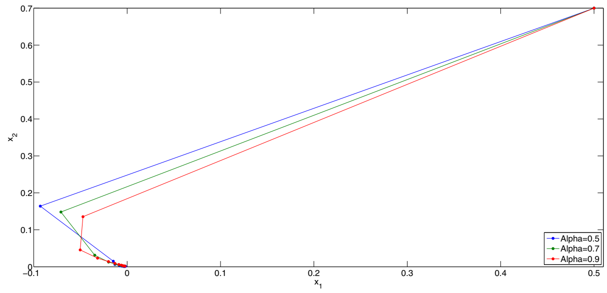

Fig. 1. Optimal trajectory for $\alpha = 0.5, 0.7, 0.9$ and $N = 10$.

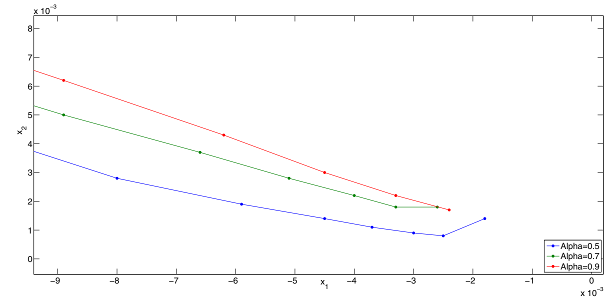

Fig. 2. Optimal trajectory for $\alpha = 0.5, 0.7, 0.9$ and $N = 10$ (zoom).

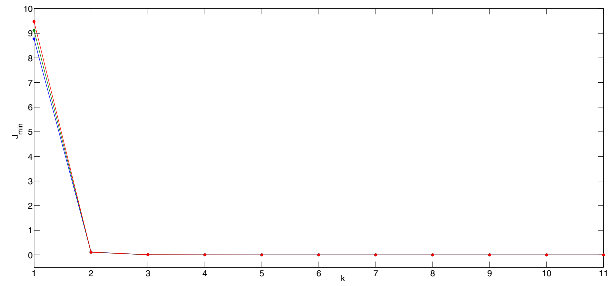

Fig. 3. The minimum values of the performance index for $\alpha = 0.5, 0.7, 0.9$ and $N = 10$.

The minimum values of the performance index are given as

$$J_1(\Sigma x_1) = 0.1193 \,, \quad J_2(\Sigma x_2) = 0.0027 \,, \tag{30}$$

and

$$J_3(u) = G(x_3) = 0.0007 \,. \tag{31}$$

The figures Fig. 1-6 show the above considerations for the system (1) with matrices (18) and the performance index (2) with matrices (19) for three different values of $\alpha = 0.5, 0.7, 0.9$, and the number of elementary sections of the optimal trajectory $N = 10$. Individual results were obtained with the help of written for that purpose computer algorithm implementing the above issues.

5. RELATION WITH INTEGER-ORDER SYSTEMS THEORY

We shall show that the above considerations for fractional discrete-time systems for $\alpha = 1$ are identical to the result for classical discrete-time systems.

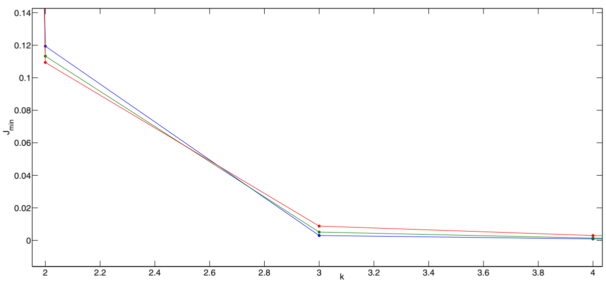

Fig. 4. The minimum values of the performance index for $\alpha = 0.5, 0.7, 0.9$ and $N = 10$ (zoom).

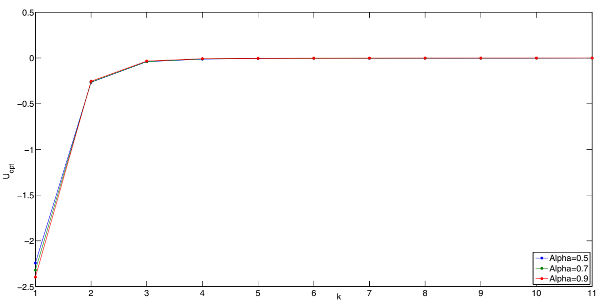

Fig. 5. Optimal control for $\alpha = 0.5, 0.7, 0.9$ and $N = 10$.

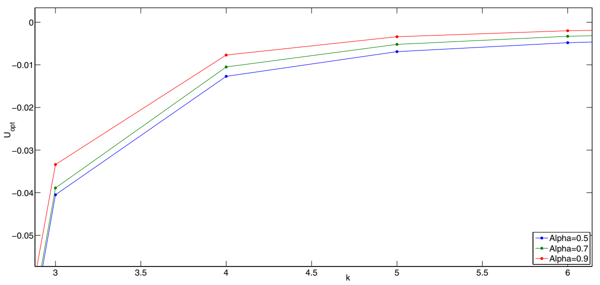

Fig. 6. Optimal control for $\alpha = 0.5, 0.7, 0.9$ and $N = 10$ (zoom).

Equations describing fractional discrete-time system (1) for $\alpha = 1$ take the form of a classical discrete-time system, i.e.

$$x_{k+1} = A_d x_k + B u_k \ , \tag{32}$$

since the coefficients d_j for $\alpha = 1$ have the form

$$d_0 = A_\alpha = A + I_n = A_d \ , \ d_j = \mathbb{O} \ , \ j = 1, \ldots, k \ . \tag{33}$$

In this case the relationship (16), for q last sections of the optimal trajectory, minimizing the performance index (2) with constraints (32) is given as

$$\begin{aligned} S_{N-q}(x_{N-q}) &= x_{N-q}^T Q x_{N-q} \\ &+ \sum_{l=0}^{q-2} \left\{ \left[V_{N-q}^{Q_{l,1}} d_0 x_{N-q} \right]^T Q \left[V_{N-q}^{Q_{l,1}} d_0 x_{N-q} \right] \right\} \\ &+ \sum_{r=0}^{q-1} \left\{ \left[V_{N-q}^{R_{r,1}} d_0 x_{N-q} \right]^T R \left[V_{N-q}^{R_{r,1}} d_0 x_{N-q} \right] \right\} \\ &+ \left[V_{N-q}^{S_1} d_0 x_{N-q} \right]^T S \left[V_{N-q}^{S_1} d_0 x_{N-q} \right] . \end{aligned} \tag{34}$$

where

$$V_{N-q}^{Q_{01}} = I_n + B W_{N-q}^1 \ , \quad V_{N-q}^{Q_{l,1}} = V_{N-q+1}^{Q_{l-1,1}} d_0 V_{N-q}^{Q_{01}} \ , \tag{35a}$$

$$V_{N-q}^{R_{01}} = W_{N-q}^1 \ , \quad V_{N-q}^{R_{r,1}} = V_{N-q+1}^{R_{r-1,1}} d_0 V_{N-q}^{Q_{01}} \ , \tag{35b}$$

$$V_{N-1}^{S_1} = I_n + B W_{N-1}^1 \ , \quad V_{N-q}^{S_1} = V_{N-q+1}^{S_1} d_0 V_{N-q}^{Q_{01}} \ . \tag{35c}$$

Substituting repeatedly (35), for the indexes $N-q, \ldots, N-1$, we can write the equation (34) as

$$\begin{aligned} S_{N-q}(x_{N-q}) &= x_{N-q}^T \left\{ Q + \left(V_{N-q}^{R_{01}} d_0 \right)^T R \left(V_{N-q}^{R_{01}} d_0 \right) \right. \\ &+ \left(V_{N-q}^{Q_{01}} d_0 \right)^T \left[Q + \left(V_{N-q+1}^{R_{01}} d_0 \right)^T R \left(V_{N-q+1}^{R_{01}} d_0 \right) \right. \\ &+ \left(V_{N-q+1}^{Q_{01}} d_0 \right)^T \left[\cdots + \left(V_{N-2}^{Q_{01}} d_0 \right)^T \left[Q + \left(V_{N-1}^{R_{01}} d_0 \right)^T R \right. \right. \\ &\times \left(V_{N-1}^{R_{01}} d_0 \right) + \left. \left(V_{N-1}^{Q_{01}} d_0 \right)^T S \left(V_{N-1}^{Q_{01}} d_0 \right) \right] \left(V_{N-2}^{Q_{01}} d_0 \right) \Big] \cdots \Big] \\ &\times \left. \left(V_{N-q-1}^{R_{01}} d_0 \right) \right] \left(V_{N-q}^{R_{01}} d_0 \right) \Big\} x_{N-q} \ . \end{aligned}$$

Denoting as $S = S_N$ we can write the above relationship in the form

$$\begin{aligned} S_{N-q} &= x_{N-q}^T \left\{ Q + \left(V_{N-q}^{R_{01}} d_0 \right)^T R \left(V_{N-q}^{R_{01}} d_0 \right) \right. \\ &\quad \left. + \left(V_{N-q}^{Q_{01}} d_0 \right)^T S_{N-q+1} \left(V_{N-q}^{R_{01}} d_0 \right) \right\} x_{N-q} \ , \end{aligned} \tag{36}$$

which, after adoption of proper notation is the same as shown in Naidu (2002); Lewis and Syrmos (1995).

The control sequence (17), which minimize $J_{N-q}(u)$, in this case is given as

$$u_{N-q} = W_{N-q}^1 d_0 x_{N-q} \ , \tag{37}$$

where

$$\begin{aligned} W_{N-q}^1 &= - \left\{ R + R^T + B^T \left[Q + Q^T \right] B \right. \\ &+ \sum_{w=0}^{q-3} \left[\left(V_{N-q+1}^{Q_{w,1}} d_0 B \right)^T \left[Q + Q^T \right] \left(V_{N-q+1}^{Q_{w,1}} d_0 B \right) \right] \\ &+ \sum_{z=0}^{q-2} \left[\left(V_{N-q+1}^{R_{z,1}} d_0 B \right)^T \left[R + R^T \right] \left(V_{N-q+1}^{R_{z,1}} d_0 B \right) \right] \\ &+ \left. \left(V_{N-q+1}^{S_1} d_0 B \right)^T \left[S + S^T \right] \left(V_{N-q+1}^{S_1} d_0 B \right) \right\}^{-1} \\ &\times \left\{ B^T \left[Q + Q^T \right] \right. \\ &+ \sum_{w=0}^{q-3} \left[\left(V_{N-q+1}^{Q_{w,1}} d_0 B \right)^T \left[Q + Q^T \right] \left(V_{N-q+1}^{Q_{w,1}} d_0 \right) \right] \\ &+ \sum_{z=0}^{q-2} \left[\left(V_{N-q+1}^{R_{z,1}} d_0 B \right)^T \left[R + R^T \right] \left(V_{N-q+1}^{R_{z,1}} d_0 \right) \right] \\ &+ \left. \left(V_{N-q+1}^{S_1} d_0 B \right)^T \left[S + S^T \right] \left(V_{N-q+1}^{S_1} d_0 \right) \right\} , \end{aligned}$$

Substituting repeatedly (35), for the indexes $N-q, \ldots, N-1$, we can write the above relationship as

$$\begin{aligned} W_{N-q}^1 &= - \left\{ R + R^T + B^T \left[\left(Q + Q^T \right) \right. \right. \\ &+ \left(V_{N-q+1}^{R_{01}} d_0 \right)^T \left(R + R^T \right) \left(V_{N-q+1}^{R_{01}} d_0 \right) + \left(V_{N-q+1}^{Q_{01}} d_0 \right)^T \\ &\times \left[\left(Q + Q^T \right) + \left(V_{N-q+2}^{R_{01}} d_0 \right)^T \left(R + R^T \right) \left(V_{N-q+2}^{R_{01}} d_0 \right) \right. \end{aligned}$$

$$+\left(V^{Q_{01}}_{N-q+2}d_0\right)^T\left[\cdots+\left(V^{Q_{01}}_{N-2}d_0\right)\left[(Q+Q^T)+\left(V^{R_{01}}_{N-1}d_0\right)^T\right.\right.$$
$$\times\left(R+R^T\right)\left(V^{R_{01}}_{N-1}d_0\right)+\left(V^{Q_{01}}_{N-1}d_0\right)^T\left(S+S^T\right)\left(V^{Q_{01}}_{N-1}d_0\right)\Big]$$
$$\left.\left.\times\ \left(V^{Q_{01}}_{N-2}d_0\right)\right]\cdots\right]\left(V^{Q_{01}}_{N-q+1}d_0\right)\Big]\left(V^{Q_{01}}_{N-q+1}d_0\right)\Big]B\Big\}^{-1}$$
$$\times\Big\{B^T\left[(Q+Q^T)+\left(V^{R_{01}}_{N-q+1}d_0\right)^T\left(R+R^T\right)\left(V^{R_{01}}_{N-q+1}d_0\right)\right.$$
$$+\left(V^{Q_{01}}_{N-q+1}d_0\right)^T\left[(Q+Q^T)+\left(V^{R_{01}}_{N-q+2}d_0\right)^T\left(R+R^T\right)\right.$$
$$\times\left(V^{R_{01}}_{N-q+2}d_0\right)+\left(V^{Q_{01}}_{N-q+2}d_0\right)^T\left[\cdots+\left(V^{Q_{01}}_{N-2}d_0\right)\right.$$
$$\times\left[(Q+Q^T)+\left(V^{R_{01}}_{N-1}d_0\right)^T\left(R+R^T\right)\left(V^{R_{01}}_{N-1}d_0\right)\right.$$
$$\left.+\left(V^{Q_{01}}_{N-1}d_0\right)^T\left(S+S^T\right)\left(V^{Q_{01}}_{N-1}d_0\right)\right]$$
$$\left.\left.\times\ \left(V^{Q_{01}}_{N-2}d_0\right)\right]\cdots\right]\left(V^{Q_{01}}_{N-q+1}d_0\right)\Big]\left(V^{Q_{01}}_{N-q+1}d_0\right)\Big]B\Big\}\ .$$

Denoting as $S = S_N$ and taking into account (36) we can write the above relationship in the form

$$W^1_{N-q} = -\left\{\left(R+R^T\right)+B^T\left(S_{N-q+1}+S^T_{N-q+1}\right)B\right\}^{-1} \times B^T\left(S_{N-q+1}+S^T_{N-q+1}\right)\ , \tag{38}$$

which, after adoption of proper notation is the same as shown in Naidu (2002); Lewis and Syrmos (1995).

6. CONCLUSION

Dynamical programming problem for fractional discrete-time systems with quadratic performance index has been formulated and solved. A new method for numerical computation of optimal dynamic programming problem has been presented. The efficiency of the method has been demonstrated on numerical example and illustrated by graphs. A link to the classical theory has been demonstrated. The differences between the fractional and classical (standard) systems theory have been shown. A computer algorithm for solving dynamic programming problems with quadratic performance index for fractional discrete-time systems has been tested for different cases of coefficient alpha. Detailed description of a computer algorithm can be found in Dzielinski and Czyronis (2013).

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