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Boundary Energy-Shaping Control of the Shallow Water Equation

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Abstract: The aim of this paper is to apply new results on the boundary stabilisation via energy-shaping of distributed port-Hamiltonian systems to a nonlinear PDE, i.e. a slightly simplified formulation of the shallow water equation. Usually, stabilisation of non-zero equilibria via energy-balancing has been achieved by looking at, or generating, a set of structural invariants (Casimir functions), in closed-loop. This approach is not successful in case of the shallow water equation because at the equilibrium the regulator is supposed to supply an infinite amount of energy (dissipation obstacle). In this paper, it is shown how to construct a controller that behaves as a state-modulated boundary source and that asymptotically stabilises the desired equilibrium. The proposed approach relies on a parametrisation of the dynamics provided by the image representation of the Dirac structure associated to the distributed port-Hamiltonian system. In this way, the effects of the boundary inputs on the state evolution are explicitly shown, and as a consequence the boundary control action that maps the open-loop system into a target one characterised by the desired stability properties, i.e. by a "new" Hamiltonian with an isolated minimum at the equilibrium, is determined.

Keywords: distributed port-Hamiltonian systems, passivity-based control, stability of distributed parameter systems

1. INTRODUCTION

This paper deals with the energy-based boundary control of a simplified version of the shallow water equation formulated as a distributed port-Hamiltonian system, van der Schaft and Maschke [2002], Macchelli and Maschke [2009]. For such class of infinite dimensional systems, this task has been usually accomplished by looking at, or generating, a set of Casimir functions in closed-loop that robustly (i.e., independently from the Hamiltonian function) relates the state of the infinite dimensional port-Hamiltonian system with the state of the controller. The controller is a finite dimensional port-Hamiltonian system which is interconnected to the boundary of the distributed parameter system. The shape of the closed-loop energy function is changed by choosing the Hamiltonian of the controller for example to introduce a minimum in a desired configuration. Examples can be found e.g. in Rodriguez et al. [2001], Macchelli and Melchiorri [2004, 2005], Pasumarthy and van der Schaft [2007], Macchelli [2012a,b]. The result is an energy-balancing passivity-based controller that is not able to deal with equilibria that require an infinite amount of supplied energy in steady state, i.e. with the "dissipation obstacle."

The limits of the energy-Casimir method are intrinsic, and due to the fact that Casimir functions are invariants that do not depend on the particular Hamiltonian and on the resistive structure, i.e. they are completely determined by the Dirac structure of the system, van der Schaft [2000], Pasumarthy and van der Schaft [2007]. The class of controllers can be enlarged beyond the dissipation obstacle by focusing on the trajectories that correspond to a particular Hamiltonian, rather than on the geometric structure of the system only. Then, the regulator is developed to map the open-loop trajectories into the trajectories of a target system with (at least) a different Hamiltonian and, clearly, characterised by the desired stability properties. This is the same concept adopted for finite dimensional in case of stabilisation with state-modulated sources in Ortega et al. [2001], or with the more general IDA-PBC control technique in Ortega et al. [2002].

The starting point are the definition of Dirac structures on Hilbert spaces, and their kernel and image representations proposed in Iftime and Sandovici [2011]. The latter representation, in particular, provides a natural way to parametrize the dynamics of the system and to relate the effect of boundary inputs on the evolution of the state. As in the lumped parameter case, the control action is then determined to map the open-loop dynamical system into a "target" one, which is characterised by the same Dirac structure, but with a different Hamiltonian, now selected in order to have an isolated minimum at the desired equilibrium. Asymptotic stability is obtained by (boundary) damping injection, if needed. In this way, the controller can be interpreted as a state-modulated source, thus able to deal with equilibria that require a non-zero supplied power in steady state. The methodology has been presented in Macchelli [2013a,b] with reference to a simple linear example, namely a lossless transmission line with RLC load. This paper shows that the same techniques are applicable also to nonlinear PDEs. Asymptotic stability for the closed-loop system is then proved via La Salle's Invariance Principle arguments, see Luo et al. [1999].

This paper is organised as follows. In Sect. 2, a brief background on the distributed port-Hamiltonian formulation of the shallow water equation, and on Dirac structures on Hilbert spaces is given. Then, the boundary energy-shaping control is illustrated in Sect. 3. At first, in Sect. 3.1, the case in which friction forces are neglected is considered: the result is an energy-balancing controller for which an interpretation in terms of Casimir functions is given. The general case is treated in Sect. 3.2, for which an explicit expression of the control law is provided in the simplified case of linear friction forces. Concluding remarks are in Sect. 4.

2. BACKGROUND

2.1 Shallow water equation in port-Hamiltonian form

Let us consider a rectangular open channel with a single, flat reach, with length L and unitary width, which is delimited by upstream and downstream gates, and terminated by an hydraulic outfall. Moreover, it is assumed that the fluid has a unitary density. For simplicity, we are in fact considering a simpler situation than the one treated in Hamroun et al. [2010]. However, all the results discussed here can be easily extended to cope with more general cases. The dynamics of the system are described by the so-called shallow water equations, whose port-Hamiltonian formulation has been extensively discussed e.g. in Pasumarthy et al. [2008], Hamroun et al. [2010].

In this respect, denote by $Z = [0, L]$ the spatial domain, and by $q(z,t) > 0$ and $p(z,t)$ the infinitesimal volume and kinetic momentum density, respectively. These are the state (energy) variables. Note that, due to the unitary width and fluid density assumptions, these quantities are numerically equal to the height of the fluid in the channel and to its velocity. The distributed port-Hamiltonian formulation of the shallow water equations is

$$\begin{aligned}\frac{\partial}{\partial t}\begin{pmatrix} q \\ p \end{pmatrix} &= \\ &= \left\{\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\frac{\partial}{\partial z} - \begin{pmatrix} 0 & 0 \\ 0 & D(q,p) \end{pmatrix}\right\}\begin{pmatrix} \delta_q H(q,p) \\ \delta_p H(q,p) \end{pmatrix} \end{aligned} \tag{1}$$

where δ denotes the variational derivative (e.g., refer to van der Schaft and Maschke [2002]), and H is the total energy of the fluid, that is given by

$$H(q,p) = \frac{1}{2}\int_0^L \left(qp^2 + gq^2\right) \mathrm{d}z$$

being g the gravity acceleration. Moreover, in (1), $D(q,p) \geq 0$ is the dissipation term associated to the friction forces, usually modelled by highly nonlinear empirical constitutive formula (e.g., Manning–Strickler). Note that

$$\frac{\delta H}{\delta q}(q,p) = \frac{1}{2}p^2 + gq =: P \qquad \frac{\delta H}{\delta p}(q,p) = qp =: Q$$

are the co-energy variables, which are equal to the hydrodynamic pressure P and water flow Q, respectively.

The channel whose dynamic is described by the PDE (1) exchanges power with the environment through a couple of power ports defined in $z = 0$ and $z = L$:

$$\begin{aligned}\big(f_0(t), e_0(t)\big) &= \left(\frac{\delta H}{\delta p}(0,t),\, \frac{\delta H}{\delta q}(0,t)\right) \\ \big(f_L(t), e_L(t)\big) &= \left(-\frac{\delta H}{\delta p}(L,t),\, \frac{\delta H}{\delta q}(L,t)\right)\end{aligned} \tag{2}$$

With simple computations, it is possible to verify that the following energy-balance relation is satisfied:

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= -\int_0^L D(q,p)\left(\frac{\delta H}{\delta p}\right)^2 \mathrm{d}z + e_0 f_0 + e_L f_L \\ &\leq e_0 f_0 + e_L f_L\end{aligned} \tag{3}$$

2.2 A class of distributed port-Hamiltonian systems

The distributed port-Hamiltonian system (1) with boundary ports (2) belongs to the following class of systems:

$$\frac{\partial x}{\partial t}(t,z) = \left\{P_1\frac{\partial}{\partial z} + \big[P_0 - G_0(x(t,z))\big]\right\}\frac{\delta H}{\delta x}(x(t,z)) \tag{4}$$

Such class generalizes what has been presented in Le Gorrec et al. [2005] as far as the linear case is concerned. Here, the spatial domain is $Z = [a, b]$ and $x \in L_2(a, b; \mathbb{R}^n)$ denotes the state (energy) variable. Moreover, $P_1 = P_1^{\mathrm{T}} > 0$, $P_0 = -P_0^{\mathrm{T}}$, and $G_0(x) = G_0^{\mathrm{T}}(x) \geq 0$, while H is the total energy of the system, that is not necessarily quadratic in the energy variables. Note that the entries in G_0 can be non-linear.

To define a distributed port-Hamiltonian system, the PDE (4) has to be "completed" with a boundary port. In this respect, the boundary port variables associated to (4) are the vectors f_∂, $e_\partial \in \mathbb{R}^n$ defined by

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix}}_{=:R}\begin{pmatrix} \delta_x H(b) \\ \delta_x H(a) \end{pmatrix} \tag{5}$$

which turn out to be a linear combination of the restrictions on the boundary of the spatial domain of the co-energy variables. To have a so-called boundary control system e.g. in the sense of Curtain and Zwart [1995], inputs and outputs have to be defined. From Le Gorrec et al. [2005], a simple procedure to have system (4) in impedance form is the following. Let W and $\tilde{W}$ a pair of $n \times 2n$ full rank real matrices, such that $(W^{\mathrm{T}}\ \tilde{W}^{\mathrm{T}})$ is invertible, and

$$W\Sigma W^{\mathrm{T}} = 0 \qquad W\Sigma\tilde{W}^{\mathrm{T}} = I \qquad \tilde{W}\Sigma\tilde{W}^{\mathrm{T}} = 0$$

being

$$\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

The (boundary) input u and output y can be defined as

$$u(t) = W\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \qquad y(t) = \tilde{W}\begin{pmatrix} f_\partial(t) \\ e_\partial(t) \end{pmatrix} \tag{6}$$

and it is easy to prove that the following energy balance equation is satisfied:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -\int_a^b G_0(x)\left(\frac{\delta H}{\delta x}(x)\right)^2 \mathrm{d}z + y^{\mathrm{T}}u \leq y^{\mathrm{T}}u$$

Note the similarities with (3), as expected.

As discussed in Macchelli [2013a,b] in the linear case, the distributed port-Hamiltonian system (4) is characterised by a constant Dirac structure $\mathcal{D}$ on the space of flows

$$\mathcal{F} = \mathcal{F}_S \times \mathcal{F}_R \times \mathcal{F}_C$$

with $\mathcal{F}_S = L_2(a, b; \mathbb{R}^n)$, $\mathcal{F}_R = L_2(a, b; \mathbb{R}^r)$, and $\mathcal{F}_C = \mathbb{R}^n$, being $r = \operatorname{rank} G_0$, supposed constant. Refer e.g. to van der Schaft [2000] for a rigorous definition in the lumped parameter case. For simplicity, it is assumed that the space of efforts $\mathcal{E}$ is equal to the space of flows, i.e. $\mathcal{E} \equiv \mathcal{F}$. Here, (f_S, e_S) represets the energy-storage port, (f_R, e_R) the dissipative port, and $(f_C, e_C) \equiv (y, u)$ the control port, that is assumed with effort-in causality, and is responsible for the interaction between system and "environment," e.g. a (boundary) controller. The Dirac structure admits the kernel representation

$$\mathcal{D} = \Big\{ (f, e) \in \mathcal{F} \times \mathcal{E} \ \mid \ Ff + Ee = 0 \Big\} \tag{7}$$

where $F : \mathcal{F} \to \Lambda$ and $E : \mathcal{E} \to \Lambda$ the linear operators

$$F = (F_S \ F_R \ F_C) \qquad E = (E_S \ E_R \ E_C) \tag{8}$$

where

$$\begin{gathered} F_S = \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad F_R = \begin{pmatrix} 0 \\ I \\ 0 \\ 0 \end{pmatrix} \qquad F_C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ I \end{pmatrix} \\ E_S = \begin{pmatrix} P_1 \dfrac{\partial}{\partial z} + P_0 \\ -G_R^{\mathrm{T}} \\ -W R \mathcal{B} \\ -\tilde{W} R \mathcal{B} \end{pmatrix} \quad E_R = \begin{pmatrix} G_R \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad E_C = \begin{pmatrix} 0 \\ 0 \\ I \\ 0 \end{pmatrix} \end{gathered} \tag{9}$$

being

$$\mathcal{B}(e) = \begin{pmatrix} e(b) \\ e(a) \end{pmatrix}, \quad e \in L_2(a, b; \mathbb{R}^n)$$

and G_R a matrix of proper dimensions such that $\operatorname{rank} G_R = r$, and there exists a function $\bar{G}(x) \geq 0$ for which $G_0(x) = G_R^{\mathrm{T}} \bar{G}(x) G_R$. As discussed in Iftime and Sandovici [2011], Λ is a "supporting" space that is required to be isometrically isomorphic to $\mathcal{F}$. The simplest choice is

$$\Lambda = L_2(a, b; \mathbb{R}^n) \times L_2(a, b; \mathbb{R}^r) \times \{0\} \times \mathbb{R}^n \tag{10}$$

where $\{0\} \subset \mathbb{R}^n$ is the set that contains the null vector of $\mathbb{R}^n$. From (10), we have that

$$\begin{aligned} \operatorname{dom}(F \ E) = \Big\{ & (f, e) \in \mathcal{F} \times \mathcal{E} \ \mid \ e_S \text{ abs. continuous}, \\ & \frac{\partial e_S}{\partial z} \in L_2(a, b; \mathbb{R}^n), \text{ and } e_C = W R \mathcal{B}(e_S) \Big\} \end{aligned}$$

while it is quite easy to prove that

$$\operatorname{dom} \begin{pmatrix} F^* \\ E^* \end{pmatrix} = \Big\{ \lambda \in \Lambda \ \mid \ \lambda_u = W R \mathcal{B}(\lambda_S) \Big\} \tag{11}$$

where $\lambda = (\lambda_S, \lambda_R, 0, \lambda_u)$, and $F_S^* = F_S^{\mathrm{T}}$, $F_R^* = F_R^{\mathrm{T}}$, $F_C^* = F_C^{\mathrm{T}}$, $E_R^* = E_R^{\mathrm{T}}$, and

$$\begin{aligned} E_S^* &= \left(-P_1 \frac{\partial}{\partial z} - P_0 \ -G_R \ 0 \ 0 \right) \\ E_C^* &= \left(\tilde{W} R \mathcal{B} \ 0 \ 0 \ 0 \right) \end{aligned}$$

More details in Macchelli [2013b]. Note that from (7) and (8), the distributed port-Hamiltonian system (4) is obtained once the following behaviour at the energy-storage and dissipative ports is imposed:

$$f_S = -\frac{\partial x}{\partial t} \qquad e_S = \frac{\delta H}{\delta x} \qquad e_R = -\bar{G}(x) f_R$$

3. BOUNDARY CONTROL VIA ENERGY-SHAPING

In this section, energy-shaping methodologies are applied to the boundary stabilisation of the shallow water equation (1). The controller acts on the boundary ports (2) with the following causality:

$$u(t) = \begin{pmatrix} f_0(t) \\ e_L(t) \end{pmatrix} \qquad y(t) = \begin{pmatrix} e_0(t) \\ f_L(t) \end{pmatrix} \tag{12}$$

This choice is coherent with the most general case (6), thanks to a proper choice of the matrices W and $\tilde{W}$. To stabilise (1), it is required to determine a state-feedback law in the form

$$u = \beta(q, p) + u' \tag{13}$$

being u' an auxiliary input signal, such that the equilibrium $(q_\star, p_\star)$ is asymptotically stable. Such equilibrium is solution of

$$\begin{aligned} \frac{\partial}{\partial z} \frac{\delta H}{\delta p}(q_\star, p_\star) &= 0 \\ \frac{\partial}{\partial z} \frac{\delta H}{\delta q}(q_\star, p_\star) + D(q_\star, p_\star) \frac{\delta H}{\delta p}(q_\star, p_\star) &= 0 \end{aligned} \tag{14}$$

or, equivalently, of

$$\frac{\partial}{\partial z} Q_\star = 0 \qquad \frac{\partial}{\partial z} P_\star + D(q_\star, p_\star) Q_\star = 0 \tag{15}$$

where $Q_\star = q_\star p_\star$ and $P_\star = \frac{1}{2} p_\star^2 + g q_\star$. Another problem, treated e.g. in Hamroun et al. [2010] but in the finite dimensional case only, is to stabilise the channel around a desired water flow $\delta_p H(q_\star, p_\star) = Q_\star$, somehow independently from the hydrodynamic pressure $P_\star$, which is not constant on Z in case internal friction forces are present, i.e. when $D(q, p) \neq 0$.

With reference to the more general case (4), the control problem can be stated as follows: determine the (boundary) control action $u = \beta(x) + u'$ that maps the port-Hamiltonian system (4) into

$$\frac{\partial x}{\partial t}(t, z) = \left\{ P_1 \frac{\partial}{\partial z} + \big[P_0 - G_0(x(t, z)) \big] \right\} \frac{\delta H_d}{\delta x}(x(t, z)) \tag{16}$$

i.e., into a desired systems that is characterised by the same Dirac structure and resistive relation of the original one, but with a different Hamiltonian $H_d(x) = H(x) + H_a(x)$. By following Macchelli [2013a,b] with just minor modifications, the class of functions H_a that can be employed in the energy-shaping procedure, together with the corresponding control action β, are solution of:

$$\begin{pmatrix} 0 \\ \dfrac{\delta H_a}{\delta x}(x) \\ 0 \\ -\beta(x) \end{pmatrix} = \begin{pmatrix} E_S^* \\ F_S^* \\ \bar{G}(x) E_R^* + F_R^* \\ F_C^* \end{pmatrix} \lambda(x) \tag{17}$$

where λ is an element of (11). The physical interpretation of this relation is that we are looking for a state-dependent control action β that is able to "generate" part of the dynamics that is missing in the open-loop system and that is preventing to match the desired one.

In the remaining part of this section, (17) is solved with reference to the shallow water equation (1), where the boundary input is given as in (12). In this respect, the matrices appearing in the associated Dirac structure defined by (8) and (9) can be easily computed.

3.1 First case: $D(q, p) = 0$

If the internal friction forces are neglected, the dissipative port (f_R, e_R) in the Dirac structure is not present. Beside the trivial simplification in the operators defined in (9), the supporting space Λ introduced in (10) is now

$$\Lambda = L_2(0, L; \mathbb{R}^2) \times \{0\} \times \mathbb{R}^2 \tag{18}$$

where $\{0\} \subset \mathbb{R}^2$. Given $\lambda = (\lambda_q, \lambda_p, 0, 0, \lambda_{u0}, \lambda_{uL})$, in this particular case (11) implies that $\lambda_{u0} = \lambda_p(0)$ and $\lambda_{uL} = \lambda_q(0)$. Consequently, from (17) we have that

$$\frac{\partial}{\partial z}\begin{pmatrix}\lambda_q\\ \lambda_p\end{pmatrix} = 0 \qquad \begin{pmatrix}\delta_q H_a\\ \delta_p H_a\end{pmatrix} = \begin{pmatrix}\lambda_q\\ \lambda_p\end{pmatrix} \qquad -\beta = \begin{pmatrix}\lambda_p(0)\\ \lambda_q(L)\end{pmatrix}$$

that implies

$$\begin{aligned}\frac{\partial}{\partial z}\frac{\delta H_a}{\delta q}(q,p) &= 0\\ \frac{\partial}{\partial z}\frac{\delta H_a}{\delta p}(q,p) &= 0\\ \beta(q,p) &= -\begin{pmatrix}\delta_p H_a(q,p)\,|_{z=0}\\ \delta_q H_a(q,p)\,|_{z=L}\end{pmatrix}\end{aligned} \tag{19}$$

This means that the class of functions H_a is such that

$$H_a(q,p) = \hat{H}_a\big(\xi_1(q,p), \xi_2(q,p)\big) \tag{20}$$

with

$$\xi_1(q) = \int_0^L q(z)\,\mathrm{d}z \qquad \xi_2(p) = \int_0^L p(z)\,\mathrm{d}z \tag{21}$$

and $\hat{H}_a$ that can be freely chosen.

From (14) or (15), it is easy to find out that the equilibrium configuration is given by

$$q(t,z) = q_\star > 0 \qquad\qquad p(t,z) = p_\star \tag{22}$$

which means constant water level and velocity along the channel or, equivalent, constant hydrodynamic pressure $P_\star$ and water flow $Q_\star$. In order to have in closed-loop a port-Hamiltonian system (16) with Hamiltonian H_d with a minimum in (22), a possible choice of $\hat{H}_a$ in (20) is

$$\begin{aligned}\hat{H}_a(\xi_1,\xi_2) = &\frac{1}{2}K_1(\xi_1 - \xi_{1\star})^2 + \frac{1}{2}K_2(\xi_2 - \xi_{2\star})^2\\ &- \underbrace{\left(\frac{1}{2}p_\star^2 + g q_\star\right)}_{\equiv P_\star}\xi_1 - \underbrace{q_\star p_\star}_{\equiv Q_\star}\xi_2\end{aligned} \tag{23}$$

where $\xi_{1\star}$ and $\xi_{2\star}$ are the values of ξ_1 and ξ_2 at the equilibrium, i.e. $\xi_{1\star} = L q_\star$, and $\xi_{2\star} = L p_\star$, while K_1, K_2 are two positive gains. Note the similarities with Hamroun et al. [2010], where a finite-element approximation of the channel dynamics has been used.

In this respect, the finite dimensional formulation of the shallow water equation adopted in Hamroun et al. [2010] follows from the general discretisation technique for distributed port-Hamiltonian systems discussed in Golo et al. [2004]. It has been shown in Macchelli [2011] that for the "full-order" system, i.e. the distributed parameter one, there exists an energy-shaping controller based on the energy-Casimir method if and only if the same kind of controller exists for the corresponding spatially discretised system. This is clearly true also for the shallow water equation. In this respect, let us consider the following control system in port-Hamiltonian form:

$$\begin{cases}\dot{\xi} = J_C \dfrac{\partial H_C}{\partial \xi}(\xi) + u_C\\ y_C = \dfrac{\partial H_C}{\partial \xi}(\xi)\end{cases} \tag{24}$$

where now $\xi \in \mathbb{R}^2$ denotes the state variable of the controller, $J_C = -J_C^{\mathrm{T}}$ and H_C is the to-be-assigned Hamiltonian. Let us interconnect (24) at the boundary ports (12) of the shallow water equation in power conserving way, i.e. $u = -y_C + u'$ and $u_C = y$, being u' an auxiliary input, as in (13). The energy-Casimir method requires to select J_C in (24) such that there exists a set of functions $C(\xi, q, p)$ that are constant along system trajectories for every possible Hamiltonians H and H_C. In particular, we are looking for invariants in the form $C(\xi, q, p) = \xi - F(q,p)$, being F some functional to be determined. If it is the case, the closed-loop Hamiltonian becomes $H_{CL}(q,p) = H(q,p) + H_C(F(q,p) + \kappa)$, with κ some constant that depends on the initial conditions only, i.e. it is at the end just function of the state variables of the plant, and it can be properly shaped by acting on the controller Hamiltonian H_C.

It is quite easy to check that Casimir functions are not present in closed-loop if $J_C = 0$. From a physical point of view, this result is obvious. With this choice, in fact, the boundary controller (24) consists of two separate systems, each required to provide a constant power flow in steady state. So, they are not energy-balancing controllers. So, it is necessary to couple these regulators and allow for an internal power flow at the controller side. This can be achieved by choosing

$$J_C = \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}$$

which implies that the closed loop system is characterised by the following Casimir functions:

$$\begin{aligned}C_1(\xi_1, q) &= \xi_1 - \int_0^L q(z)\,\mathrm{d}z\\ C_2(\xi_2, p) &= \xi_2 - \int_0^L p(z)\,\mathrm{d}z\end{aligned}$$

Note the similarities with (21). As a consequence, the class of functions H_C that can be employed in the energy shaping are the same as in (20), and also the control action is exactly the same determined in (19) by relying on energy-balancing considerations. In both the cases, the closed-loop system is lossless, so only simple stability has been achieved e.g. in the sense of Swaters [2000]. However, asymptotic stability can be obtained by damping injection at the boundary. This point is not investigated here due to space limitations, but the rationale can be deduced from the case treated in the next section.

3.2 Second case: $D(q,p) \neq 0$

To introduce the effect of the friction forces, the dissipative port (f_R, e_R) in the Dirac structure (7) must be present and "terminated" on a proper resistive element. If in (9) we assume that $G_R = 1$ since $r = 1$, we can write that $e_R = -D(q,p) f_R$. At first, it is immediate to prove that, if the aim is to stabilise the channel around some desired water flow value $Q_\star = q_\star p_\star$ without any particular requirement on the hydrodynamic pressure, the control action (13) with β given in (19) is able to accomplish this task, provided that $K_2 = 0$ in (23). In fact, due to the boundary control action, the closed-loop system satisfies the PDE (1), but with the shaped Hamiltonian $H_d = H + H_a$ in place of H, and with the boundary input

$$u' = \begin{pmatrix} \delta_p H_d(0) \\ \delta_q H_d(L) \end{pmatrix} = \begin{pmatrix} Q(0) - Q_\star \\ P(L) - P_\star + K_1(\xi_1 - \xi_{1\star}) \end{pmatrix}$$

that is equal to zero. Then, from the energy-balance relation (3), we have that energy decreases until in steady state $\delta_p H_d(q,p) = 0$ on Z, which implies that the steady state configuration with $Q = qp = Q_\star$ is reached. A similar result holds with $K_2 \neq 0$, but only if further boundary damping is injected through the auxiliary input u' in (13).

Due to dissipation, the control action obtained in the previous section is not able to stabilise the channel with desired water flow $Q_\star(z)$ and hydraulic pressure $P_\star(z)$ or, equivalently, at some $q_\star(z)$ and $p_\star(z)$ solution of (14) or (15). With the image representation of the Dirac structure in mind, the supporting space Λ now is

$$\Lambda = L_2(0, L; \mathbb{R}^2) \times L_2(0, L; \mathbb{R}) \times \{0\} \times \mathbb{R}^2$$

where as in (18) we have that $\{0\} \subset \mathbb{R}^2$. Similarly to the previous section, given $\lambda = (\lambda_q, \lambda_p, \lambda_R, 0, 0, \lambda_{u0}, \lambda_{uL})$, from (11), we have that in this particular case $\lambda_{u0} = \lambda_p(0)$, and $\lambda_{uL} = \lambda_q(L)$. Then, from (17), there exists an energy shaping control action in the form (13) if there exists H_a and β such that (17) holds for some $\lambda \in \Lambda$, i.e.:

$$\frac{\partial}{\partial z}\begin{pmatrix} \lambda_q \\ \lambda_p \end{pmatrix} = \begin{pmatrix} \lambda_R \\ 0 \end{pmatrix} \qquad \begin{pmatrix} \lambda_q \\ \lambda_p \end{pmatrix} = \begin{pmatrix} \delta_q H_a \\ \delta_p H_a \end{pmatrix}$$

$$D(q,p)\lambda_p = \lambda_R \qquad \beta = -\begin{pmatrix} \lambda_p(0) \\ \lambda_q(L) \end{pmatrix}$$

that implies

$$\begin{aligned} &\frac{\partial}{\partial z}\frac{\delta H_a}{\delta q}(q,p) = -D(q,p)\frac{\delta H_a}{\delta p}(q,p) \\ &\frac{\partial}{\partial z}\frac{\delta H_a}{\delta p}(q,p) = 0 \\ &\beta(q,p) = -\begin{pmatrix} \delta_p H_a(q,p)\mid_{z=0} \\ \delta_q H_a(q,p)\mid_{z=L} \end{pmatrix} \end{aligned} \tag{25}$$

Solving (25) in the general case, or with the empiric expressions associated to the viscous friction term $D(q,p)$ is very difficult. To simplify the problem *a lot*, let us assume that the friction forces are just proportional to the water flow, i.e. that $D(q,p) = \bar{D} > 0$. Under this assumption and if written in terms of the co-energy variables, the equilibrium configuration takes the following form

$$Q_\star(z) = \bar{Q}_\star \qquad P_\star(z) = -\bar{D}\bar{Q}_\star z + \bar{P}_\star \tag{26}$$

where $\bar{Q}_\star$, $\bar{P}_\star$ are some real constants.

From the second relation in (25), we obtain that

$$\frac{\delta H_a}{\delta p}\big(q(z), p(z)\big) = \kappa_p$$

being κ_p some constant function Z, ant then from the first one we have that

$$\frac{\delta H_a}{\delta q}\big(q(z), p(z)\big) = -\kappa_p \bar{D} z + \kappa_q$$

with κ_q another constant function on Z. Consequently, in a slightly different manner than in (20) a class of admissible H_a takes the form

$$H_a(q,p) = \hat{H}_a\big(\xi(q,p)\big)$$

where now

$$\xi(q,p) = \int_0^L \left[q(z)\left(\gamma - \bar{D}z\right) + p(z)\right] \mathrm{d}z$$

with γ a constant that is determined later on. Stability of the desired equilibrium either expressed in terms of hydrodynamic pressure $P_\star(z)$ and water flow $Q_\star(z)$ defined in (26), or in terms of the corresponding water level $q_\star(z)$ and velocity $p_\star(z)$ profiles is achieved by properly selecting $\hat{H}_a$. Similarly to (23), a possible choice is

$$\hat{H}_a(\xi) = \frac{1}{2}\left(\xi - \xi_\star\right)^2 - \Gamma\xi \tag{27}$$

with K a positive gain, and $\xi_\star$ the value of ξ at the desired equilibrium. Stability of $(Q_\star, P_\star)$ is obtained by selecting $\Gamma = \bar{Q}_\star$ in (27), and $\gamma = \bar{P}_\star/\bar{Q}_\star$ in (26). Finally, the associated control action β follows from the last relation in (25), i.e.:

$$\beta(q,p) = \begin{pmatrix} -K\big(\xi(q,p) - \xi_\star\big) + \bar{Q}_\star \\ -K\big(\xi(q,p) - \xi_\star\big) - \bar{D}\bar{Q}_\star L + \bar{P}_\star \end{pmatrix}$$

To enforce asymptotic stability of (26), the auxiliary input u' in (13) is used to introduce (boundary) damping in the system. With (12) in mind, the "new" boundary port for the closed-loop system is

$$(u', y') = \left(\begin{pmatrix} \delta_p H_d(0) \\ \delta_q H_d(L) \end{pmatrix}, \begin{pmatrix} \delta_q H_d(0) \\ -\delta_p H_d(L) \end{pmatrix} \right)$$

that has to be terminated over a dissipative element, i.e.

$$u' = \Pi y', \qquad \Pi = \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix} \tag{28}$$

with $\pi_1, \pi_2 > 0$. Under the assumptions of existence of solutions, and of pre-compactness of the orbits, asymptotic stability is a consequence of La Salles Invariance Principle, Luo et al. [1999]. More precisely, from energy-balancing considerations, we have that the total Hamiltonian H_d decreases until a steady state configuration characterised by

$$\frac{\delta H_d}{\delta p}(q,p) = 0 \text{ on } Z \qquad u' = y' = 0 \tag{29}$$

is reached. The first condition in (29) means that

$$Q(z) - \bar{Q}_\star + K\left(\xi\big(q(z), p(z)\big) - \xi_\star\right) = 0 \tag{30}$$

on Z, and also that q reaches a constant (as function of time) profile. Time differentiation of (30) implies that also ξ is constant in steady state, and consequently also p and P. Moreover, again from (30) and due to the fact that ξ is constant in time and independent from z, $Q(z) = \tilde{Q}$, with $\tilde{Q}$ a constant to be determined later on. On Z, we have also that

$$\begin{aligned} &\frac{\delta H_d}{\delta q}\big(q(z), p(z)\big) = \\ &\quad = K\left(\frac{\bar{P}_\star}{\bar{Q}_\star} - \bar{D}z\right)\left(\xi\big(q(z), p(z)\big) - \xi_\star\right) \\ &\qquad + P(z) + \bar{Q}_\star \bar{D} z - \bar{P}_\star = 0 \end{aligned} \tag{31}$$

This is a consequence of the fact that in steady state $\frac{\partial p}{\partial t} = 0$, and of the second condition in (29) enforced by the damping injection. From (30) and (31), it is immediate to determine that the steady state profile of $P(z)$ is related to $Q(z) \equiv \tilde{Q}$ as follows:

$$P(z) = -\tilde{Q}\bar{D}z + \bar{P}_\star$$

Finally, from (28) and (30), we can deduce that in steady state $\xi = \xi_\star$ provided that $\pi_1, \pi_2 > 0$, i.e. only if full-boundary dissipation is present. This fact immediately implies that $\tilde{Q} = \bar{Q}_\star$ and, finally that $P(z) = -\bar{Q}_\star \bar{D} z + \bar{P}_\star \equiv P_\star(z)$ in steady state. So, the only invariant solution compatible with $\dot{H}_d = 0$ is the equilibrium (26), which turns out to be asymptotically stable from La Salle's Invariance Principle considerations.

4. CONCLUSIONS

In this paper, novel techniques devoted to the boundary stabilisation via energy-shaping of distributed port-Hamiltonian systems have been applied to a nonlinear PDE, namely a simplified version of the shallow water equation. The method relies on the parametrisation of the system trajectories provided by the image representation of the Dirac structure associated to the system, and the control action is determined in order to map the open-loop dynamic into a target one with the desired stability properties. For the shallow water equation, stability has been obtained at first in case no internal dissipation is present, i.e. when viscous friction forces are set equal to zero. In this case, the controller is an energy-balancing kind of, and the equivalence between the proposed approach and the energy-Casimir method is shown. The stabilisation problem is then solved also when (linear) internal friction forces are present. In this situation, the controller behaves as a state modulated source able to properly shape the energy function of the system. Asymptotic stability is obtained via damping injection, and proved thanks to La Salle's arguments.

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