FIR Digital Filter Design by Sampled-Data H^∞ Discretization *

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Abstract: FIR (finite impulse response) digital filter design is a fundamental problem in signal processing. In particular, FIR approximation of analog filters (or systems) is ubiquitous not only in signal processing but also in digital implementation of controllers. In this article, we propose a new design method of an FIR digital filter that optimally approximates a given analog filter in the sense of minimizing the H^∞ norm of the sampled-data error system. By using the lifting technique and the KYP (KalmanYakubovichPopov) lemma, we reduce the H^∞ optimization to a convex optimization described by an LMI (linear matrix inequality). We also extend the method to multi-rate and multi-delay systems. A design example is shown to illustrate the effectiveness of the proposed method.

1. INTRODUCTION

In this article, we consider a fundamental problem in signal processing, namely FIR (finite impulse response) approximation of *analog* filters. FIR digital filters are preferred to IIR (infinite-impulse response) digital filters because of the following merits:

- FIR filters are always stable.
- They can be easily implemented in digital systems.
- They are free from problems of IIR filters such as limit cycles caused by quantization.

On the other hand, there are a considerable design methods for IIR digital filters, e.g., Butterworth, Chebyshev and Elliptic, to name a few (see Oppenheim and Schafer [2009]). To obtain an FIR digital filter that approximates a given IIR *digital* filter, approximation methods with an appropriate optimization have been proposed in Kootsookos et al. [1992], Yamamoto et al. [2003] for example. These methods are available if we are given a target IIR digital filter.

In practice, a target filter (or a system) to be approximated by an FIR digital filter may be at first given by an *analog* filter. An RLC filter (an electrical circuit consisting of resistors, inductors, and capacitors) is one example and a PID (proportional-integral-derivative) controller is another. To obtain an FIR digital filter that mimics such an analog filter, one might go through two steps:

(1) Compute an IIR digital filter that approximates the original analog filter via step-invariant transformation or bilinear (or Tustin) transformation (see Chen and Francis [1995]),

(2) Approximate the obtained IIR digital filter to an FIR one by truncation of the impulse response, or more sophisticated methods as in Kootsookos et al. [1992], Yamamoto et al. [2003].

Obviously, it is desirable if an FIR digital filter is obtained *directly* from the original analog filter. For this purpose, we propose a direct design method of FIR digital filters based on the theory of sampled-data H^∞ control. We have proposed a design method in Nagahara and Yamamoto [2013] via sampled-data H^∞ control theory, which gives the IIR digital filter that approximates a given analog filter with the H^∞ performance index. In this article, we extend this work to FIR digital filter design. A key idea is to use the KYP (Kalman-Yakubovich-Popov) lemma that reduces the H^∞ optimization problem to an optimization described in an LMI (linear matrix inequality). We also extend the result to multi-rate systems that consists an upsampler and a fast hold, and to multi-delay systems that appear in the Smith predictor proposed in Smith [1957], or multipath propagation in wireless communications (see Goldsmith [2005] for example). A design example is shown to illustrate the effectiveness of our methods.

2. MATHEMATICAL NOTATION AND REVIEW

Throughout this article, we use the following notation. $L^2[0, \infty)$ is the Lebesgue space consisting of all square integrable real functions on $[0, \infty)$. $L^2[0, \infty)$ is sometimes abbreviated to L^2. The L^2 norm of $f \in L^2$ is defined by

$$\|f\|_2 := \sqrt{\int_0^\infty |f(t)|^2 dt}.$$

The symbol t denotes the argument of time, s the argument of Laplace transform, and z the argument of Z transform. These symbols are used to indicate whether a signal or a system is of continuous-time or discrete-time;

for example, $y(t)$ is a continuous-time signal, $F(s)$ is a continuous-time system, $K(z)$ is a discrete-time system. The operator e^{-ls} with nonnegative integer l denotes continuous-time delay (or shift) operator: $(\mathrm{e}^{-ls}y)(t) = y(t-l)$. $\mathcal{S}_h$ and $\mathcal{H}_h$ denote the ideal sampler and the zero-order hold respectively with sampling period $h > 0$. We denote the imaginary number $\sqrt{-1}$ by j.

A transfer function with state-space matrices A, B, C, D is denoted by

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \begin{cases} C(sI-A)^{-1}B + D, & \text{(continuous-time)} \\ C(zI-A)^{-1}B + D, & \text{(discrete-time)} \end{cases}$$

For this notation, we have the following formulae:

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \pm \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & \pm B_2 \\ \hline C_1 & C_2 & D_1 \pm D_2 \end{array}\right], \quad (1)$$

$$\left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] \times \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1 C_2 & A_1 & B_1 D_2 \\ \hline D_1 C_2 & C_1 & D_1 D_2 \end{array}\right]. \quad (2)$$

For a stable discrete-time system $G(z)$, its discrete-time H^∞ norm is defined by

$$\|G\|_\infty := \max_{\theta \in [0,\pi]} \sigma_{\max}(G(e^{j\theta})),$$

where $\sigma_{\max}(\cdot)$ is the largest singular value of the argument matrix. The well-known KYP (Kalman-Yakubovich-Popov) lemma (see Anderson [1967], Rantzer [1996], Tuqan and Vaidyanathan [1998], Nagahara [2011]) characterizes the H^∞ norm of a discrete-time system by a linear matrix inequality (LMI):

Lemma 1. (KYP lemma). Let A, B, C, D be a minimal realization of a stable discrete-time transfer function $G(z)$. Let $\gamma > 0$. Then the following are equivalent conditions:

(1) $\|G\|_\infty = \left\|\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]\right\|_\infty < \gamma$.

(2) There exists a positive definite matrix X such that

$$\begin{bmatrix} A^\top X A - X & A^\top X B & C^\top \\ B^\top X A & B^\top X B - \gamma I & D^\top \\ C & D & -\gamma I \end{bmatrix} < 0.$$

This lemma is a key to derive a computationally efficient method for FIR digital filter design.

3. PROBLEM FORMULATION

Assume that a transfer function $K_c(s)$ of an analog filter is given. We suppose that $K_c(s)$ is a stable[1], real-rational, proper transfer function. Our objective is to implement this analog system in a digital system. To do this, let us consider a digital system shown in Fig. 1, where $K(z)$ is an FIR digital filter of length M described by

$$K(z) = \sum_{k=0}^{M-1} a_k z^{-k}, \quad (3)$$

$\mathcal{S}_h$ is an ideal sampler with a fixed sampling period $h > 0$ that converts a continuous-time signal $u(t)$ to a discrete-time signal $v[n]$ as
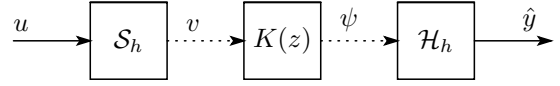
Fig. 1. Digital system $\mathcal{K}$ consisting of ideal sampler $\mathcal{S}_h$, digital filter $K(z)$, and zero-order hold $\mathcal{H}_h$ with sampling period h.

$$v[n] = (\mathcal{S}_h u)[n] = u(nh), \quad n = 0, 1, 2, \ldots,$$

and $\mathcal{H}_h$ is a zero-order hold that produces a continuous-time signal $\hat{y}(t)$ from a discrete-time signal $\psi[n]$ as

$$\hat{y}(t) = \sum_{n=0}^\infty \psi[n]\phi(t-nh), \quad t \in [0, \infty),$$

where $\phi(t)$ is a box function defined by

$$\phi(t) = \begin{cases} 1, & \text{if } t \in [0, h), \\ 0, & \text{otherwise.} \end{cases}$$

Our problem is to obtain the FIR digital filter coefficients $a_0, a_1, \ldots, a_{M-1}$ in (3) so that the digital system

$$\mathcal{K} := \mathcal{H}_h K \mathcal{S}_h = \mathcal{H}_h \left(\sum_{k=0}^{M-1} a_k z^{-k}\right) \mathcal{S}_h$$

mimics the input/output behavior of the analog filter $K_c(s)$.

Let $\alpha(t)$ denote the impulse response (or the inverse Laplace transform) of $K_c(s)$. The digital filter $K(z)$ (or the filter coefficients $a_0, a_1, \ldots, a_{M-1}$) is designed to produce a continuous-time signal $\hat{y}$ after the zero-order hold $\mathcal{H}_h$ that approximates the *delayed* output

$$y(t) = (\alpha * u)(t-l) = \int_0^{t-l} \alpha(\tau)u(t-l-\tau)d\tau$$

of $K_c(s)$ for an input u. A positive delay time l may improve the approximation performance when l is large enough as discussed in e.g., Nagahara et al. [2011], Yamamoto et al. [2012]. We assume that l is an integer multiple of h, that is, $l = mh$, where m is a nonnegative integer. Then, to avoid a trivial solution (i.e., $K(z) = 0$), we should assume some *a priori* information for the input signal u. As used in Nagahara et al. [2011], Yamamoto et al. [2012], we adopt the following signal subspace of $L^2[0, \infty)$ to which the input signals belong:

$$FL^2 := \{Fw : w \in L^2[0, \infty)\},$$

where F is a linear system with a stable, real-rational, strictly proper transfer function $F(s)$. This transfer function, $F(s)$, defines the *analog characteristic* of the input signals in the frequency domain.

In summary, our discretization problem is formulated as follows:

Problem 1. Given target filter $K_c(s)$, analog characteristic $F(s)$, sampling period h, and delay step m, find the filter coefficients $a_0, a_1, \ldots, a_{M-1}$ of $K(z)$ given by (3) that minimizes

$$\|\mathcal{E}(K)\|_\infty = \left\|\left(\mathrm{e}^{-mhs}K_c - \mathcal{H}_h K \mathcal{S}_h\right)F\right\|_\infty$$

$$= \sup_{\substack{w \in L^2 \\ w \neq 0}} \frac{\left\|\left(\mathrm{e}^{-mhs}K_c - \mathcal{H}_h K \mathcal{S}_h\right)Fw\right\|_2}{\|w\|_2}.$$

The corresponding block diagram of the error system

$$\mathcal{E}(K) := \left(\mathrm{e}^{-mhs}K_c - \mathcal{H}_h K \mathcal{S}_h\right)F \quad (4)$$

---

[1] If $K_c(s)$ has no poles on the imaginary axis, the proposed method can be used for unstable $K_c(s)$ as follows: factorize it as $K_c(s) = K_s(s)K_{as}(s)$, where $K_s(s)$ is stable and $K_{as}(s)$ is anti-stable, and then discretize $K_s(s)$ and $K_{as}(-s)$ by the proposed method.
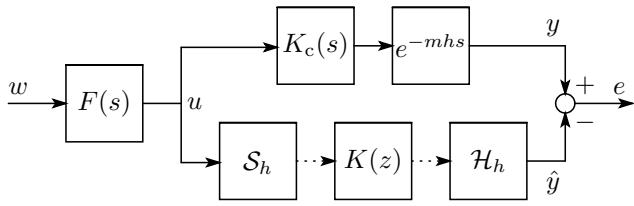
Fig. 2. Error system $\mathcal{E}(K)$.

is shown in Fig. 2.

4. FIR FILTER DESIGN VIA SAMPLED-DATA H^∞ OPTIMIZATION

In this section, we give a design formula to numerically compute the H^∞-optimal filter coefficients of Problem 1 via *fast sample/hold* approximation as used in Nagahara and Yamamoto [2013], and the KYP lemma described in Lemma 1.

We first define the discrete-time lifting of a discrete-time system by

$$
\begin{aligned}
&\mathbf{lift}\left(\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], N\right) \\
&:= \mathbf{L}_N \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \mathbf{L}_N^{-1} \\
&= \left[\begin{array}{c|cccc} A^N & A^{N-1}B & A^{N-2}B & \dots & B \\ \hline C & D & 0 & \dots & 0 \\ CA & CB & D & & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ CA^{N-1} & CA^{N-2}B & CA^{N-3}B & \dots & D \end{array}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{L}_N &:= (\downarrow N)\left[\begin{array}{cccc} 1 & z & \cdots & z^{N-1} \end{array}\right]^T, \\
\mathbf{L}_N^{-1} &:= \left[\begin{array}{cccc} 1 & z^{-1} & \cdots & z^{-N+1} \end{array}\right](\uparrow N).
\end{aligned}
$$

In this definition, $\downarrow N$ and $\uparrow N$ are respectively a down-sampler and an upsampler (see e.g., Vaidyanathan [1993]) defined as

$$
\uparrow N : \{x[k]\}_{k=0}^\infty \mapsto \{x[0], \underbrace{0, \dots, 0}_{N-1}, x[1], 0, \dots\},
$$

$$
\downarrow N : \{x[k]\}_{k=0}^\infty \mapsto \{x[0], x[N], x[2N], \dots\}.
$$

These operators give the fast sample/hold approximation E_N of the sampled-data error system $\mathcal{E}(K)$ given in (4) as

$$
E_N(z) = \left(z^{-mN}K_N(z) - H_N K(z)S_N\right)F_N(z), \quad (5)
$$

where

$$
\begin{aligned}
K_N(z) &= \mathbf{lift}\left(\mathcal{S}_{h/N}K_c\mathcal{H}_{h/N}, N\right), \\
F_N(z) &= \mathbf{lift}\left(\mathcal{S}_{h/N}F\mathcal{H}_{h/N}, N\right), \\
H_N &= \underbrace{[1, \dots, 1]}_{N}^\top, \quad S_N = [1, \underbrace{0, \dots, 0}_{N-1}].
\end{aligned} \quad (6)
$$

Note that $K_N(z)$ and $F_N(z)$ are finite-dimensional linear time-invariant systems as shown in Nagahara and Yamamoto [2013]. Define

$$
A_K := \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix}, \quad B_K := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad (7)
$$

$$
C_K := [\, a_{M-1} \; a_{M-2} \; \dots \; a_1 \,], \quad D_K := a_0
$$

Then $\{A_K, B_K, C_K, D_K\}$ is a minimal realization of FIR filter $K(z)$ in (3), that is,

$$
K(z) = \sum_{k=0}^{M-1} a_k z^{-k} = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]. \quad (8)
$$

Substituting (8) into (5), we obtain

$$
E_N(z) = T_1(z) + Q(z)T_2(z),
$$

where

$$
T_1(z) = z^{-mN}K_N(z)F_N(z) =: \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & 0 \end{array}\right], \quad (9)
$$

$$
T_2(z) = S_N F_N(z) =: \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & 0 \end{array}\right], \quad (10)
$$

$$
Q(z) = -H_N \sum_{k=0}^{M-1} a_k z^{-k} = \left[\begin{array}{c|c} A_K & B_K \\ \hline H_N C_K & H_N D_K \end{array}\right].
$$

Then using (1) and (2), we have

$$
\begin{aligned}
E_N(z) &= \left[\begin{array}{ccc|c} A_1 & 0 & 0 & B_1 \\ 0 & A_2 & 0 & B_2 \\ 0 & B_K C_2 & A_K & 0 \\ \hline C_1 & H_N D_K C_2 & H_N C_K & 0 \end{array}\right] \\
&=: \left[\begin{array}{c|c} A & B \\ \hline C(a) & D(a) \end{array}\right],
\end{aligned} \quad (11)
$$

where a is the coefficient vector defined by

$$
a = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{M-1} \end{bmatrix} \in \mathbb{R}^M. \quad (12)
$$

Note that the coefficient vector a to be designed is independent of A and B of $E_N(z)$ as in (11). This property is fundamental for the derivation of LMI optimization. Finally, our problem (Problem 1) is reduced to finding $a \in \mathbb{R}^M$ that minimizes the H^∞ norm of $E_N(z)$, which can be described as an LMI from the KYP lemma (Lemma 1) as follows.

Problem 2. Find $a \in \mathbb{R}^M$ that minimizes γ subject to

$$
\begin{bmatrix} A^\top X A - X & A^\top X B & C(a)^\top \\ B^\top X A & B^\top X B - \gamma I & D(a)^\top \\ C(a) & D(a) & -\gamma I \end{bmatrix} < 0, \quad (13)
$$
$$
X > 0.
$$

This problem can be efficiently solved via numerical optimization software such as SDPT3 (see Toh et al. [1999], SeDuMi (see Sturm [1999]), or cvx (see Grant and Boyd [2011, 2008]) on MATLAB.

In summary, we have derived a computationally efficient method with an LMI in (13) for FIR approximation of analog filters based on fast sample/hold approximation in (5) and the KYP lemma (Lemma 1).
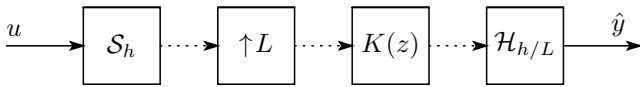
Fig. 3. Multi-rate system $\mathcal{K}_{\mathrm{mr}}$ consisting of ideal sampler $\mathcal{S}_h$, upsampler $\uparrow L$, digital filter $K(z)$, and fast hold $\mathcal{H}_{h/L}$.
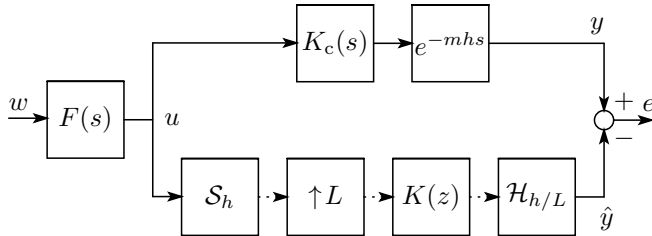

Fig. 4. Multi-rate error system $\mathcal{E}_{\mathrm{mr}}(K)$.

5. EXTENSION TO MULTI-RATE SYSTEMS

In this section, we extend the result in the previous section to multi-rate systems as proposed in Nagahara and Yamamoto [2013]. That is, we consider the following multi-rate signal processing system

$$\mathcal{K}_{\mathrm{mr}} := \mathcal{H}_{h/L} K(z)(\uparrow L)\mathcal{S}_h = \mathcal{H}_{h/L} \left(\sum_{k=0}^{M-1} a_k z^{-k} \right) (\uparrow L)\mathcal{S}_h,$$

where L is an integer greater than or equal to 2. The block diagram of multi-rate $\mathcal{K}_{\mathrm{mr}}$ is shown in Fig. 3. The objective here is to find an FIR digital filter $K(z)$ given by (3) that minimizes the H^∞ norm of the multi-rate error system $\mathcal{E}_{\mathrm{mr}}(K)$ defined by

$$\mathcal{E}_{\mathrm{mr}}(K) := \left(e^{-mhs} K_{\mathrm{c}} - \mathcal{H}_{h/L} K(\uparrow L)\mathcal{S}_h \right) F. \quad (14)$$

The corresponding block diagram of the error system $\mathcal{E}_{\mathrm{mr}}(K)$ is shown in Fig. 4. Then, our problem is described as follows:

Problem 3. Given target filter $K_{\mathrm{c}}(s)$, analog characteristic $F(s)$, sampling period h, delay step m, and upsampling ratio L, find the filter coefficients $a_0, a_1, \ldots, a_{M-1}$ of $K(z)$ given by (3) that minimizes

$$\|\mathcal{E}_{\mathrm{mr}}(K)\|_\infty = \left\| \left(e^{-mhs} K_{\mathrm{c}} - \mathcal{H}_{h/L} K(\uparrow L)\mathcal{S}_h \right) F \right\|_\infty \quad (15)$$

As in the single-rate case discussed in Section 4, we use the method of fast sample/hold approximation. Assume that $N = Lp$ for some positive integer p. Then, the fast sample/hold approximation of $\mathcal{E}_{\mathrm{mr}}$ is given by

$$E_{\mathrm{mr},N}(z) = \left(K_N(z)z^{-mN} - \widetilde{H}_N \widetilde{K}(z)S_N \right) F_N(z), \quad (16)$$

where $K_N(z)$, $F_N(z)$ and S_N are given in (6), and

$$\widetilde{H}_N = \mathbf{blkdiag}\{\underbrace{\mathbf{1}_p, \mathbf{1}_p, \ldots, \mathbf{1}_p}_{L}\}, \quad \mathbf{1}_p = [\underbrace{1, \ldots, 1}_{p}]^\top,$$

$$\widetilde{K}(z) = \mathbf{lift}\left(K(z), L\right)[1, \underbrace{0, \ldots, 0}_{L-1}]^\top = \left[\begin{array}{c|c} \widetilde{A}_K & \widetilde{B}_K \\ \hline \widetilde{C}_K & \widetilde{D}_K \end{array} \right],$$

$$\widetilde{A}_K := A_K^L, \quad \widetilde{B}_K := A_K^{L-1} B_K,$$

$$\widetilde{C}_K := \begin{bmatrix} C_K \\ C_K A_K \\ \vdots \\ C_K A_K^{L-1} \end{bmatrix}, \quad \widetilde{D}_K := \begin{bmatrix} D_K \\ C_K B_K \\ \vdots \\ C_K A_K^{L-2} B_K \end{bmatrix}. \quad (17)$$

Note that matrices A_K, B_K, C_K, and D_K in (17) are defined in (7). Substituting (8) into (16), we obtain

$$E_{\mathrm{mr},N}(z) = T_1(z) + \widetilde{Q}(z)T_2(z),$$

where $T_1(z)$ and $T_2(z)$ are given by (9) and (10) respectively, and

$$\widetilde{Q}(z) = -\widetilde{H}_N \sum_{k=0}^{M-1} a_k z^{-k} = \left[\begin{array}{c|c} \widetilde{A}_K & \widetilde{B}_K \\ \hline \widetilde{H}_N \widetilde{C}_K & \widetilde{H}_N \widetilde{D}_K \end{array} \right]$$

Then, using (1) and (2), we have

$$E_{\mathrm{mr},N}(z) = \left[\begin{array}{ccc|c} A_1 & 0 & 0 & B_1 \\ 0 & A_2 & 0 & B_2 \\ 0 & \widetilde{B}_K C_2 & \widetilde{A}_K & 0 \\ \hline C_1 & \widetilde{H}_N \widetilde{D}_K C_2 & \widetilde{H}_N \widetilde{C}_K & 0 \end{array} \right]$$

$$=: \left[\begin{array}{c|c} \widetilde{A} & \widetilde{B} \\ \hline \widetilde{C}(a) & \widetilde{D}(a) \end{array} \right],$$

where a is the coefficient vector given by (12) which is to be designed.

Finally, our problem (Problem 3) is also reduced to finding $a \in \mathbb{R}^M$ that minimizes the H^∞ norm of $E_{\mathrm{mr},N}(z)$, which can be described as an LMI from the KYP lemma (Lemma 1) as follows.

Problem 4. Find $a \in \mathbb{R}^M$ that minimizes γ subject to

$$\begin{bmatrix} \widetilde{A}^\top X \widetilde{A} - X & \widetilde{A}^\top X \widetilde{B} & \widetilde{C}(a)^\top \\ \widetilde{B}^\top X \widetilde{A} & \widetilde{B}^\top X \widetilde{B} - \gamma I & \widetilde{D}(a)^\top \\ \widetilde{C}(a) & \widetilde{D}(a) & -\gamma I \end{bmatrix} < 0,$$

$$X > 0.$$

This problem can be also efficiently solved via numerical optimization software such as SDPT3, SeDuMi, or cvx on MATLAB.

6. DISCRETIZATION OF MULTI-DELAY SYSTEMS

In this section, we consider discretization of multi-delay systems of the following type:

$$K_{\mathrm{c}}(s) = \sum_{i=1}^{\mu} e^{-m_i hs} G_i(s) \quad (18)$$

where $G_i(s)$ is a stable transfer function and m_i is a non-negative integer. This system appears in e.g. the Smith predictor

$$K_{\mathrm{c}}(s) = (1 - e^{-mhs})G(s),$$

for controlling time delay systems proposed in Smith [1957], whose discretization is important for implementing the controller on a digital system. Another example is a mathematical model of multipath propagation in wireless communications (see e.g., Goldsmith [2005]).

Let $K_i(z)$ be the H^∞-optimal FIR digital filter (the optimal solution to Problem 4) with $K_{\mathrm{c}}(s) = e^{-m_i hs} G_i(s)$ and $m = 0$. Then the following filter

$$\bar{K}(z) = \sum_{i=1}^{\mu} K_i(z) \quad (19)$$

is a sub-optimal FIR approximation of $K_{\mathrm{c}}(s)$ given by (18) as shown in the following lemma.

Lemma 2. Fix non-negative integer L and positive number h. Let γ_i be the value of the H^∞ norm of $\mathcal{E}_{\mathrm{mr}}$ defined

in (15) with $m = 0$, $K_c(s) = e^{-m_i hs} G_i(s)$, and $K(z) = K_i(z)$. Then, the H^∞ norm of $\mathcal{E}_{mr}(\bar{K})$ with $m = 0$ and $K_c(s)$ defined in (18) satisfies

$$\|\mathcal{E}_{mr}(\bar{K})\|_\infty \le \sum_{i=1}^{\mu} \gamma_i. \qquad (20)$$

Proof. First we have
$$\mathcal{E}_{mr}(\bar{K}) = (K_c - \mathcal{H}_{h/L}\bar{K}(\uparrow L)\mathcal{S}_h)F$$
$$= \left(\sum_{i=1}^{\mu} e^{-L_i s}G_i - \mathcal{H}_{h/L}\sum_{i=1}^{\mu} K_i(\uparrow L)\mathcal{S}_h\right)F$$
$$= \sum_{i=1}^{\mu}(e^{-L_i s}G_i - \mathcal{H}_{h/L}K_i(\uparrow L)\mathcal{S}_h)F.$$

It follows that
$$\|\mathcal{E}_{mr}(\bar{K})\|_\infty \le \sum_{i=1}^{\mu} \left\|(e^{-L_i s}G_i - \mathcal{H}_{h/L}K_i(\uparrow L)\mathcal{S}_h)F\right\|_\infty$$
$$= \sum_{i=1}^{\mu} \gamma_i.$$

$\square$

This lemma suggests if $\gamma_1, \ldots, \gamma_p$ are sufficiently small, the sum filter in (19) is a good approximation of multi-delay system $K_c(s)$ given in (18) thanks to inequality (20).

7. DESIGN EXAMPLE

We here show a design example of FIR digital filter design to illustrate the effectiveness of the proposed design method. We assume that the target analog filter is given by

$$K_c(s) = \frac{0.0031623(s^2 + 1.33)(s^2 + 1.899)}{(s^2 + 0.3705s + 0.1681)(s^2 + 0.1596s + 0.7062)}$$
$$\times \frac{(s^2 + 10.31)}{(s^2 + 0.03557s + 0.9805)},$$

which is a 6-th order elliptic filter with 3 dB passband peak-to-peak ripple, 50 dB stopband attenuation, and 1 (rad/sec) cut-off frequency. This filter is computed with MATLAB command `ellip(6,3,50,1,'s')`. We set sampling period $h = 1$ (sec), upsampling ratio $L = 2$ (i.e., we consider a multi-rate system), and delay step $m = 5$. The analog characteristic $F(s)$ is chosen as

$$F(s) = \frac{1}{s + 1}.$$

The fast sample/hold approximation factor N is chosen as $N = 6$. We fix the FIR filter length M to be 32.

Under these parameters, we design the H^∞-optimal FIR digital filter based on the LMI optimization described in Problem 4. Also, we design the H^∞-optimal IIR digital filter based on Nagahara and Yamamoto [2013], and we truncate the H^∞-optimal impulse response to obtain an FIR digital filter of length 32. Fig. 5 shows the frequency response of the obtained digital filters. The truncated FIR filter shows a large differences in low frequencies while the H^∞-optimal FIR filter shows differences at high frequencies.

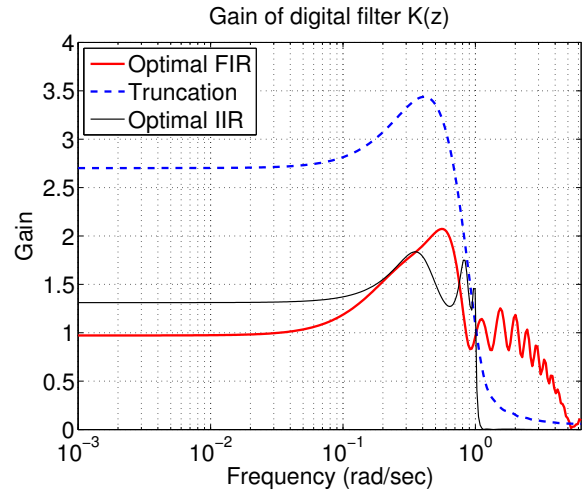Figs. 6 and 7 show the impulse responses (filter coefficients) of the H^∞-optimal FIR filter and the truncated

Fig. 5. Digital filters: H^∞-optimal FIR filter (solid thick line), truncation of H^∞-optimal IIR filter (dashed line), and H^∞-optimal IIR filter (solid thin line).
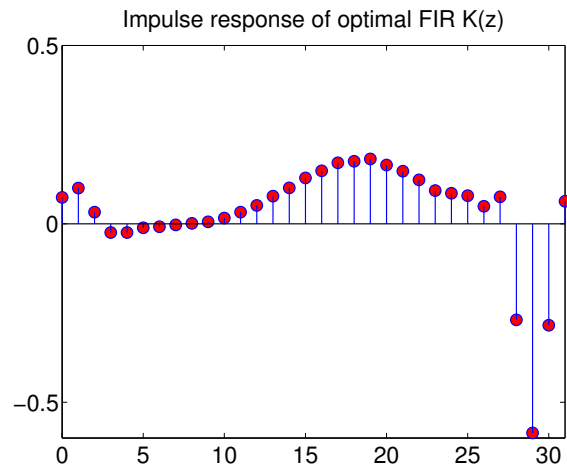

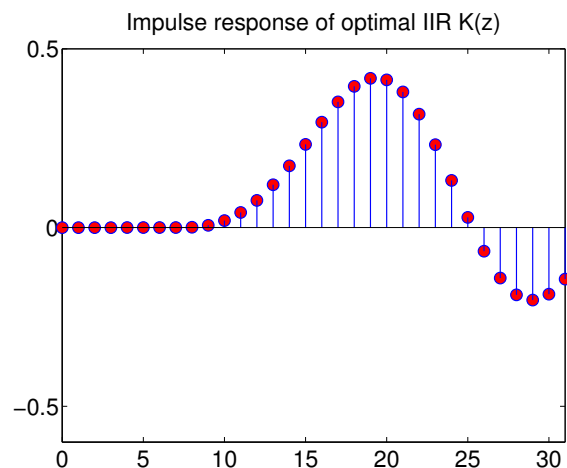Fig. 6. Impulse response of H^∞-optimal FIR filter.

Fig. 7. Impulse response of H^∞-optimal IIR filter.

FIR filter, respectively. The H^∞-optimal FIR filter has non-trivial values around $k = 0$ and $k = 31$, which can not be obtained by just truncation as shown in Fig. 7.
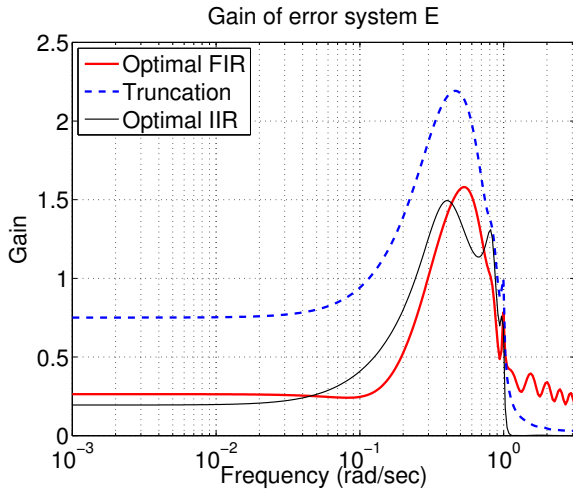
Fig. 8. Gain frequency response of sampled-data error system $\mathcal{E}_{\mathrm{mr}}$: H^∞-optimal FIR filter (solid thick line), truncation of H^∞-optimal IIR filter (dashed line), and H^∞-optimal IIR filter (solid thin line).

To see the difference of performance between the H^∞-optimal FIR filter and the truncated FIR filter, we show the gain frequency response of the sampled-data error system $\mathcal{E}_{\mathrm{mr}}$ defined in (14) in Fig. 8. The truncated FIR filter shows a large approximation error at low frequencies and results in a larger H^∞ norm of the error system, while the H^∞-optimal filter shows a tolerable performance for all frequencies. This is a merit of the use of H^∞ optimization.

8. CONCLUSION

In this paper, we have proposed a method for the design of an FIR digital filter that optimally approximates a given analog filter with a sampled-data H^∞ performance index. The design is described as an optimization with LMI, which can be efficiently solved by numerical optimization softwares. We also extend the proposed method to multi-rate and multi-delay systems. A design example has shown the effectiveness of the proposed method. Future works include

- design of FIR filters with gain minimization on a subset of the frequency range by the generalized KYP lemma proposed by Iwasaki and Hara [2005] as used in Nagahara and Yamamoto [2012].
- multiplierless implementation of H^∞-optimal FIR digital filters as discussed in Samueli [1989], Macleod and Dempster [2005].

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