The Conservative Expected Value: A New Measure with Motivation from Stock Trading via Feedback[*]

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Abstract: The probability distribution for profits and losses associated with a feedback-based stock-trading strategy can be highly skewed. Accordingly, when this random variable has a large expected value, it may be a rather unreliable indicator of performance. That is, a large profit may be exceedingly improbable even though its expected value is high. In addition, the lack of confidence in the underlying stock price model contributes to lack of reliability in the expected value for profits and losses. Motivated by these issues, in this paper, we propose a new measure, called the *Conservative Expected Value* (CEV), which discounts the "ordinary" expected value. Once the CEV is defined, it is calculated for some classical probability distributions and a few of its important properties are established.

Keywords: Financial Markets, Stochastic Systems, Uncertain Dynamical Systems, Robustness

1. INTRODUCTION

This paper is motivated by our work to date on "skewing effects" related to the use of feedback when trading in financial markets; e.g., see [1] and [2]. Suffice it to say, when a feedback control is used to modify an investment position, the resulting probability distribution for profits and losses can be highly skewed. For example, if $K > 0$ is the gain of a linear stock-trading controller, the resulting skewness $S(K)$ for profits and losses can increase dramatically with K and can easily become so large as to render many existing forms of risk-return analysis of questionable worth. Said another way, the long tail of the resulting highly-skewed distribution can lead to a large expected profit but the probability of an "adequate" profit may be quite small. Another negative associated with high skew is that there can be a significant probability of large drawdown in an investor's account; e.g., see [3].

In addition to the negatives related to skewness, another factor which complicates the expected profit-loss prediction is that the model used for the stock price may not be reliable, particularly, in turbulent markets. The issue of "distrust" in the price model combined with the possibility of misleading results due to skewness suggests that a discounting procedure should be introduced to obtain a "conservative" expected value.

1.1 Motivating Example

To provide a concrete illustration of the issues raised above, we consider a stock-trading strategy based on the linear feedback controller given in papers such as [1] and [4]. The amount invested $I(t)$, at time t, is given by

$$I(t) = I_0 + Kg(t),$$

where I_0 is the initial investment, K is the feedback gain and $g(t)$ is the cumulative gain-loss up to time t. When

Geometric Brownian Motion (GBM) is used to drive the stock prices, the random variable $g(t)$ turns out to be a shifted and scaled log-normal distribution which can be highly skewed with an expected value which may be misleading in terms of the prospect for success.

To illustrate the scenario above, suppose time $t = 1$ represents one year and assume GBM process parameters $\mu = 0.25$ and $\sigma = 0.5$, where μ is the annualized drift and σ is the annualized volatility. Furthermore, assume initial investment $I_0 = 1$ representing one dollar and feedback gain $K = 4$. Then, via a simple modification of the results in [4], the probability density function for the gains and losses, $g(t)$, at $t = 1$, is given by

$$f(x) = \frac{1}{\sqrt{\frac{\pi}{2}}(1 + 4x)} e^{-\frac{\left(\log(1+4x)+1\right)^2}{8}}$$
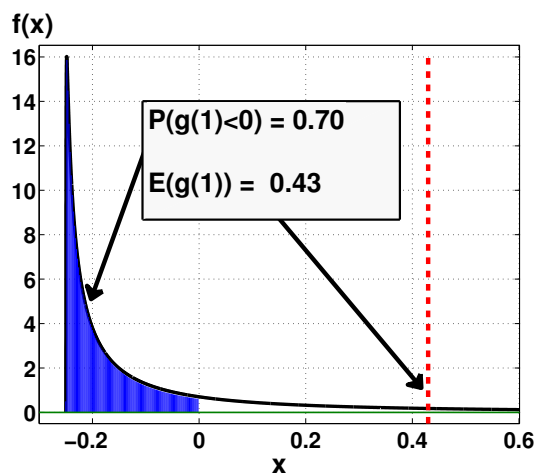
for $x > -0.25$; see Figure 1.

Fig. 1. Trading Profit-Loss: The Probability Density Function

As seen in the figure, the expected value is $E[g(1)] \approx 0.43$ which is shown via the vertical dashed line. This expected value represents a raw return of 43% on an investment of one dollar. However, as seen in the figure, the probability of loss, the shaded area, is $p_{\text{LOSS}} \approx 0.70$. In other words, the expected return is quite attractive but it is highly probable that a losing trade will occur.

1.2 Skewness Considerations

The "pathology" above can be explained by the large skewness, found to be $S = 414$, of the probability density function $f(x)$ of the random variable $g(1)$. To get a sense of how large this degree of skewness is, it is instructive to compare it against an exponentially distributed random variable which is known to be highly skewed with $S = 2$ or a uniform distribution with no skewness at all.

In the view of high degree of right-sided skewness of the distribution for $g(1)$, the large expected value provides an "unduly optimistic" assessment of the "bet at hand." For many traders, the high value of this expected pay-off provides insufficient compensation for the fact that it is overwhelmingly likely that a loss will occur. To address this issue, in this paper, we introduce a procedure which discounts the long tail of such highly-skewed distributions. This discounting process leads to a conservative alternative to the classical expected value, which we called the *Conservative Expected Value* (CEV). Using this discounting process, as seen in Section 2, for the motivating example above, we obtain CEV ≈ -0.12. This negative value indicates an expectation of loss from a conservative perspective. Comparing this new measure to the classical expected value, $\mathbb{E}[g(1)] \approx 0.43$, shows how the long tail of the distribution is discounted.

1.3 Other Considerations and the CEV

In addition to the possibility of high skewness of a probability distribution, the uncertainty of the underlying model can dramatically impact an analysis. For example, in turbulent markets such as those experienced in the crash of 2008-2009, celebrated models based on Geometric Brownian Motion failed miserably when the volatility dramatically increased. Similar shortcomings of various other models in describing observed market prices motivates the search for a conservative measure of expected value to robustify predictions against model uncertainty; see [5] for discussion of robustness in a macro-economic context.

As previously mentioned, to address the high skewness and possible model uncertainty, in this paper, we introduce the *Conservative Expected Value* (CEV) for a random variable X. In a financial context, involving unreliable models, we do not ascribe high credibility to large profits which are highly improbable. It is important to note that the CEV is defined for the class of random variables with finite leftmost support point. That is, we are addressing random variables for which the worst case is bounded. In fact, the finite leftmost support point requirement above is satisfied in our papers involving linear feedback in financial markets, see [1]–[4]. Another example involving finite leftmost support point is a random variable modelling the lifetime of a component in a system.

By way of further motivation for the CEV definition, in the financial literature, it is a routine procedure to pick a target value γ for the acceptable profit or loss and declare "win" for outcomes larger than γ and a "loss" for smaller outcomes. Taking a conservative perspective, for a given target value γ for a random variable X, the first step in CEV analysis is to shift the probability mass associated with all possible losses, $\{x : x \leq \gamma\}$, to the worst-possible loss, the leftmost support point. Also the probability mass associated with the outcomes which are declared as "wins" are all shifted to the smallest possible value for a win, namely the target value $x = \gamma$. We call this process "Bernoullizing." Motivation for this mass-shifting process is based on "distrust" in the assumed distribution.

The Bernoulli random variable obtained by mass shifting as described above; call it X_γ, provides a conservative lower bound on performance which discounts long tails. For any given target value γ, it is easy to see that the expected value of the resulting Bernoulli random variable X_γ is smaller than the expected value of the original random variable X. By picking a target value $\gamma = \gamma^*$ which leads to the largest expected value for X_γ, we avoid excess conservatism and obtain the *Conservative Expected Value* (CEV). More specifically any target value $\gamma < \gamma^*$ is deemed inefficient in the following sense: The pair $(\gamma, \mathbb{E}(X_\gamma))$ is dominated by $(\gamma^*, \mathbb{CEV}(X))$. By finding the pair $(\gamma^*, \mathbb{CEV}(X))$ we can identify a range of *inefficient* target values, γ, and exclude them from the risk-return evaluation.

1.4 Related Literature

It is important to note the distinction between the CEV and so-called "risk-adjusted performance measures" in the finance literature. Whereas the CEV only discounts the expected value, classical risk-adjusted measures also account for the spread indicators such as variance; see [6] for a detailed survey. We note that some of the risk-adjusted performance metrics such as the Sharpe Ratio [7], which are based solely on expected value and variance, have been questioned for not taking higher order moments into account. In this regard, some performance measures are defined to address the effect of these moments; e.g., see, [8] and [9]–[11].

Finally, it is instructive to mention a related but yet different line of research called Prospect Theory in Behavioral Finance; e.g., see [12]. This theory describes how a rational individual follows a two-stage process called "editing" and "evaluation." These two phases have a lot in common with what is proposed in the calculation of the CEV since both methods consist of finding a threshold and simplifying the original random variable. Once the distribution is simplified both methods evaluate the profitability of the resulting random variable.

1.5 Remainder of Paper

The remainder of the paper is organized as follows: In Section 2, the Conservative Expected Value is formally defined for a general random variable X with finite leftmost support point. In Section 3, the CEV is calculated for some of the classical probability distributions. Then in Section 4, some of the most important properties of the CEV are established. Finally in Section 5, a discussion of possible research directions is provided.

2. THE CONSERVATIVE EXPECTED VALUE

In this section, the *Conservative Expected Value* is formally defined. The motivation and main steps associated with the calculation below were given earlier in Subsection 1.3.

2.1 The CEV Definition

Let X be a random variable with cumulative distribution function $F_X(x)$ and finite leftmost support point

$$\alpha_X \doteq \inf\{x : x \in \mathbb{R} \text{ such that } F_X(x) > 0\}.$$

Then, given $\gamma \in \mathbb{R}$ and Bernoulli random variable

$$X_\gamma \doteq \begin{cases} \alpha_X & \text{with probability } F_X(\gamma); \\ \gamma & \text{with probability } 1 - F_X(\gamma), \end{cases}$$

the *Conservative Expected Value* of X is defined to be

$$\mathbb{CEV}(X) \doteq \sup_\gamma \mathbb{E}(X_\gamma).$$

2.2 Remarks on the Definition

The definition of CEV can be written in terms of the cumulative distribution function, $F_X(\cdot)$; that is,

$$\begin{aligned} \mathbb{CEV}(X) &= \sup_\gamma \mathbb{E}(X_\gamma) \\ &= \sup_\gamma \alpha_X F_X(\gamma) + \gamma(1 - F_X(\gamma)) \\ &= \sup_\gamma \gamma + (\alpha_X - \gamma) F_X(\gamma). \end{aligned}$$

For the case, $\gamma < \alpha_X$, we see that $X_\gamma = \gamma$ with probability one and moreover $\mathbb{E}(X_{\alpha_X}) > \gamma$. Hence, in the analysis of the supremum entering into the CEV definition, attention can be restricted to $\gamma \geq \alpha_X$. Finally, since the probability masses of X are moved to the left to create X_γ, as shown in Lemma 4.1, we have $\mathbb{CEV}(X) \leq \mathbb{E}(X)$.

2.3 Motivating Example Revisited

Recalling the motivating example given in the introduction and its probability density function $f(x)$, to obtain the CEV, we first find the cumulative distribution function. Via a straightforward calculation, we obtain

$$F(x) = \Phi\left(\frac{\log(1 + 4x) + 1}{2}\right)$$

for $x \geq -0.25$, where Φ is the cumulative distribution function for the standard normal random variable $\mathcal{N}(0, 1)$. As noted earlier, the probability distribution associated with profits and losses was found to be highly-skewed; i.e., $S = 414$. The expected gain-loss was 43% and was deemed insufficient in the presence of large probability of loss, $p_{\text{LOSS}} \approx 0.7$. We now calculate

$$\mathbb{CEV}(g(1)) = \sup_\gamma \mathbb{E}[g(1)_\gamma] = \sup_\gamma \left\{\gamma + (\alpha_{g(1)} - \gamma) F(\gamma)\right\}$$

$$= \sup_\gamma \left\{\gamma - (0.25 + \gamma)\Phi\left(\frac{\log(1 + 4\gamma) + 1}{2}\right)\right\}.$$

A line-search using $\mathbb{E}[g(1)_\gamma]$ leads to maximizer $\gamma^* \approx 1.8$ and we obtain $\mathbb{CEV}(g(1)) \approx -0.12$, which compares to $\mathbb{E}[g(1)] \approx 0.43$. To summarize, after discounting the long tail, the negative sign of CEV is a warning that the classical expected value may be unduly optimistic.

3. COMPUTING CEV: EXAMPLES

The CEV is now calculated for various well-known probability distributions. These examples demonstrate that the CEV can differ dramatically from $\mathbb{E}(X)$.

3.1 Uniform Distribution

Suppose X is uniformly distributed on $[0, 1]$. Then, noting that for $\gamma \in [0, 1]$, $X_\gamma = 0$ with probability γ and $X_\gamma = 1$ with probability $1 - \gamma$, a straightforward calculation leads to expected value

$$\mathbb{E}(X_\gamma) = \begin{cases} \gamma - \gamma^2; & 0 \leq \gamma \leq 1; \\ 0; & \gamma > 1. \end{cases}$$

Hence, $\mathbb{E}(X_\gamma)$ is maximized at $\gamma = 0.5$ with resulting conservative expected value given by $\mathbb{CEV}(X) = 0.25$ which compares with $\mathbb{E}(X) = 0.5$. This result can be generalized to a random variable distributed uniformly over $[\alpha_X, b]$. For this case, we obtain

$$\mathbb{CEV}(X) = \frac{3\alpha_X + b}{4}$$

which compares to $\mathbb{E}(X) = (\alpha_X + b)/2$.

3.2 Bernoulli Random Variable

With random variable $X = 0$ with probability p and $X = 1$ with probability $1 - p$, for $\gamma \geq 0$, a straightforward calculation leads to

$$\mathbb{E}(X_\gamma) = \begin{cases} \gamma(1 - p); & 0 \leq \gamma < 1; \\ 0; & \gamma \geq 1. \end{cases}$$

Now, the supremum in the CEV definition is reached as $\gamma \to 1$ and we obtain

$$\mathbb{CEV}(X) = 1 - p = \mathbb{E}(X).$$

That is, for the "extreme" case of a Bernoulli random variable, no discounting of the classical expected value results.

3.3 Modified Log-Normal Random Variable

The motivating stock-trading example in Section 1 of this paper can be generalized with arbitrary values for the parameters I_0, K, μ, σ and t. That is, beginning with probability density function

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}(I_0 + Kx)} e^{-\frac{\left(\log(1 + \frac{Kx}{I_0}) + 0.5K^2\sigma^2 t - \mu Kt\right)^2}{2K^2\sigma^2 t}}$$

with $\alpha_X = -I_0/K$ and calculating the cumulative distribution $F_X(x)$, we arrive at

$$\mathbb{E}(X_\gamma) = \gamma - \left(\frac{I_0}{K} + \gamma\right)\Phi\left(\frac{\frac{1}{K}\log(\frac{I_0 + K\gamma}{I_0}) + 0.5K\sigma^2 - \mu t}{\sigma\sqrt{t}}\right)$$

where Φ is the cumulative distribution function for the standard normal random variable $\mathcal{N}(0, 1)$. The supremum of $\mathbb{E}(X_\gamma)$ above gives $\mathbb{CEV}(X)$ and is found via a single-variable optimization problem which can easily be solved by a line-search over $\gamma \in [-I_0/K, \infty)$. Then, we can compare $\mathbb{CEV}(X)$ with the classical expected value

$$\mathbb{E}(X) = \frac{I_0}{K}\left[e^{\mu Kt} - 1\right].$$

3.4 Weibull Random Variable

Consider the random variable X having cumulative distribution function

$$F_X(x) = 1 - e^{-(\lambda x)^\alpha},$$

with $\alpha, \lambda > 0$ and for $x \geq 0$. A straightforward calculation leads to

$$\mathbb{E}(X_\gamma) = \gamma e^{-(\lambda\gamma)^\alpha}$$

for $\gamma \geq 0$. Then, setting the derivative to zero gives $\gamma^* = \alpha^{1/\alpha}/\lambda$, and the CEV is obtained as

$$\mathbb{CEV}(X) = \frac{\alpha^{1/\alpha}}{\lambda} e^{-\alpha}.$$

This compares with the classical expected value

$$\mathbb{E}(X) = \frac{1}{\lambda}\Gamma\left(\frac{1}{\alpha}+1\right).$$

We can consider the *percentage discounting* of CEV relative to the classical expected value $\mathbb{E}(X)$; i.e., let

$$\mathrm{PD}(X) \doteq \frac{\mathbb{E}(X) - \mathbb{CEV}(X)}{\mathbb{E}(X)} = 1 - \frac{\alpha^{1/\alpha}e^{-\alpha}}{\Gamma\left(\frac{1}{\alpha}+1\right)}.$$

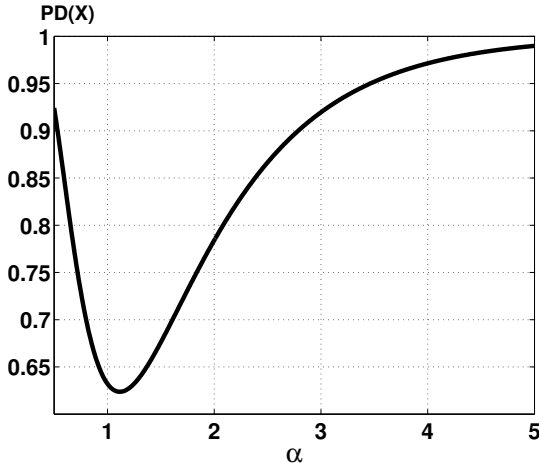A plot of $\mathrm{PD}(X)$ versus α is provided in Figure 2.

Fig. 2. Percentage Discounting for Weibull Random Variable

The lack of monotonicity of $\mathrm{PD}(X)$ with respect to α is interesting to note. The discounting of $\mathbb{E}(X)$ is heavy for small and large values of α. The non-monotonic behavior of $\mathrm{PD}(X)$ is mainly due to the fact that neither the expected value nor the CEV are monotonic functions of α. A Rayleigh random variable, another special case of Weibull random variable is similarly analyzed.

3.5 Pareto Random Variable
For $\alpha_X > 0$ and $\beta > 1$, we consider the cumulative distribution function for random variable X given by

$$F_X(x) = 1 - \left(\frac{\alpha_X}{\gamma}\right)^\beta.$$

We calculate

$$\mathbb{E}(X_\gamma) = \alpha_X\left[1-\left(\frac{\alpha_X}{\gamma}\right)^\beta\right] + \gamma\left(\frac{\alpha_X}{\gamma}\right)^\beta$$

for $\gamma \geq \alpha_X$. Then, taking the derivative of $\mathbb{E}(X_\gamma)$ with respect to γ and setting it to zero, we obtain

$$\gamma^* = \left(1 + \frac{1}{\beta-1}\right)\alpha_X.$$

This leads to

$$\mathbb{CEV}(X) = \frac{\beta\alpha_X}{\beta-1}\left[1 + \frac{-1+\left(1-\frac{1}{\beta}\right)^\beta}{\beta}\right]$$

which can be compared to $\mathbb{E}(X) = \beta\alpha_X/(\beta-1)$. Using the two formulae above, the percentage discounting by the CEV in this example is

$$\mathrm{PD}(X) = \frac{1 - \left(1-\frac{1}{\beta}\right)^\beta}{\beta}.$$

This discounting is monotonically decreasing in β.

4. PROPERTIES OF CEV

In this section, some of the basic properties of the CEV are established. In the lemma below, simple bounds on the $\mathbb{CEV}(X)$ are given. The tightness of these bounds is discussed immediately following the lemma.

4.1 Lemma (Bounds on the CEV)
Let X be a random variable with finite leftmost support point α_X. Then

$$\frac{median(X) + \alpha_X}{2} \leq \mathbb{CEV}(X) \leq \mathbb{E}(X).$$

Proof: Since $\mathbb{E}(X_\gamma) \leq \mathbb{E}(X)$ for all γ, taking the supremum over γ immediately leads to $\mathbb{CEV}(X) \leq \mathbb{E}(X)$. For the lower bound, we consider the special choice $\gamma = median(X)$. Then the expected value of the resulting Bernoulli random variable achieves the lower bound above. This completes the proof. $\square$

4.2 Remarks on CEV Bounds
The lower bound in Lemma 4.1 is achieved when X is uniformly distributed. When X is a Bernoulli random variable the upper bound is achieved; see Section 3 for the derivations. In the following theorem, it shown that the $\mathbb{CEV}(X)$ has an affine linearity property.

4.3 Theorem (Affine Linearity)
Given constants $a \geq 0$ and $b \in \mathbb{R}$, for a random variable X with finite leftmost support point α_X, the CEV satisfies

$$\mathbb{CEV}(aX + b) = a\mathbb{CEV}(X) + b.$$

Proof: The proof is broken in two parts; First, it is proved that $\mathbb{CEV}(aX) = a\mathbb{CEV}(X)$ for given $a \geq 0$ and then it is shown $\mathbb{CEV}(X + b) = \mathbb{CEV}(X) + b$ for any $b \in \mathbb{R}$. Combining these two will complete the proof. For the first part, consider the random variable $Y \doteq aX$. Indeed, proceeding from the definition,

$$\mathbb{CEV}(Y) \doteq \sup_\gamma \mathbb{E}(Y_\gamma) = \sup_\gamma\{\gamma + (\alpha_Y - \gamma)F_Y(\gamma)\}.$$

Now substituting $F_Y(\gamma) = F_X(\gamma/a)$ and noting that Y has leftmost support point $\alpha_Y = a\alpha_X$, we obtain

$$\mathbb{CEV}(Y) = \sup_\gamma\{\gamma + (a\alpha_X - \gamma)F_X(\gamma/a)\}.$$

Using the change of variables $\theta = \gamma/a$ gives

$$\mathbb{CEV}(Y) = \sup_\theta\{a\theta + a(\alpha_X - \theta)F_X(\theta)\} = a\mathbb{CEV}(X).$$

For the second part of the proof, consider the random variable $Z \doteq X + b$. Now

$$\mathbb{CEV}(Z) \doteq \sup_\gamma \mathbb{E}(Z_\gamma) = \sup_\gamma\{\gamma + (\alpha_Z - \gamma)F_Z(\gamma)\}.$$

Then substituting $F_Z(\gamma) = F_X(\gamma - b)$ and noting Z has leftmost support point $\alpha_Z = \alpha_X + b$, we obtain

$$\mathbb{CEV}(Z) = \sup_\gamma\{\gamma + (\alpha_X + b - \gamma)F_X(\gamma - b)\}.$$

Using the change of variables $\theta = \gamma - b$ gives

$$\mathbb{CEV}(Z) = b + \sup_\theta\{\theta + (\alpha_X - \theta)F_X(\theta)\}$$

$$= b + \mathbb{CEV}(X). \square$$

4.4 Average of i.i.d Random Variables
In the theorem to follow, we consider the average $\mathcal{X}_n$ of n independent and identically distributed (i.i.d.) random variables X_k, and show that the $\mathbb{CEV}(\mathcal{X}_n)$ tends to the common expected value, $\mu = \mathbb{E}(X_k)$, as $n \to \infty$.

4.5 Theorem (Average of i.i.d Random Variables)
For positive integers k, let X_k be a sequence of i.i.d. random variables with finite mean $\mathbb{E}(X_k) = \mu$, finite variance σ^2 and finite leftmost support point, $\alpha_{X_k} = \alpha_X$. Then, with partial sum averages given by

$$\mathcal{X}_n \doteq \frac{1}{n}\sum_{k=1}^{n} X_k,$$

it follows that

$$\lim_{n\to\infty} \mathbb{CEV}(\mathcal{X}_n) = \mu.$$

Proof: For each n, note that α_X must be the leftmost support point of $\mathcal{X}_n$; that is, $\alpha_{\mathcal{X}_n} = \alpha_X$. Now, using Theorem 4.3 gives $\mathbb{CEV}(\mathcal{X}_n - \alpha_X) = \mathbb{CEV}(\mathcal{X}_n) - \alpha_X$ and hence, without loss of generality, we assume that $\alpha_X = 0$ in the remainder of the proof which implies $\mu \geq 0$. Now, along the sequence $\mathcal{X}_n$, recalling Lemma 4.1,

$$\mathbb{CEV}(\mathcal{X}_n) \leq \mathbb{E}(\mathcal{X}_n) = \mu.$$

Next, we construct a lower bound for $\mathbb{CEV}(\mathcal{X}_n)$ using a one-sided Chebyshev inequality. Indeed, since $\mathcal{X}_n$ has finite mean μ and bounded variance $\sigma_n^2 = \sigma^2/n$, for $\epsilon > 0$ and each n, the Chebyshev inequality

$$P(\mathcal{X}_n \leq (1-\epsilon)\mu) \leq \frac{\sigma_n^2}{\sigma_n^2 + \epsilon^2\mu^2}$$

is satisfied. Hence, for any $\gamma \in [0,\mu)$, letting $\epsilon = (\mu - \gamma)/\mu$ and noting that $\epsilon > 0$, via the inequality above, we obtain

$$P(\mathcal{X}_n > \gamma) \geq \frac{(\mu-\gamma)^2}{\sigma_n^2 + (\mu-\gamma)^2}.$$

Using this inequality leads to a lower bound for the CEV. That is, using $\alpha_X = 0$, a straightforward calculation yields

$$\mathbb{CEV}(\mathcal{X}_n) = \sup_{\gamma}\; \gamma P(\mathcal{X}_n > \gamma) \geq \sup_{\gamma\in[0,\mu)}\; \gamma\frac{(\mu-\gamma)^2}{\sigma_n^2 + (\mu-\gamma)^2}.$$

For large enough n, noting that $\mu > (1/n)^{0.25}$, for the specific choice $\gamma = \mu - (1/n)^{0.25}$,

$$\sup_{\gamma}\gamma\frac{(\mu-\gamma)^2}{\sigma_n^2 + (\mu-\gamma)^2} \geq \left(\mu - \left(\frac{1}{n}\right)^{0.25}\right)\cdot\frac{\frac{1}{\sqrt{n}}}{\sigma_n^2 + \frac{1}{\sqrt{n}}}$$

Since μ is an upper bound for $\mathbb{CEV}(\mathcal{X}_n)$ and further noting that $\sigma_n^2 = \sigma^2/n$; for large enough n we can combine the inequalities above to obtain

$$\mu \geq \mathbb{CEV}(\mathcal{X}_n) \geq \left(\mu - \left(\frac{1}{n}\right)^{0.25}\right)\frac{1}{\frac{\sigma^2}{\sqrt{n}}+1}.$$

Now letting $n \to \infty$, it is easy to show that the right-hand side tends to μ and hence $\mathbb{CEV}(\mathcal{X}_n)$ tends to μ. $\square$

4.6 Convexity Property of the CEV
Consider a random variable X whose probability density function is a convex combination of the probability density functions of n random variables, $X_1, X_2, \ldots, X_n$; i.e.,

$$f_X(x) = \sum_{i=1}^{n}\lambda_i f_{X_i}(x).$$

where $\lambda_i \geq 0$, $\sum_{i=1}^{n}\lambda_i = 1$ and f_{X_i} is the probability density function for X_i. To illustrate how the situation above arises, consider the case for the random variable describing the output of a system which can switch among n different states. Suppose, the state is modelled by a random variable θ such that $P(\theta = i) = \lambda_i$, for values

of $i = 1, 2, \ldots, n$. Further assume that the output of the system, X, conditioned on the state is modelled by a set of random variables X_i; that is,

$$f_X(x|\theta = i) \doteq f_{X_i}(x).$$

This implies that, the probability density function for X is a convex combination of the f_{X_i} given above. In the lemma below, an upper bound on the CEV of X is given in terms of the convex combination of the $\mathbb{CEV}(X_i)$.

4.7 Lemma (Convexity Property of the CEV)
Let the probability density function f_X of the random variable X be the convex combination above of the probability density functions f_{X_i} of the n random variables, $X_1, X_2, \ldots, X_n$. Then X has a conservative expected value satisfying

$$\mathbb{CEV}(X) \leq \sum_{i=1}^{n}\lambda_i\mathbb{CEV}(X_i).$$

Proof: Without loss of generality, we assume that

$$\alpha_{X_1} \leq \alpha_{X_2} \leq \ldots \leq \alpha_{X_n}.$$

Using the definition of X, it is easy to show that $\alpha_X = \alpha_{X_1}$. Now we calculate

$$\mathbb{CEV}(X) = \sup_{\gamma}\gamma + (\alpha_X - \gamma)F_X(\gamma)$$

$$= \sup_{\gamma}\gamma + (\alpha_X - \gamma)\sum_{i=1}^{n}\lambda_i F_{X_i}(\gamma)$$

$$\leq \sum_{i=1}^{n}\sup_{\gamma}\lambda_i(\gamma + (\alpha_{X_i} - \gamma)F_{X_i})$$

$$= \sum_{i=1}^{n}\lambda_i CEV(X_i). \quad \square$$

4.8 Finiteness of the CEV
This section is concluded with a discussion of the conditions under which the CEV is finite. We begin with the simple observation that $\mathbb{CEV}(X) \leq \mathbb{E}(X)$ implies that $\mathbb{CEV}(X)$ is finite whenever $\mathbb{E}(X)$ is finite. For the case when $\mathbb{E}(X) = \infty$, the CEV can be either finite or infinite; see examples below.

Infinite Expected Value with Finite CEV: This example is known as St. Petersburg Paradox; see [13] for details. Consider the random variable X with probability density function given by $X = 2^k$ with probability $p = 1/2^{k+1}$ for non-negative integers k. Then, it immediately follows that $\mathbb{E}(X) = \infty$. Now, to calculate $\mathbb{CEV}(X)$, noting that $\alpha_X = 1$, we obtain

$$\mathbb{E}(X_\gamma) = P(X \leq \gamma) + \gamma P(X > \gamma) = 1 + \frac{\gamma - 1}{2^{k+1}}$$

for $\gamma \in [2^k, 2^{k+1})$ and non-negative integers k. For every value of k, $\mathbb{E}(X_\gamma)$ varies linearly from $1.5 - 1/2^{k+1}$ to $2 - 1/2^{k+1}$. By letting $k \to \infty$, it is easy to show that $\mathbb{CEV}(X) = 2$.

Infinite CEV: Consider the random variable X with probability density function $f_X(x) = 1/(2x^{\frac{3}{2}})$ for $x \geq 1$. Then a straightforward calculation leads to

$$\mathbb{E}(X_\gamma) = \sqrt{\gamma} - \frac{1}{\sqrt{\gamma}} - 1$$

for $\gamma \geq 1$. Now as $\gamma \to \infty$, we obtain $\mathbb{E}(X_\gamma) \to \infty$. Hence,
$$\mathbb{CEV}(X) = \sup_\gamma \mathbb{E}(X_\gamma) = \infty.$$

Note that $P(X > \gamma) = 1/\sqrt{\gamma}$ is tending to zero as $\gamma \to \infty$ but not as fast as γ is tending to infinity. Since $\mathbb{CEV}(X) \leq \mathbb{E}(X)$, it must also be the case that $\mathbb{E}(X) = \infty$. The following lemma gives a necessary and sufficient condition for the finiteness of the CEV.

4.9 Lemma (Finitieness of the CEV)
For random variable X, the condition
$$\mathbb{CEV}(X) < \infty$$
is satisfied if and only if
$$\limsup_{\gamma \to \infty} \gamma(1 - F_X(\gamma)) < \infty.$$

Proof: First, assuming $\mathbb{CEV}(X) < \infty$, there exists an $M < \infty$ such that for every γ, we have
$$\mathbb{E}(X_\gamma) = \gamma + (\alpha_X - \gamma)F_X(\gamma) < M.$$
Hence for all γ,
$$\gamma(1 - F_X(\gamma)) < M - \alpha_X F_X(\gamma) < M + |\alpha_X|$$
and we obtain
$$\limsup_{\gamma \to \infty} \gamma(1 - F_X(\gamma)) \leq M + |\alpha_X| < \infty.$$
Now suppose $\limsup_{\gamma \to \infty} \gamma(1 - F_X(\gamma)) < \infty$. Then there exist an $M < \infty$ and a $\gamma_M < \infty$ such that for all $\gamma > \gamma_M$,
$$\gamma(1 - F_X(\gamma)) < M.$$
Using the definition of CEV yields
$$\mathbb{CEV}(X) = \sup_\gamma \mathbb{E}(X_\gamma)$$
$$= \max\Big(\sup_{\gamma \in (-\infty, \gamma_M)} \mathbb{E}(X_\gamma), \sup_{\gamma \in [\gamma_M, \infty)} \mathbb{E}(X_\gamma)\Big);$$
$$\leq \max\big(\gamma_M, M + |\alpha_X|\big) < \infty$$
which completes the proof. $\square$

5. CONCLUSION AND FUTURE WORK

In this paper, the Conservative Expected Value was introduced. Its motivation in terms of financial markets, large skewness and model distrust was discussed. The calculation of the CEV was demonstrated for a number of well-known probability distributions and a random variable corresponding to the gains and losses of a feedback-based stock-trading strategy. Some of the important properties of the CEV were established. Recalling that the CEV is defined for random variables with finite leftmost support point, it is natural to consider the possibility of an extension of the definition to include random variables with unbounded support. It is also of interest to develop an alternative for variance; i.e., a *Conservative Variance*.

We envision use of the CEV in a number of applications. In finance, when the degree of distrust in a model is large, one can replace the ordinary expected value with the CEV in the analysis. For example, one can consider "modified" Sharpe and Sortino Ratios by replacing the expected value by the CEV. Another possible application of CEV is in Control Theory. If one considers a stochastic linear system which involves uncertain parameters, for example, see [14], instead of analysing various quantities using the expected value, the CEV can be used to robustify against lack of accuracy in the underlying model.

Finally, in the study of system reliability, for example, see [15], various aspects of performance are modelled by random variables. The "mean time between failures" (MTBF) is frequently used and the corresponding random variable is usually assumed to be highly-skewed; e.g., exponentially distribution is a commonly used model. Motivated by this large skewness and possible model distrust, the CEV may be an appropriate alternative to the classical expected value.

REFERENCES

[1] Malekpour, S. and B. R. Barmish, "How Useful are Mean-Variance Considerations in Stock Trading via Feedback Control?," Proceedings of the IEEE Conference on Decision and Control, pp. 2110–2115, 2012.

[2] Barmish, B. R., J. A. Primbs, S. Malekpour and S. Warnick, "On the Basics for Simulation of Feedback-Based Stock Trading Strategies: An Invited Tutorial Session," Proceedings of the IEEE Conference on Decision and Control, pp. 7181–7186, 2013.

[3] Malekpour, S. and B. R. Barmish, "A Drawdown Formula for Stock Trading Via Linear Feedback in a Market Governed by Brownian Motion," Proceedings of the European Control Conference. pp. 87–92, 2013.

[4] Barmish, B. R. and J. A. Primbs, "On Arbitrage Possibilities Via Linear Feedback in an Idealized Brownian Motion Stock Market," Proc. of the IEEE Conference on Decision and Control, pp. 2889–2894, 2011.

[5] Hansen, L. P. and T. J. Sargent, Robustness, Princeton University Press, 2011.

[6] Le Sourd, V., "Performance Measurement for Traditional Investment: Literature Survey," Financial Analysts Journal, vol. 58, no. 4, pp. 36–52, 2007.

[7] Sharpe, W. F., "Mutual Fund Performance," Journal of Business, vol. 39, no. 1, pp. 119–138, 1966.

[8] Keating, C. and W. F. Shadwick, "A Universal Performance Measure," Journal of Performance Measurement, vol. 6, no. 3, pp. 59–84, 2002.

[9] Sortino, F. A. and L. N. Price, "Performance Measurement in a Downside Risk Framework," The Journal of Investing, vol. 3, no. 3, pp. 59–64, 1994.

[10] Hwang S. and S. E. Satchell, "Modelling Emerging Market Risk Premia Using Higher Moments," International Journal of Finance and Economics, vol. 4, no. 4, pp. 271–296, 1999.

[11] Ziemba, W. T., "The Symmetric Downside-Risk Sharpe Ratio," The Journal of Portfolio Management, vol. 32, no. 1, pp. 108–122, 2005.

[12] Kahneman, D. and A. Tversky, "Prospect Theory: An Analysis Of Decision Under Risk," Econometrica, vol. 47, no. 2, pp. 263–291, 1979.

[13] Bernoulli, D., "Exposition of a New Theory on the Measurement of Risk," Econometrica: Journal of the Econometric Society, vol. 22, no. 1, pp.23–36, 1954.

[14] Kushner, H., Introduction to Stochastic Control, Brown University, Providence RI Division of Applied Mathematics, 1971.

[15] Høyland, A. and M. Rausand, System Reliability Theory Models and Statistical Methods. John Wiley and Sons, 2004.