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Observation and Observers for Systems from Delay Convoluted Observation ⋆

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Abstract: This paper analyzes finite dimensional linear time-invariant systems with observation of a delay, where that delay satisfies a particular implicit relation with the state variables, rendering the entire problem nonlinear. The objective is to retrieve the state variables from the measured delay. The first contribution involves the direct inversion of the delay, the second is the design of a finite dimensional observer, and the third presents properties of the delay - state relation. Realistic examples treat vehicles with ultrasonic position sensors.

Keywords: Nonlinear systems, delay systems, observers and observation

1. INTRODUCTION

This paper revisits the "soft landing" problem, posed in Walther [2007]. The problem is one where the dynamics is linear and finite dimensional, but the observed quantity (either directly or indirectly) is a delay, which is itself dependent on the present and the delayed state. In the simple version of the soft landing problem, the dynamics are given by a second order system with state variables position, x and velocity v. The objective is to obtain an estimate of the full system state from these delay observations (the "delay-inversion"). For instance, if a robot is to avoid hitting a wall, x is the distance to this wall (in a one-dimensional configuration space), and v the rate of change. The state vector $\xi = [x, v]^\top$ satisfies in this case simple Newtonian dynamics $\dot{x} = v,\ \dot{v} = u$. If the problem is one of soft landing, say on the ocean floor, the second state equation should include the relevant viscous friction: $\dot{v} = -kv + u$. In both cases, a sound wave is used to detect the position. As the speed of sound is finite, this means that only some past position can be measured. Precisely how far in the past depends again on the state itself. That is where the convoluted implicit relation between output and state-dependent delay appears, rendering the problem more difficult (e.g., see Hartung et al. [2006]), even in toy systems (Verriest [2012, 2013]). Moreover the state-dependent delay model leads to inconsistencies if the rate of change of the delay exceeds 1, as expounded in Verriest [2010, 2011].
In Ahmed and Verriest [2013] the Newtonian system with observation model $\gamma\tau(t) = x(t) + x(t - \tau(t))$, where γ is the speed of sound, was studied. An asymptotic estimate was provided by a new type of observer (using "delay-injection"). This observer is itself a system with time-varying delay, for which only sufficient conditions for convergence are easily obtained. In this paper we will show that direct inversion of the delay (i.e., the map from delay τ to output (position x) is possible in a certain subinterval. We also provide an observer which does not involve delayed dynamics, hence is finite dimensional and easier to implement. Thus motivated, a general version of this problem is posed and solved.

2. THE GENERAL PROBLEM

Let the dynamics be given by the finite dimensional system

$$\dot{\xi} = A\xi + bu, \quad y = c\xi, \tag{1}$$

where $\xi \in \mathbb{R}^n$ is the state and u and y are respectively a scalar input and output signal. However, the output is not directly measured. Only an indirect observation of y, given by the convolution

$$\tau(t) = \sum_{k=0}^{N-1} a_k y(t - k\tau(t)), \tag{2}$$

and parameterized by the vector $a = [a_1, \ldots, a_N]^\top$ is available. This may be written more compactly with the implicit form $\tau(t) = a^\top Y(t, \tau(t))$. where $Y(t, \tau(t))^\top = [y(t), y(t - \tau), \ldots, y(t - (N-1)\tau)]$.
The observability question is now: Can one retrieve the state $\xi(t)$ from knowledge of the past history of the delay $\tau(t)$ and the applied input, $u(t)$? If so, we will say that the system $\Sigma = (A, b, c)$ is *state-observable from the delay.*

3. INVERSION OF THE DELAY

Similar to the derivation of the observer for a linear system as for instance described in Kailath [1980], apply successive differentiation of the output, τ:

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$$\tau(t) = \sum_{k=0}^{N-1} a_k c\xi(t - k\tau(t))$$

$$\dot{\tau}(t) = \sum_{k=0}^{N-1} a_k \left[cA\xi(t - k\tau(t)) + cbu(t - k\tau(t))\right](1 - k\dot{\tau}(t)),$$

etc. These equations can be streamlined in matrix form. Let $\mathcal{T}$ denote the vector of successive derivatives of τ, and define for each $k = 0, 1, \ldots, N-1$, the vector $\mathcal{U}(t - k\tau)$ by

$$\mathcal{T}(t) = \begin{bmatrix} \tau \\ \dot{\tau} \\ \ddot{\tau} \\ \vdots \\ \tau^{(n-1)} \end{bmatrix}, \quad \mathcal{U}(t - k\tau) = \begin{bmatrix} u(t-k\tau) \\ \dot{u}(t-k\tau) \\ \ddot{u}(t-k\tau) \\ \vdots \\ u^{(n-1)}(t-k\tau) \end{bmatrix}$$

Let the matrix of powers and derivatives of $(1 - k\dot{\tau})$ be denoted by $\mathbb{T}_k(\tau)$,

$$\mathbb{T}_k(\tau) = \begin{bmatrix} 1 & & & \\ 0 & 1-k\dot{\tau} & & \\ 0 & -k\ddot{\tau} & (1-k\dot{\tau})^2) & \\ \vdots & \vdots & \ddots & \\ 0 & -k\tau^{(n-1)} & \ldots & (1-k\dot{\tau})^{n-1} \end{bmatrix}.$$

Then in compact format:

$$\mathcal{T}(t) = \sum_{k=0}^{N-1} a_k \left[\mathbb{T}_k(\tau)\mathbf{O}(\Sigma)\xi(t - k\tau) + \mathbf{T}(\Sigma)\mathcal{U}(t - k\tau)\right],$$

where $\mathbf{O}(\Sigma)$ and $\mathbf{T}(\Sigma)$ are respectively the observability and Toeplitz matrix of the LTI system $\Sigma = (A, b, c)$.

$$\mathbf{O}(\Sigma) = \begin{bmatrix} c \\ cA \\ cA^2 \\ \vdots \\ cA^{n-1} \end{bmatrix}, \quad \mathbf{T}(\Sigma) = \begin{bmatrix} 0 & & & \\ cb & 0 & & \\ cAb & cb & 0 & \\ \vdots & & \ddots & \ddots \\ cA^{n-2}b & \ldots & cb & 0 \end{bmatrix}.$$

But the solution of the system state equation gives

$$\xi(t - k\tau) = \mathrm{e}^{-Ak\tau}\,\xi(t) - J_k(\{u\}_{t-k\tau}^{t}),$$

where $J_k(\cdot)$ denotes the convolution integral. Hence

$$\mathcal{T}(\tau) + \sum_{k=0}^{N-1} a_k \left[\mathbb{T}_k\mathbf{O}(\Sigma)J_k\left(\{u\}_{t-k\tau}^{t}\right) - \mathbf{T}(\Sigma)\mathcal{U}(t - k\tau)\right]$$
$$= \left[\sum_{k=0}^{N-1} a_k\mathbb{T}_k\mathbf{O}(\Sigma)\,\mathrm{e}^{-Ak\tau}\right]\xi(t). \quad (3)$$

It follows from (3) that the state, $\xi(t)$, can be retrieved from the input and delay history if the matrix

$$\mathbf{O}_1(\Sigma, a, \tau(t)) \overset{\text{def}}{=} \left[\sum_{k=0}^{N-1} a_k\mathbb{T}_k\mathbf{O}(\Sigma)\,\mathrm{e}^{-Ak\tau(t)}\right]$$

is nonsingular for all t.
Denoting the sum which depends on u simply by $\mathcal{A}(\{u\}, \tau))$, the inversion is

$$y(t) = c\mathbf{O}_1^{-1}(\Sigma, a, \tau)\left[\mathcal{T}(\tau) + \mathcal{A}(\{u\}, \tau)\right].$$

Theorem 1. Observability of the system Σ with output y is necessary for state-observability from τ.

Proof: By contradiction using the PBH-test. □
The above then proves:

Theorem 2. The system Σ is state observable from the delay τ if the matrix $\mathbf{O}_1(\Sigma, a, \tau(t))$ is nonsingular for all t.

In sections 3, 4 and 5, we reconsider the soft-landing problem in the one-dimensional configuration space (which corresponds to a second order system) for two realistic observation models.

4. EXAMPLE 1

Consider a mobile unit (MU) of mass m, moving in a viscous fluid with friction coefficient α. Let the mass emit a continuous time-stamped signal $s(t)$. By the latter it is meant that if the signal $s(t)$ is transmitted at time t_x, and observed at a later time t, after propagating with a speed γ for a time $t - t_x$, the transmission time t_x can be detected. Consider now the following (passive) problem: Suppose that the signal $s(t_x)$ is emitted by the MU, when it is at position $x(t_x)$ and detected by a stationary observer located at the origin at time t. Since the signal has traveled for a distance $\gamma(t - t_x) = x(t_x)$, it reveals an earlier position of the MU to this stationary observer. In this example we assume that the receiver sits at an impenetrable wall so that we may assume that $x(\cdot) \geq 0$. This could model the (one-dimensional) vertical motion of a submersible, with the detector at the bottom of the ocean. Letting $t - t_x = \tau(t)$, this gives

$$x(t - \tau(t)) = \gamma\tau(t),$$

which corresponds with $a^\top = \left[0, \frac{1}{\gamma}\right]$ in the general model.

4.1 Exact inversions

It is fairly simple to derive $\tau(\cdot)$ from knowledge of $x(\cdot)$. See Figure 1. Let $x(t)$ be given in $[t_0, t_1]$, with $x(t_0) = x(t_1) = 0$. Consider the point A with coordinates $(t, x(t)/\gamma)$ on the graph of x/γ. Construct the line with slope -1 through A, which intersects the time axis in B, with coordinates $(t + x(t)/\gamma, 0)$. The horizontal line through A and the vertical line through B intersect in C, which has the coordinates $(t + x(t)/\gamma, x(t)/\gamma)$ and lies on the graph of τ. In fact, if $t' = t + x(t)/\gamma$, then $\tau(t') = x(t)/\gamma$, which gives a parameterized form of the graph of τ. Moreover, if x is differentiable, then two neighboring points on the graph of x/γ, say $(t, x(t)/\gamma)$ and $(t + \mathrm{d}t, (x(t) + \dot{x}(t)\mathrm{d}t)/\gamma)$, map

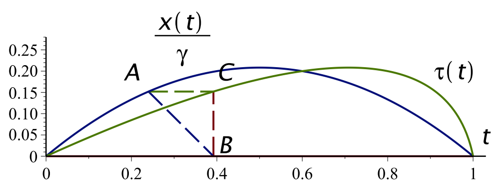

Fig. 1. Constructing $\tau(\cdot)$ from $x(\cdot)$

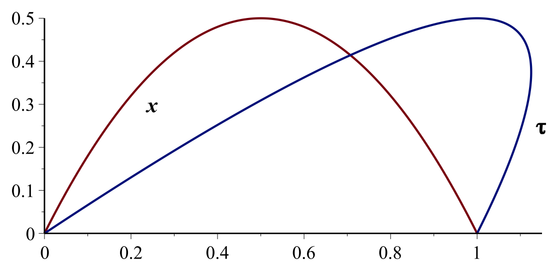

Fig. 2. Non-unique $\tau(\cdot)$ if $\dot{x} < -1$.

to $(t + x(t)/\gamma, x(t)/\gamma) = (t', \tau(t'))$ and $(t + \mathrm{d}t + (x(t) + \dot{x}(t)\,\mathrm{d}t)/\gamma) = (t' + \mathrm{d}t', \tau(t') + \dot{\tau}(t')\mathrm{d}t')$. This implies

$$\dot{\tau}(t') = \frac{\dot{x}(t)}{\gamma + \dot{x}(t)}. \tag{4}$$

Imposing the causality constraint $\dot{\tau} < 1$, see Verriest [2011], implies then a constraint on the feasible functions x, namely $\dot{x} > -\gamma$. Indeed, it can be seen that when this constraint is violated, a unique τ cannot be constructed. See Figure 2. For $t \geq 1$, two compatible delay values occur.

The observation problem is actually the reverse of the above. It is desired to reconstruct x from observations of the delay τ. For this the previous graphical construction can be inverted. Let the delay $\tau(\cdot)$ be specified and strictly positive in the interval (t_0, t_1), and assume it satisfies the causality constraint. Then for $t_0 \leq t \leq t_1$, the parameterized point $(t - \tau(t), \gamma\tau(t))$ lies on the graph of x (Figure 3). Point A has coordinates $(t, \tau(t))$. The line AB has slope 1, so that B has coordinates $(t - \tau(t), 0)$. The vertical through B intersects the horizontal through A to give C with coordinates $(t - \tau(t), \tau(t))$. The length of BD is γ times the length of BC, thus D has coordinates $(t - \tau(t), \gamma\tau(t))$ and therefore lies on the graph of x. Finally, note that $x(\cdot)$ can only be determined in the interval $(t_0 - \tau(t_0), t_1 - \tau(t_1))$.

4.2 Analyticity

Theorem 3. In (4): x is analytic $\Leftrightarrow$ τ is analytic.

Proof: (1.) If $\tau(\cdot)$ is analytic, then $t - \tau(t)$ is analytic. Suppose that x were not analytic, then $x(t - \tau(t))$ would also not be analytic, contradicting analyticity of $\gamma\tau(t)$.

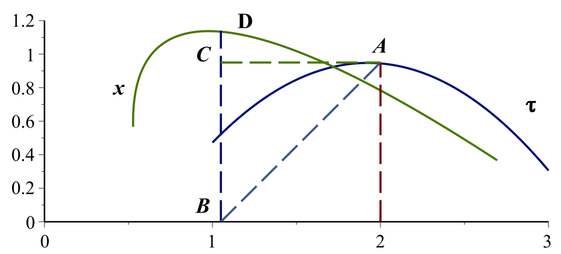

Fig. 3. Construction of x from τ (for $\gamma = 1.2$).

(2.) If $x(\cdot)$ is analytic, then let $x(t') = \tau(t)$, $\quad t' = t - \tau(t)$. Thus $t = t' + \frac{1}{\gamma}x(t')$, so that t is an analytic function of t'. Since $\tau(t(t')) = \frac{1}{\gamma}x(t')$ is an analytic function of t', and $t(t')$ is analytic it *must* follow that $\tau(\cdot)$ is analytic. The latter follows by contradiction: Suppose that $\tau(\cdot)$ were not analytic, then $\tau(t(t')) = \frac{1}{\gamma}x(t')$ is not analytic. $\square$

Note that x is generated by a finite dimensional linear time invariant ODE. Hence if the driving force u is an analytic function of time, so will be x, and by the theorem therefore also the delay τ.

4.3 State observability from the delay

Let's temporarily leave this delay model, and see how the state equations generate the observations in this model. With the state defined as $\xi = [x, v]^\top$, where x is position and v the velocity, the state space realization is given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}, \quad c = [1, 0].$$

Let's first look at the dynamics without the delay, assuming that the position is directly observed: $y(t) = x(t)$. The observability matrix $\mathbf{O}(A, c) = I$, and thus

$$x(t) = y(t), \quad v(t) = \dot{y}(t),$$

irrespective the applied force u. In fact, by taking a second derivative of the observation, the unknown input can be found by differentiation:

$$u(t) = m\left(\ddot{y}(t) + \alpha\dot{y}(t)\right).$$

Thus the state as well as the input are observable from y. Note that this delay-free case corresponds to the limit of a model where the speed of the MU is much smaller than the propagation speed of the signal ($\gamma = c$). Indeed, $x(t - \tau(t)) \approx x(t) - \dot{x}(t)\tau(t)$, and thus

$$x(t) \approx c\left(1 + \frac{\dot{x}(t)}{c}\right)\tau(t) \approx c\tau(t).$$

With the delay incorporated in the model, it is easily seen from (1) and (2) that

$$\begin{aligned} x(t - \tau(t)) &= \gamma\tau(t) \\ v(t - \tau(t)) &= \gamma\frac{\dot{\tau}(t)}{1 - \dot{\tau}(t)}. \end{aligned}$$

Hence, only *past* values of the state can be detected. Since $\tau(t)$ is detected it is known precisely at which past time these state values are known. The dynamical equations also yield the input value from

$$u(t - \tau(t)) = m\left(\dot{v}(t - \tau(t)) + \alpha v(t - \tau(t))\right).$$

The chain rule gives

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t - \tau(t)) = \dot{v}(t - \tau(t))\,(1 - \dot{\tau}(t)). \tag{5}$$

But the he left hand side of (5) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\gamma\dot{\tau(t)}}{1 - \dot{\tau(t)}} = \frac{\gamma\ddot{\tau}}{1 - \dot{\tau}} + \frac{\gamma\dot{\tau}\ddot{\tau}}{(1 - \dot{\tau})^2} = \frac{\gamma\ddot{\tau}}{(1 - \dot{\tau})^2}$$

So: $\dot{v}(t - \tau(t)) = \frac{\gamma\ddot{\tau}}{(1 - \dot{\tau})^3}$, and

$$u(t-\tau(t)) = m\gamma\frac{\ddot{\tau}(t)+\alpha\dot{\tau}(t)(1-\dot{\tau}(t))^2}{(1-\dot{\tau}(t))^3}.$$

If the input force, $u(t)$, is known for $t \geq 0$, but not the initial state, then at time $t > 0$, the state $[x(t-\tau(t)), v(t-\tau(t)]^\top$ is detected. This can be integrated forward to get

$$x(t) = x(t-\tau(t)) + \frac{1}{2m}\int_{t-\tau(t)}^{t}(t-s)\,\mathrm{e}^{-\alpha(t-s)}\,u(s)\,\mathrm{d}s$$
$$= \gamma\tau(t) + \frac{1}{2m}\int_{t-\tau(t)}^{t}(t-s)\,\mathrm{e}^{-\alpha(t-s)}\,u(s)\,\mathrm{d}s$$

$$v(t) = v(t-\tau(t)) + \frac{1}{m}\int_{t-\tau(t)}^{t}\mathrm{e}^{-\alpha(t-s)}\,u(s)\,\mathrm{d}s$$
$$= \gamma\frac{\dot{\tau}(t)}{1-\dot{\tau}(t)} + \frac{1}{m}\int_{t-\tau(t)}^{t}\mathrm{e}^{-\alpha(t-s)}\,u(s)\,\mathrm{d}s.$$

Since $\tau(t)$ is measured without error, $\dot{\tau}(t)$ is known, and the above integrals are computable at time t.

4.4 *A Finite dimensional asymptotic observer*

In practical situations, measurements cannot be perfect. Hence the observed $\tau(t)$ may be imbedded in a wildly fluctuating perturbation $w(t)$, which may be deterministically or stochastically modeled. In either case differentiation is impractical. The way out is then to use a *dynamic observer.* Since the dynamic model is finite dimensional, the basic simulator with *delay error injection* is

$$\dot{\xi}(t) = \eta(t) + \ell_x(\tau(t)-\hat{\tau}(t)) \tag{6}$$
$$\dot{\eta}(t) = -\alpha\eta(t) + \frac{1}{m}u(t-\tau(t)) + \ell_v(\tau(t)-\hat{\tau}(t)), \tag{7}$$

where

$$\hat{\tau}(t) = \frac{1}{\gamma}\xi(t). \tag{8}$$

Note however that it is necessary to drive this observer with the delayed input. Subtracting these equations from the *delayed* dynamical model, evaluated at $t-\tau(t)$, and setting

$$\begin{bmatrix}\tilde{x}(t-\tau(t))\\ \tilde{v}(t-\tau(t))\end{bmatrix} = \begin{bmatrix}x(t-\tau(t))\\ v(t-\tau(t))\end{bmatrix} - \begin{bmatrix}\xi(t)\\ \eta(t)\end{bmatrix},$$

we get the error model

$$\dot{\tilde{x}} = \tilde{v} - \frac{\ell_x}{\gamma}\tilde{x}, \quad \dot{\tilde{v}} = -\alpha\tilde{v} - \frac{\ell_v}{\gamma}\tilde{x},$$

evaluated at $t-\tau(t)$. Hence if the observer gain $\ell = [\ell_x, \ell_v]^\top$ is chosen so that the observer dynamical matrix

$$\begin{bmatrix} -\dfrac{\ell_x}{\gamma} & 1\\ -\alpha-\dfrac{\ell_v}{\gamma} & 0\end{bmatrix},$$

which has characteristic polynomial, $s^2 + \frac{\ell_x}{\gamma}s + \alpha + \frac{\ell_v}{\gamma}$, is Hurwitz, the error will converge to zero. Consequently, the observer (6,7) is an asymptotic estimator of the *past* state, $[\hat{x}(t-\tau(t)), \hat{v}(t-\tau(t))] = [\xi(t), \eta(t)]$. A prediction step then completes the observer for the present state

$$\hat{x}(t) = \xi(t) + \frac{1}{2m}\int_{t-\tau(t)}^{t}(t-s)\,\mathrm{e}^{-\alpha(t-s)}\,u(s)\,\mathrm{d}s$$
$$\hat{v}(t) = \eta(t) + \frac{1}{m}\int_{t-\tau(t)}^{t}\mathrm{e}^{-\alpha(t-s)}\,u(s)\,\mathrm{d}s.$$

The error goes also asymptotically to zero if $u(\cdot)$ is perfectly known. In the other case, bounds are easily obtained for the integrals in the above expression.

5. EXAMPLE 2

Consider now the system from Example 1, but with the sonar device (transmitter and receiver) located on the mobile unit (MU). This corresponds to the special case $a^\top = \left[\frac{1}{\gamma}, \frac{1}{\gamma}\right]$. Consider thus

$$x(t) + x(t-\tau(t)) = \gamma\tau(t). \tag{9}$$

Without any knowledge of the dynamics involved, what can now be inferred from the observation model (9)?

5.1 *Causality*

First consider the simple limiting case: $\tau(t) = t - t_0$, for some $t \in (t_1, t_2)$ with $t_1 \geq t_0$ in order to maintain causality. Substitution in equation (9) leads to

$$x(t) = -x(t_0) + \gamma(t-t_0), \quad t \in (t_1, t_2).$$

Note that if $t_0 = t_1$, i.e., $\tau(t_0) = 0$, it follows from the above that also $x(t_0) = 0$.

The limit case can thus only occur when $x(\cdot)$ is a straight line with slope γ. This is equivalent to $\dot{\tau} = 1$, this truly being the limit case for causal behavior.

Let $\tau(t) \geq 0$ be given in (t_0, t_1) and assume it satisfies the causality constraint $\dot{\tau}(t) < 1$. Differentiating (9) gives

$$\dot{x}(t) + \dot{x}(t-\tau(t))(1-\dot{\tau}(t)) = \gamma\dot{\tau}(t).$$

The causality constraint imposes

$$\dot{\tau}(t) = \frac{\dot{x}(t) + \dot{x}(t-\tau(t))}{\gamma + \dot{x}(t-\tau(t))} < 1.$$

For $\gamma + \dot{x}(t-\tau(t)) > 0$, this inequality yields

$$\dot{x}(t) < \gamma.$$

Hence $|\dot{x}| < \gamma$ implies consistent (causal) behavior. The physical meaning is that the MU should not move faster than the speed of sound.

5.2 *Obtaining τ from x.*

Consider the forward problem: Determine $\tau(t)$, satisfying (9), from full knowledge of $x(t)$ in the interval (t_1, t_2). We shall assume that u is also perfectly known in this case.

From time t_1' on, where $t_1' - t_1 = \tau(t_1')$, the delay $\tau(\cdot)$ is well defined. Reorganize the equation as

$$x(\theta') = -x(\theta) + \gamma(\theta'-\theta), \quad \tau(\theta') = \theta' - \theta.$$

The construction is as follows. From a point $(\theta, -x(\theta))$ draw the line with slope γ. This line intersects the curve $x(t)$ in a point with horizontal coordinate θ'. The delay at θ' is then $\tau(\theta') = \theta' - \theta$. See Figure 4.

Point B′ has coordinates $(t, -x(t))$. The line B′C′ has slope γ, and intersects the curve $x(\cdot)$ in C′, so that C′ has

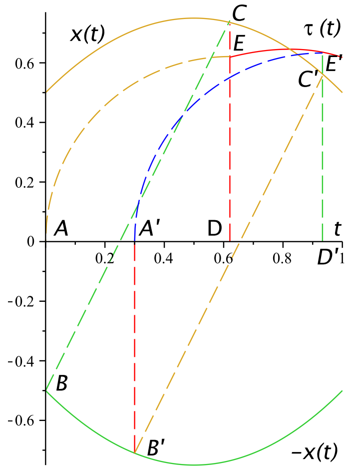

Fig. 4. Construction of τ from x (for $\gamma = 2$).

coordinates $(t', x(t'))$. The vertical through C$'$ intersects the time axis in D$'$. The delay at t' is then the length A$'$D$'$=E$'$D$'$. This creates the point E$'$ with coordinates $(t', \tau(t'))$. Likewise, ABCDE gives the construction for the first time for which τ can be derived.
An alternative construction (Figure 5) of the same follows from

$$t' - t = \tau(t'), \quad x(t') - \gamma t' = -x(t) - \gamma t.$$

Plot the graphs of $\pm x(t) - \gamma t$. Let point B have coordinates $(t, -x(t) - \gamma t)$. The horizontal through B intersects $x(s) - \gamma s$ in C with coordinates $(t', x(t') - \gamma t')$. The delay at t' is therefore $\tau(t') = t' - t$.

5.3 Obtaining x from τ

Finally, consider the converse construction of $x(t)$ from $\tau(t)$.
Assume that $\tau(\cdot)$ is known in the interval (t_0, t_1), with $\tau(t_0) = \tau(t_1) = 0$. As discussed, this implies that $x(t_0) = x(t_1) = 0$, and if $\tau(t) > 0$, for some $t \in (t_0, t_1)$, then $x(t) > 0$. Consider figure 6. At time t, the delay $\tau(t)$ is known (point B). The line through B with slope 1,

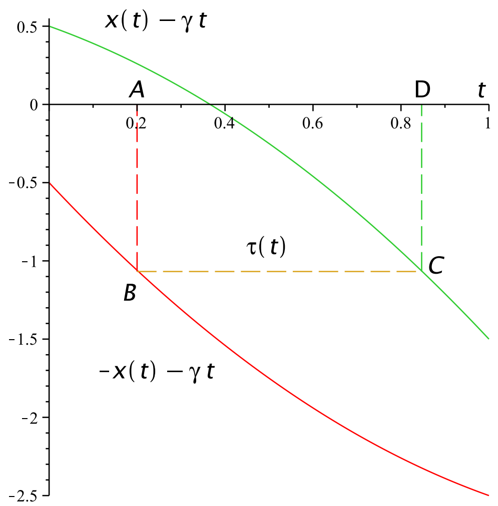

Fig. 5. Alternative construction of τ from x (for $\gamma = 2$).

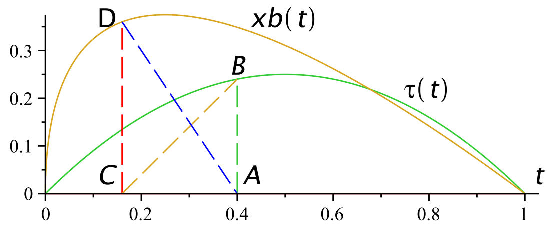

Fig. 6. Construction of x from τ (for $\gamma = 1.5$).

intersects the time axis in point C, determining the time $t - \tau(t)$. It holds that

$$x(t - \tau(t)) + x(t) = \gamma \tau(t)$$

Hence since $x(t) \geq 0$, it holds that

$$x(t - \tau(t)) \leq \gamma \tau(t).$$

Through point A construct the line with slope $-\gamma$. This line intersects the vertical through C in point D. Hence, it follows that $x(t - \tau(t))$ must be constrained to the interval CD. Since this construction can be performed for all $t \in (t_0, t_1)$, an *upper bound* for $x(t)$, the line $xb(t)$, is obtained. The same construction holds when $\tau(t_0)$ and $\tau(t_1)$ are nonzero. See Figure 7, for $\gamma = 0.5$. In this case the interval where the upper bound is known differs from the interval where the delay τ is known.

Can one actually obtain the exact values of x from τ? Consider again Figure 7. In order to determine the value of $x(t)$ at t_A, one needs to know x at time t_C. We only know this value is constrained to the interval CD, but otherwise we may assume it to be 'free'. Thus the construction defines a mapping of x in the interval CA, to x in an interval starting at AA$'$, where A$'$ is the time at which the parallel to CB intersects the graph of τ. This mapping is given by

$$\forall \theta \in (t_C, t_A), \forall x \in (0, x_b(\theta)) : \ (\theta, x) \to (\theta', -x + \gamma(\theta' - \theta)),$$

where θ' is the explicit function, say $\theta' = T(\theta)$, associated with the implicit relation $\theta' - \theta = \tau(\theta')$. By the implicit function theorem, this explicit function will exist (and be unique) if $\dot{\tau} \neq 1$. But this is holds in view of the causality requirement we had imposed on the problem.

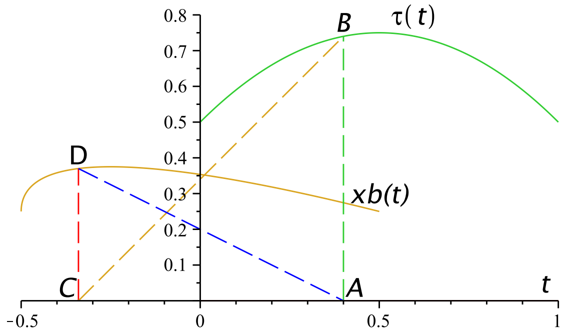

Fig. 7. Upper bound for x from τ (for $\gamma = 0.5$).

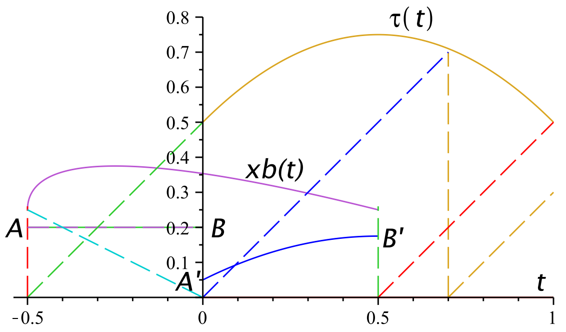

Fig. 8. A discontinuous candidate for the function $x(t)$.

It follows that many initializations exist which will give a consistent value for $x(t)$ over the interval. Unless we have some side information about x, no unique solution can result. What could such side information be? For the case of Figure 7, consider the initialization by $x(\theta) = 0.2$ in the interval $(-0.5, 0)$. This corresponds to the segment AB in Figure 8, and it gets mapped to A′B′. Although it satisfies the constraint $x(t) < x_b(t)$, this cannot correspond to a solution of the problem if it is known that $x(t)$ should be a continuous function, as we get a discontinuity at zero. Even if we adjust the initial data in the interval $(-0.5, 0)$ so that the continuation by the above mapping is continuous, differentiability may fail at zero. But this requisite side information is precisely what one would get from a dynamical model for $x(t)$.

The problem is simpler if $\tau(t_0) = 0$, and $\tau(t)$ is differentiable. A differentiable solution of $x(t)$ is obtained by differentiating the defining equation. Indeed let $\epsilon > 0$ be small. Then from

$$x(t_0 + \epsilon) + x(t_0 + \epsilon - \tau(t_0 + \epsilon)) = \gamma\tau(t_0 + \epsilon)$$

we get

$$x(t_0) + \epsilon\dot{x}(t_0) + x(t_0) + (\epsilon - \tau(t_0 + \epsilon))\dot{x}(t_0) = \gamma\tau(t_0 + \epsilon).$$

This yields

$$\dot{x}(t_0) = \frac{\gamma\dot{\tau}(t_0)}{2 - \dot{\tau}(t_0)}.$$

5.4 Behavior near a common zero of x and τ.

Without loss of generality let $t = 0$ be the common zero. If $x(t)$ has dominant behavior $x(t) = at^\mu$ for $\mu > 0$ and $a > 0$, then substitution in (9) gives

$$at^\mu + at^\mu\left(1 - \frac{\tau(t)}{t}\right)^\mu = \gamma\tau(t).$$

Causality imposes $\tau(t) < t$, hence the factor $\left(1 - \frac{\tau(t)}{t}\right)$ takes values in the interval $(0, 1)$. It follows then that

$$\frac{a}{\gamma}t^\mu < \tau(t) < \frac{2a}{\gamma}t^\mu.$$

Conversely, if $\tau(t)$ has dominant behavior $\tau(t) = bt^\nu$, where for causality reasons $\nu > 1$, then

$$x(t) + x(t - bt^\nu) = \gamma bt^\nu$$

from which a first order Taylor expansion gives the ODE

$$2x(t) - bt^\nu\dot{x}(t) = \gamma bt^\nu$$

But this is non-Lipshitz, so a unique solution may not be inferred. Upon substituting $x(t) = at^\mu$, one gets

$$2at^\mu - ab\mu t^{\mu+\nu-1} = \gamma bt^\nu.$$

If $\nu < 2$, the left hand side becomes negative and no conclusion can be drawn from this approximation. But if $\nu > 2$, then the second term on the left may be neglected compared to the first, leading to the viable solution $a = \gamma b/2$ and $\mu = \nu$, thus $x(t)$ behaves as $x(t) = \frac{\gamma b}{2}t^\nu$. Finally, note that a *linear* increase in both x and τ is compatible. Indeed, letting $\tau(t) = bt$ and $x(t) = at$ in (9) gives $at + a(t - bt) = \gamma bt$ from which the complementary relations

$$a = \frac{\gamma b}{2 - b}, \quad b = \frac{2a}{\gamma + a} \tag{10}$$

are exact. One can ask again, if as in example 1, analytic solutions exist

$$x(t) = \sum_{i=1}^{\infty} a_i t^i, \quad \tau(t) = \sum_{i=1}^{\infty} b_i t^i.$$

For instance, the second order approximations for x and τ in the neighborhood of a common zero (placed at t=0), $x(t) = a_1 t + a_2 t^2$ and $\tau(t) = b_1 t + b_2 t^2$ leads again to $2a_1 - a_1 b_1 = \gamma b_1$, i.e., (10) is retrieved and

$$a_2(b_1^2 - 2b_1 + 2) = (\gamma + a_1)b_2.$$

More terms can be computed, but the procedure becomes more convoluted as the accuracy increases. The existence of analytic solutions implies that the delay-inversion can be computed iteratively as a matter of principle.

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