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Partial Network Synchronization Using Diffusive Dynamic Couplings

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Abstract: Partial network synchronization is studied for a class of nonlinear oscillators interconnected through diffusive dynamic couplings. We construct diffusive dynamic couplings combining nonlinear observers and output feedback controllers. Sufficient conditions on the systems to be interconnected, on the network topology, on the observer dynamics, and on the coupling strength that guarantee (global) partial synchronization are derived. The results are illustrated by computer simulations of coupled Hindmarsh-Rose oscillators.

1. INTRODUCTION

Synchronization of dynamical systems has attracted the attention of many researchers over the last decades. One of the first results regarding synchronization of chaotic systems was presented by Fujisaka and Yamada [1983]. In their paper, it is shown that coupled chaotic systems may synchronize in spite of their high sensitivity to initial conditions. After this result, considerable interest in the notion of synchronization of general nonlinear systems has arisen. Several examples of synchronous behavior in nature, science, and engineering can be found in, for instance, Blekhman [1988], Pikovsky et al. [2001], Strogatz [2003], and references therein. In this manuscript, we study a phenomenon called *partial synchronization* or *clustering* in networks of coupled oscillators, i.e., some oscillators in the network do synchronize while others do not. The study of partial synchronization is relevant in many science and engineering practical applications. For instance in Terry et al. [1999], the authors report partial synchronization occurrence in arrays of chaotic semiconductor lasers. Rulkov [1996] studies experimental partial synchronization in networks of chaotic circuits with applications to communication systems. The clustering problem of Josephson junction arrays with applications to high-frequency electromagnetic generators is addressed in Qin and Chen [2004]. Partial synchronization of diffusively coupled oscillators has been investigated in, for instance, Belykh et al. [2000], Yanchuk et al. [2001], Pogromsky et al. [2002], and Pogromsky [2008]. In particular, Pogromsky et al. [2002] and Pogromsky [2008] derive conditions for the existence and stability of partial synchronization modes in networks of nonlinear *semipassive* oscillators with *convergent* internal dynamics. Moreover, the authors show that if a network contains certain symmetries, then these symmetries identify modes of partial synchronization. In these results, it is assumed that the variable z that renders the internal dynamics *convergent*, and for which each system is *strictly semipassive* is used in the diffusive (feedback) coupling. Therefore, if the measurable output is a different output function y, which does not have the desired stability properties, then these results cannot be directly applied. Nevertheless, if the systems are observable, there may exist a (nonlinear) observer, which reconstructs z from measurements of y. If such an observer exists, then a *diffusive dynamic coupling* (DDC), which only depends on the measurable output y could be constructed to interconnect the systems. In Murguia et al. [2013], we have started with the analysis of these ideas for the study of *full network synchronization*. Following the same approach, in this paper, we extend the ideas presented by Pogromsky [2008] to the case where the systems are interconnected through observer-based *diffusive dynamic couplings*. We derive sufficient conditions on the individual systems, on the network topology, on the observer dynamics, and on the coupling strength that guarantee *partial network synchronization*. The remainder of the paper is organized as follows. In Section 2, the notions of *semipassivity*, *convergent systems*, and some basic terminology of graph theory are introduced. The system description and the problem statement are given in Section 3. The observer structure, the proposed diffusive dynamic coupling, and sufficient conditions for boundedness of the closed loop system are introduced in Section 4. In Sections 5 and 6, we present the main result on network partial synchronization. In Section 7, an illustrative example is given. Finally, conclusions are stated in Section 8.

2. PRELIMINARIES

The Euclidian norm in $\mathbb{R}^n$ is denoted simply as $|\cdot|$, $|x|^2 = x^T x$, where ^{T} defines transposition. The notation $col(x_1, ..., x_n)$ stands for the column vector composed of the elements $x_1, ..., x_n$. This notation will be also used in case the components x_i are vectors. The induced norm of a matrix $A \in \mathbb{R}^{n\times n}$, denoted by $\|A\|$, is defined as $\|A\| = \max_{x\in\mathbb{R}^n,|x|=1} |Ax|$. The $n \times n$ identity matrix is denoted by I_n or simply I if no confusion can arise. The spectrum of a matrix A is denoted by $spec(A)$. For any two matrices A and B, the notation $A \otimes B$ (the Kronecker product) stands for the matrix composed of submatrices $A_{ij}B$, where A_{ij}, $i, j = 1, ..., n$, stands for the ijth entry of the $n\times n$ matrix A. Let $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{Y} \subset \mathbb{R}^m$. The space of

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continuous functions from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\mathcal{C}(\mathcal{X},\mathcal{Y})$. If the functions are (at least) $r \geq 0$ times continuously differentiable, then it is denoted by $\mathcal{C}^r(\mathcal{X},\mathcal{Y})$. A function $s : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, is of class $\mathcal{K}$ if it is continuous, strictly increasing and $s(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded.

2.1 Semipassive Systems

Consider the system

$$\dot{x} = f(x,u), \tag{1a}$$
$$z = h(x), \tag{1b}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, output $z \in \mathbb{R}^m$, sufficiently smooth functions $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $h : \mathbb{R}^n \to \mathbb{R}^m$.

Definition 1. Pogromsky et al. [2002]. System (1a),(1b) is called $\mathcal{C}^r$*-semipassive* if there exists a nonnegative storage function $V \in \mathcal{C}^r(\mathbb{R}^n,\mathbb{R}_{\geq 0})$ such that $\dot{V}(x,u) \leq y^T u - H(x)$, where the function $H \in \mathcal{C}(\mathbb{R}^n,\mathbb{R})$ is nonnegative outside some ball, i.e., $\exists\, \varphi > 0$ s.t. $|x| \geq \varphi \to H(x) \geq \varrho(|x|)$, for some continuous nonnegative function $\varrho(\cdot)$ defined for $|x| \geq \varphi$. If $s_1(x_i) \leq V(x_i) \leq s_2(x_i)$ for some $s_1, s_2 \in \mathcal{K}_\infty$ and the function $H(\cdot)$ is positive definite outside some ball, then the system (1a),(1b) is said to be *strictly* $\mathcal{C}^r$*-semipassive*.

2.2 Convergent Systems

Consider the system (1a) and suppose $f(\cdot)$ is Lipschitz in x, $u(\cdot)$ is piecewise continuous in t and takes values in some compact set $u \in U \subseteq \mathbb{R}^m$.

Definition 2. System (1a) is said to be *convergent* if and only if for any bounded signal $u(t)$ defined on the whole interval $(-\infty,+\infty)$ there is a unique bounded globally asymptotically stable solution $\bar{x}_u(t)$ defined in the same interval for which it holds that, $\lim_{t\to\infty} |x(t) - \bar{x}_u(t)| = 0$ for all initial conditions.

Proposition 1. Demidovich [1967] and Pavlov et al. [2004]. If there exists a positive definite symmetric matrix $P \in \mathbb{R}^{n\times n}$ such that all the eigenvalues $\lambda_i(Q)$ of the symmetric matrix

$$Q(x,u) = \frac{1}{2}\left(P\left(\frac{\partial f}{\partial x}(x,u)\right) + \left(\frac{\partial f}{\partial x}(x,u)\right)^T P\right), \tag{2}$$

are negative and separated from zero, i.e., there exists a constant $\delta \in \mathbb{R}_{>0}$ such that $\lambda_i(Q) \leq -\delta < 0$, for all $i \in \{1,...,n\}$, $u \in U$ and $x \in \mathbb{R}^n$, then system (1a) is globally exponentially *convergent*, and there exists a positive definite function $\mathcal{W} \in \mathcal{C}^2(\mathbb{R}^n,\mathbb{R}_{\geq 0})$ satisfying

$$(\nabla\mathcal{W}(x_1 - x_2))^T (f(x_1,u) - f(x_2,u)) \leq -\alpha\, |x_1 - x_2|^2,$$

for some $\alpha \in \mathbb{R}_{>0}$.

2.3 Graph Theory

Given a set of interconnected systems, the communication topology is encoded through a communication graph. The convention is that system i receives information from system j if and only if there is a directed link from node j to node i in the communication graph. Let $\mathcal{G} = (\mathcal{V},\mathcal{E},A)$ denote a weighted digraph (directed graph), where $\mathcal{V} = \{v_1, v_2, ..., v_k\}$ is the set of nodes, $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the set of edges, and A is the weighted adjacency matrix with nonnegative elements a_{ij}. The neighbors of v_i is the set of directed edges to a node v_i and it is denoted as $\mathcal{E}_i$. If the graph does not contain self-loops, it is called simple. Throughout this manuscript, it is assumed that the communication graph is *strongly connected*, i.e., for every two nodes $(i,j) \in \mathcal{V}$, there is at least one path connecting i and j. If two nodes have a directed edge in common, they are called *adjacent*. Assume that the network consists of k nodes, then the *adjacency matrix* $A \in \mathbb{R}^{k\times k} := a_{ij}$ with $a_{ij} > 0$, if $\{i,j\} \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Finally, we introduce the *degree matrix* $D \in \mathbb{R}^{k\times k} := diag\{d_1,...,d_k\}$ with $d_i = \sum_{j\in\mathcal{E}_i} a_{ij}$, and $L := D - A$, which is called the *Laplacian matrix* of the graph $\mathcal{G}$, see Bollobas [1998] for further details.

3. SYSTEM DESCRIPTION

Consider k identical nonlinear systems of the form

$$\dot{x}_i = f(x_i) + Bu_i, \tag{3a}$$
$$y_i = C_1 x_i, \tag{3b}$$
$$z_i = C_2 x_i, \tag{3c}$$

with $i \in \mathcal{I} := \{1,...,k\}$, state $x_i \in \mathbb{R}^n$, input $u_i \in \mathbb{R}^m$, measurable output $y_i \in \mathbb{R}^s$, semipassive output $z_i \in \mathbb{R}^m$, sufficiently smooth function $f : \mathbb{R}^n \to \mathbb{R}^n$, and matrices C_1, C_2, and B of appropriate dimensions. The matrix $C_2B \in \mathbb{R}^{m\times m}$ is assumed to be similar to a positive definite matrix. In addition, it is assumed that, the systems (3a),(3c) are *strictly* $\mathcal{C}^1$*-semipassive* and have relative degree one. A network is called *diffusively coupled* if the systems interact through a coupling of the form

$$u_i = \gamma \sum_{j\in\mathcal{E}_i} a_{ij}(z_j - z_i), \quad i \in \mathcal{I}, \tag{4}$$

where z_j denotes the semipassive outputs of systems j to which system i is connected, $\gamma > 0$ denotes the coupling strength, $a_{ij} \geq 0$ are the weights of the interconnections, and $\mathcal{E}_i$ is the set of neighbors of i. Moreover, since the coupling strength is encompassed in the constant γ, then it can be assumed without loss of generality that $\max_{i\in\mathcal{I}} \sum_{j\in\mathcal{E}_i} a_{ij} = 1$. Notice that the controller (4) can be written in a matrix notation as follows

$$u = -\gamma\,(L \otimes I_m)\, z, \tag{5}$$

where $L \in \mathbb{R}^{k\times k}$ denotes the Laplacian matrix, $z := col(z_1,...,z_k) \in \mathbb{R}^{km}$, and $u := col(u_1,...,u_k) \in \mathbb{R}^{km}$. Define $x := col(x_1,...,x_k) \in \mathbb{R}^{kn}$ and the linear manifold

$$\mathcal{M} := \{x \in \mathbb{R}^{kn} | x_i = x_j, \forall\, i,j \in \mathcal{I}\}.$$

The manifold $\mathcal{M}$ is called the synchronization manifold. The systems (3a),(3c),(4) are said to fully synchronize, or simply synchronize, if the synchronization manifold $\mathcal{M}$ contains an asymptotically stable subset. In a similar manner, consider the linear manifold

$$\mathcal{M}_P := \{x \in \mathbb{R}^{kn} | x_i = x_j, \text{ for some } i,j \in \mathcal{I}\}.$$

The manifold $\mathcal{M}_P$ is called a partial synchronization manifold. The systems (3a),(3c),(4) are said to partially synchronize, if the partial synchronization manifold $\mathcal{M}_P$ is invariant under the closed loop dynamics and contains an asymptotically stable subset.

The result presented here is a direct extension of the results presented by Pogromsky [2008, 2009], where sufficient conditions for partial synchronization in networks of diffusively interconnected semipassive systems are derived. These papers show that if a network contains certain symmetries, then these symmetries identify modes of partial synchronization. It is therefore interesting to extend these results to the case when the variable z_i is not available, but there exists a (nonlinear) observer which reconstructs z_i from the measurable output y_i. If such an observer exists, we can construct a *diffusive dynamic coupling* combining the observer and an estimated version of the *diffusive coupling* (4).

4. DIFFUSIVE DYNAMIC COUPLING

In this section, the structure of the observers that we consider and the *diffusive dynamic coupling* that is used to interconnect the systems are introduced. Moreover, sufficient conditions for boundedness of the closed loop system are presented.

4.1 Nonlinear Observer

Consider the k identical systems (3). Assume that for any initial condition $x_i(t_0) \in \mathbb{R}^n$ and every input signal u_i, the corresponding solution $x(t)$ is well defined and ultimately bounded for all $t \geq t_0$. Consider an observer of the form

$$\begin{aligned} \dot{\eta}_i &= l(\eta_i, y_i, u_i), \\ \hat{z}_i &= \beta(\eta_i, y_i), \end{aligned} \tag{6}$$

with observer state $\eta_i \in \mathbb{R}^p$, $p \geq n - s$, $\hat{z}_i \in \mathbb{R}^m$ denoting the estimated semipassive variable z_i, sufficiently smooth vectorfield $l : \mathbb{R}^p \times \mathbb{R}^s \times \mathbb{R}^m \to \mathbb{R}^p$ and function $\beta : \mathbb{R}^p \times \mathbb{R}^s \to \mathbb{R}^m$. Associated with the observer (6), we have the estimation error $\epsilon_i \in \mathbb{R}^m$ defined as

$$\epsilon_i := \hat{z}_i - z_i = \beta(\eta_i, y_i) - z_i. \tag{7}$$

Then, the estimation error dynamics is given by

$$\begin{aligned} \dot{\epsilon}_i = {} & \frac{\partial \beta(\eta_i, y_i)}{\partial \eta_i} l(\eta_i, y_i, u_i) \\ & + \left(\frac{\partial \beta(\eta_i, y_i)}{\partial y_i} C_1 - C_2 \right) (f(x_i) + Bu_i) . \end{aligned} \tag{8}$$

We assume that $l(\cdot)$ and $\beta(\cdot)$ are designed such that the estimation error dynamics (8) is of the form

$$\dot{\epsilon}_i = \phi(\epsilon_i, x_i), \tag{9}$$

with sufficiently smooth vectorfield $\phi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $\phi(0, x_j) = 0$. Furthermore, it is assumed that $\hat{z}_i$ uniformly asymptotically converge to z_i. This implies that the origin of the estimation error dynamics (9) (or 8) is uniformly asymptotically stable. Then, there may exist a radially unbounded function $V_0 \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}_{\geq 0})$ such that

$$(\nabla V_0(\epsilon_i))^T \phi(\epsilon_i, x_i) \leq -\kappa \left| \epsilon_i \right|^2 , \tag{10}$$

uniformly in $x_i(t)$, for some constant $\kappa \in \mathbb{R}_{>0}$. In general, it is unknown under what conditions on (3a) and (3b) the (nonlinear) observer (6) can be constructed. In this manuscript, it is assumed that the observer exists and the observation error converges uniformly asymptotically to zero. Nevertheless, we forward the interested reader to Nijmeijer and Mareels [1997] and Karagiannis et al. [2008] for existence conditions and interesting design methods of (nonlinear) observers.

4.2 Diffusive Dynamic Coupling

Let the k systems (3a),(3b) interact through a DDC of the form

$$\dot{\eta}_i = l(\eta_i, y_i, u_i), \tag{11a}$$
$$\hat{z}_i = \beta(\eta_i, y_i), \tag{11b}$$
$$u_i = \gamma \sum_{j \in \mathcal{E}_i} a_{ij} \left(\hat{z}_j - \hat{z}_i \right) , \tag{11c}$$

where $\gamma > 0$ denotes the coupling strength and $a_{ij} > 0$ are the weights of the interconnections. The interconnection is not assumed to be symmetric, i.e., a_{ij} is not necessarily equal to a_{ji}. Moreover, since the coupling strength is encompassed in the constant γ, then it can be assumed without loss of generality that $\max_{i \in \mathcal{I}} \sum_{j \in \mathcal{E}_i} a_{ij} = 1$. The dynamic coupling (11) is the combination of the nonlinear observer (6) and an estimated version of the diffusive coupling (4).

4.3 Boundedness of the Interconnected Systems

In this part, we give conditions for boundedness of the closed loop system.

Lemma 1. (Murguia et al. [2013]). Consider k identical systems (3a),(3b) on a simple strongly connected graph interconnected through the DDC (11). Assume that

- **(H4.1)** There exists a nonlinear observer (6) such that the estimation error dynamics (9) is (globally) asymptotically stable with radially unbounded Lyapunov function $V_0 \in \mathcal{C}^1(\mathbb{R}^m, \mathbb{R}_{\geq 0})$ satisfying (10).
- **(H4.2)** Each system (3a),(3c) is strictly $\mathcal{C}^1$-semipassive with radially unbounded storage function and the functions $H(x_i)$ are such that there exists $R > 0$ such that $|x_i| > R$ implies that $H(x_i) - \gamma d_i |z_i|^2 > 0$ with $d_i = \sum_{j \in \mathcal{E}_i} a_{ij}$.

Then, the solutions of the coupled system (3a),(3b),(11) exist for all $t \geq 0$ and are ultimately bounded.

5. SYMMETRIES AND INVARIANT MANIFOLDS

In this section, we extend the ideas presented by Pogromsky [2008] for the identification of partial synchronization modes to the case of *diffusive dynamic couplings*. If a given network possesses certain symmetry, this symmetry must be present in the Laplacian matrix L. In particular, the network may contain some repeated patterns when considering the arrangements of the constants a_{ij} and hence the permutation of some elements would leave the network unchanged. The matrix representation of a permutation of the set $\mathcal{I} = \{1, ..., k\}$ is a permutation matrix $\Pi \in \mathbb{R}^{k \times k}$. In the following lemma, we show that a symmetry in the network defines a linear invariant manifold for the closed loop dynamics.

Lemma 2. Consider a network of k systems (3a),(3b) interconnected through the DDC (11) with Laplacian matrix $L \in \mathbb{R}^{k \times k}$. Let $\Pi \in \mathbb{R}^{k \times k}$ be a permutation matrix. If there is a solution $X \in \mathbb{R}^{k \times k}$ to the matrix equation

$$\left(I_k - \Pi \right) L = X \left(I_k - \Pi \right) , \tag{12}$$

then the set $\ker \left(I_{k(n+m)} - \Pi \otimes I_{n+m} \right)$ defines a linear invariant manifold for the coupled systems (3a),(3b),(11).

Proof: Since $\epsilon_i = \hat{z}_i - z_i$, then the closed loop system (3a),(3b),(11) can be written in term of the estimation error ϵ_i as follows

$$\dot{\epsilon}_i = \phi(\epsilon_i, x_i), \tag{13}$$
$$\dot{x}_i = f(x_i) + \gamma B \sum_{j\in\mathcal{E}_i} a_{ij}\left(z_j - z_i + \epsilon_j - \epsilon_i\right), \tag{14}$$

with $\phi(\cdot)$ from (9). Introduce the new set of variables: the stacked state $\xi := col(\epsilon_1, x_1, ..., \epsilon_k, x_k)$, $\Xi := \Pi \otimes I_{n+m}$, $F(\xi) := col(\phi(\epsilon_1, x_1), f(x_1), ..., \phi(\epsilon_i, x_i), f(x_i))$, and

$$\tilde{B} = \begin{pmatrix} 0 & 0 \\ B & BC_2 \end{pmatrix}. \tag{15}$$

Then, using the new notation, the stacked closed loop system (3a),(3b),(11) can be written as

$$\dot{\xi} = F(\xi) - \gamma(L \otimes \tilde{B})\xi. \tag{16}$$

Assume that at some time t^* the extended state $\xi(t^*)$ satisfies $(I_{k(n+m)} - \Xi)\xi(t^*) = 0$. Then, the set $\ker\left(I_{k(n+m)} - \Xi\right)$ is invariant under the closed loop dynamics if the assumption of $(I_{k(n+m)} - \Xi)\xi(t^*) = 0$ implies $(I_{k(n+m)} - \Xi)\dot{\xi}(t^*) = 0$. Consider (16) and the solution X of the matrix equation (12). Since Π is a permutation matrix, it follows that $\Xi F(\xi(t^*)) = F(\Xi\xi(t^*))$, then

$$\begin{aligned}(I_{k(n+m)} - \Xi)\dot{\xi}(t^*) &= (I_{k(n+m)} - \Xi)F(\xi(t^*)) \\ &\quad - \gamma(I_{k(n+m)} - \Xi)(L \otimes \tilde{B})\xi(t^*) \\ &= F(\xi(t^*)) - F(\Xi\xi(t^*)) \\ &\quad - \gamma(X \otimes \tilde{B})(I_{k(n+m)} - \Xi)\xi(t^*) = 0,\end{aligned}$$

because it is assumed that $(I_{k(n+m)} - \Xi)\xi(t^*) = 0$. It follows that $(I_{k(n+m)} - \Xi)\xi(t) = 0$ for all $t \geq t^*$ and therefore the set $\ker\left(I_{k(n+m)} - \Pi \otimes I_{n+m}\right)$ defines a linear invariant manifold for the interconnected systems (3a),(3b),(11). ∎

6. PARTIAL SYNCHRONIZATION

In the previous section, conditions for the existence of linear invariant manifolds are presented. For partial synchronization to occur, we require these manifolds to contain an asymptotically stable subset. In this section, we present sufficient conditions for a linear invariant manifold to contain an asymptotically stable subset. Consider the k systems (3). Since it is assumed that the systems have relative degree one and the matrix C_2B is similar to a positive definite matrix, then it can be shown that there always exists a globally defined coordinate transformation such that systems (3a),(3c) can be written in the following *normal form*

$$\dot{\zeta}_i = q(\zeta_i, z_i), \tag{17}$$
$$\dot{z}_i = a(\zeta_i, z_i) + C_2Bu_i, \tag{18}$$

with $\zeta_i \in \mathbb{R}^{n-m}$ the state of the internal dynamics, sufficiently smooth vectorfields $q : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^{n-m}$, and $a : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^m$. For the sake of simplicity, it is assumed that $C_2B = I_m$. In the following theorem, we give sufficient conditions for partial synchronization to occur in the closed loop system.

Theorem 1. Consider k identical systems (3a),(3b) interconnected through the *diffusive dynamic coupling* (11) on a simple strongly connected graph. Suppose the conditions of *Lemma 1* and *Lemma 2* are satisfied for some matrix X and permutation matrix Π. In addition assume that

(H6.1) There is a constant $\lambda' > 0$ such that

$$\frac{1}{2}\vartheta^T\left(I - \Pi\right)^T\left(X + X^T\right)\left(I - \Pi\right)\vartheta \geq \lambda' \left|\left(I - \Pi\right)\vartheta\right|^2.$$

(H6.2) There exists a positive definite matrix $\mathcal{P} = \mathcal{P}^T$ such that the eigenvalues of the symmetric matrix

$$\mathcal{P}\left(\frac{\partial q}{\partial \zeta_i}(\zeta_i, z_i)\right) + \left(\frac{\partial q}{\partial \zeta_i}(\zeta_i, z_i)\right)^T \mathcal{P},$$

with $q(\cdot)$ from (17), are strictly negative and bounded away from zero for all $\zeta_i \in \mathbb{R}^{(n-m)}$ and $z_i \in \mathbb{R}^m$.

Then, there exist positive constants γ' and κ' such that if $\gamma > \gamma'$ and $\kappa > \kappa'$ with κ from (10), then the set $\ker\left(I_{k(n+m)} - \Pi \otimes I_{n+m}\right)$ contains a globally asymptotically stable subset.

A sketch of the proof of *Theorem 1* is presented in the appendix. Notice that if the matrices L and Π commute, then the matrix equation (12) admits a solution $X = L$. The problem of finding a $\lambda' \in \mathbb{R}_{>0}$ satisfying (H6.1) can be solved via singular value decomposition, see Pogromsky [2009]. Moreover, if $X + X^T$ commutes with Π, then λ' is the minimal eigenvalue of $\frac{1}{2}(X + X^T)$ under the restriction that the eigenvectors of $\frac{1}{2}(X + X^T)$ are taken from the set $\text{range}(I_k - \Pi)$.

7. EXAMPLE

A. Convergence and Semipassivity. Consider k identical Hindmarsh-Rose oscillators of the form

$$\dot{\zeta}_{1i} = 0.005\left(4\left(z_i + 1.618\right) - \zeta_{1i}\right), \tag{19a}$$
$$\dot{\zeta}_{2i} = -2z_i - z_i^2 - \zeta_{2i}, \tag{19b}$$
$$\dot{z}_i = -z_i^3 + 3z_i - 4.75 + 5\zeta_{2i} - \zeta_{1i} + u_i, \tag{19c}$$
$$y_i = \left(\,\zeta_{1i},\ \zeta_{2i}\,\right)^T. \tag{19d}$$

where y_i is the measurable output, state $x_i = (\zeta_{1i}, \zeta_{2i}, z_i)^T \in \mathbb{R}^3$, input $u_i \in \mathbb{R}$, and $i \in \mathcal{I} = \{1, 2, 3, 4\}$. It is shown in Neefs et al. [2010] that the system (19a)-(19c) is *strictly* $\mathcal{C}^1$-*semipassive* with input u_i, output z_i and storage function $V(x_i) = \frac{1}{2}(\frac{1}{0.005\cdot 4}\zeta_{1i}^2 + \mu\zeta_{2i}^2 + z_i^2)$ for some $\mu > 0$. Moreover, the corresponding $H(x_i)$ satisfies (H4.2) for arbitrarily large γ. Assumption (H6.2) is satisfied with $\mathcal{P} = I_2$, i.e., the internal dynamics (19a),(19b) is *convergent*. At this point, *Theorem 1* in Pogromsky [2008] could be applied to conclude that the network of coupled Hindmarsh-Rose systems may exhibit partial synchronization. However, the variable z_i is not available for feedback. The coupling variable is the measurable output y_i; therefore, the results in Pogromsky [2008] can not be used. Nevertheless, if there exists a nonlinear observer which estimates z_i from y_i, then the DDC (11) could be constructed and therefore *Theorem 1* may be used to study partial synchronization.

B. Nonlinear Observer.

Proposition 2. Consider k systems of the form

$$\dot{\eta}_i = -0.005\kappa_1\left(\frac{4}{\kappa_2}\beta + 4.472 - y_{1i}\right) \quad (20)$$
$$+ \kappa_2\left(-\frac{\beta^3}{\kappa_2^3} + \frac{3\beta}{\kappa_2} - 4.75 + 5y_{2i} - y_{1i} + u_i\right),$$
$$\hat{z}_i = \frac{1}{\kappa_2}\beta(y_i, \eta_i), \quad (21)$$

with state $\eta_i \in \mathbb{R}$, and function $\beta(y_i, \eta_i) = \kappa_1 y_{1i} + \eta_i$. Then, there exist positive constants $\kappa_1, \kappa_2 \in \mathbb{R}_{>0}$ such that for all initial conditions $\eta_i(t_0) \in \mathbb{R}$ the following holds $\lim_{t\to\infty}(\kappa_1 y_{1i} + \eta_i - \kappa_2 z_i) = 0$.

Proof: Define the estimation error $\epsilon_i := \kappa_1 y_{1i} + \eta_i - \kappa_2 z_i$. Then, the estimation error dynamics is given by

$$\dot{\epsilon}_i = -\left(\frac{\kappa_1}{5\kappa_2} - 3 + \frac{1}{\kappa_2}\left(\epsilon_i + \frac{3\kappa_2}{2}z_i\right)^2 + \frac{3\kappa_2}{4}z_i^2\right)\epsilon_i. \quad (22)$$

Consider the positive definite Lyapunov function $V_0 = \frac{1}{2}\epsilon_i^2$, then $\dot{V}_0 \leq -(\frac{\kappa_1}{5\kappa_2} - 3)\epsilon_i^2$. Therefore, for $\kappa_1 > 15\kappa_2$, the origin of (22) is globally uniformly asymptotically stable.

C. Dynamic Diffusive Coupling. Combining the observer (20),(21) and an estimated version of (4), the dynamic coupling (11c) is then given by

$$u_i = \frac{\gamma}{\kappa_2}\sum_{j\in\mathcal{E}_i} a_{ij}\left(\kappa_1 y_{1j} - \kappa_1 y_{1i} + \eta_j - \eta_i\right). \quad (23)$$

D. Network Topology. Consider a network of four bidirectionally coupled systems in ring configuration, see also Steur et al. [2012]. The network is strongly connected and simple. The associated Laplacian matrix is given by

$$L = \frac{1}{3}\begin{pmatrix} 3 & -1 & 0 & -2 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 3 & -1 \\ -2 & 0 & -1 & 3 \end{pmatrix}. \quad (24)$$

Note that the above Laplacian commutes with the following permutation matrices

$$\Pi_1 = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}, \Pi_2 = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix}, \Pi_3 = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix},$$

i.e., $L\Pi_i = \Pi_i L$ for all $i \in \{1,2,3\}$, and hence $X = L$ is a solution to the equations $(I_4 - \Pi_i)L = X(I_4 - \Pi_i)$. It follows that the sets $\ker(I_4 - \Pi_i \otimes I_4)$ define linear invariant manifolds for the closed loop system. Moreover, since L is symmetric and L and Π_i commute, then λ' in (H6.1) can be estimated as the minimal eigenvalue of L under the restriction that the eigenvectors of L are taken from the set $\mathrm{range}(I_k - \Pi_i)$, see Pogromsky et al. [2002]. Let λ_i be an eigenvalue of L and μ_i the corresponding eigenvector, then $\lambda_1 = 0, \lambda_2 = \frac{2}{3}, \lambda_3 = \frac{4}{3}, \lambda_4 = 2$, and

$$\mu_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \mu_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \mu_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \mu_4 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}.$$

Then, after some straightforward computations, it follows that $\lambda' = \lambda_2$ for both Π_1 and Π_2, and $\lambda' = \lambda_3$ for Π_3. Therefore, all the assumptions stated in *Theorem 1* are satisfied, and it can be concluded that for sufficiently large γ, κ_1, and κ_2 the sets $\ker(I_4 - \Pi_i \otimes I_4)$ contain globally asymptotically stable subsets. Note that the conditions for partial synchronization of Π_1 and Π_2 are the same. It may be that multiple partial synchronization manifolds coexist and also their conditions for being stable might coincide. It follows that to observe partial synchronization, it is necessary that the values of γ, κ_1, and κ_2 for which a partial manifold is stable do not coincide with those for which the full synchronization manifold is stable. Particularly, in this example, the only partial synchronization manifold that can be observed is the corresponding to the set $\ker(I_4 - \Pi_3 \otimes I_4)$, i.e., $x_1 = x_4 \neq x_2 = x_3$.

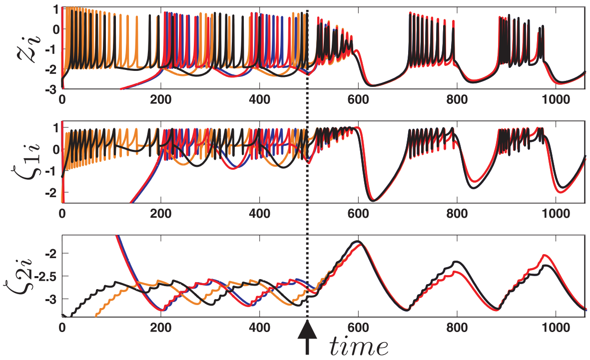

Figure 1. States responses. The controller is turned-on at $time = 500$[ms].

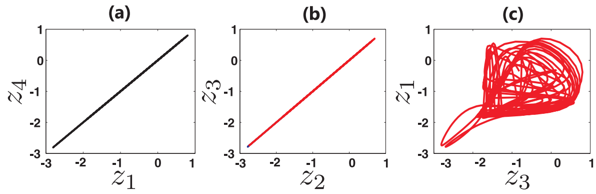

Figure 2. (a) Synchronization of neurons 1 and 4. (b) Synchronization of neurons 2 and 3. (c) No synchronization between neurons 1 and 3.

E. Numerical Results. Figure 2 and Figure 3 depict simulation results of the network of four Hindmarsh-Rose oscillators with coupling constant $\gamma = 1$, $\kappa_1 = 10$, and $\kappa_2 = 0.07$. In Figure 2, the top panel shows the z_i states of the four oscillators and the ζ_{1i} and ζ_{2i} states are depicted in the bottom ones. The controller is turned on at $time = 500$[ms].

8. CONCLUSION

We have presented a methodology for studying the emergence of partial network synchronization for a class of nonlinear oscillators interconnected through observer-based *diffusive dynamic couplings*. It has been shown that symmetries in the network define linear invariant manifolds, which, when being attracting, define modes of partial synchronization. Sufficient conditions on the systems to be interconnected, on the network topology, on the observer dynamics, and on the coupling strength that guarantee (global) partial synchronization have been derived.

REFERENCES

N. Belykh, V. Belykh, and M. Hasler. Hierarchy and stability of partially synchronous oscillations of diffusively coupled dynamical systems. *Physical Review E*, 62:6332–6345, 2000.

I. Blekhman. *Synchronization in science and technology.* ASME, New York, 1988.

B. Bollobas. *Modern Graph Theory.* Springer-Verlag, New York, 1998.

B.P. Demidovich. *Lectures on Stability Theory.* Moscow. In Russian, 1967.

H. Fujisaka and T. Yamada. Stability theory of synchronized motion in coupled-oscillator systems. *Progress of Theoretical Physics*, 69:32–37, 1983.

D. Karagiannis, D. Carnevale, and A. Astolfi. Invariant manifold based reduced-order observer design for nonlinear systems. *IEEE Trans. Automat. Contr*, 53:2602–2614, 2008.

C. Murguia, R.H.B. Fey, and H. Nijmeijer. Network synchronization by dynamic diffusive coupling. *Int. J. Bifurcation Chaos*, 4:1350076–1350088, 2013.

P. Neefs, E. Steur, and H. Nijmeijer. Network complexity and synchronous behavior an experimental approach. *Int. J. of Neural Systems*, 20:233–247, 2010.

H. Nijmeijer and M. Mareels. An observer looks at synchronization. *IEEE Transactions on Circuit and Systems-I*, 44:295–319, 1997.

A. Pavlov, A. Pogromsky, N. van de Wouw, and H. Nijmeijer. Convergent dynamics, a tribute to Boris Pavlovich Demidovich. *Syst. Control Lett.*, 52:257, 2004.

A. Pikovsky, M. Rosenblum, and J. Kurths. *Synchronization: A universal concept in nonlinear Science.* Cambridge University Press, Cambridge, 2001.

A. Pogromsky. A partial synchronization theorem. *Chaos*, 18:037107, 2008.

A. Pogromsky. Erratum: a partial synchronization theorem, chaos 18, 037107 (2008). *Chaos*, 19:049901, 2009.

A. Pogromsky, G. Santoboni, and H. Nijmeijer. Partial synchronization: From symmetry towards stability. *Physica D*, 172:65–87, 2002.

W. Qin and G. Chen. Coupling schemes for cluster synchronization in coupled josephson equations. *Physica D*, 197:375–391, 2004.

N.F. Rulkov. Images of synchronized chaos: Experiments with circuits. *Chaos*, 6, 1996.

E. Steur, T. Oguchi, C. van Leeuwen, and H. Nijmeijer. Partial synchronization in diffusively time-delay coupled oscillator networks. *Chaos*, 22:043144, 2012.

S. Strogatz. *SYNC. The emerging science of spontaneous Order.* Hyperion, New York, 2003.

J.R. Terry, K.S. Thornburg, D.J. DeShazer, G.D. VanWiggeren, S. Zhu, P. Ashwin, and R. Roy. Synchronization of chaos in an array of three lasers. *Physical Review E*, 59:4036–4043, 1999.

S. Yanchuk, Y. Maistrenko, and E. Mosekilde. Partial synchronization and clustering in a system of diffusively coupled chaotic oscillators. *Mathematics and Computers in Simulation*, 54:491–508, 2001.

Appendix A. SKETCH OF PROOF OF THEOREM 1

If there exists a solution X of the matrix equation $(I_k - \Pi)\, L = X\,(I_k - \Pi)$ for a given permutation matrix Π, then the set $\ker\left(I_{k(n+m)} - \Pi \otimes I_{n+m}\right)$ defines a linear invariant manifold for the coupled systems (3a),(3b),(11), see *Lemma 2.* Define the stacked estimation error $\epsilon := col(\epsilon_1, ..., \epsilon_k) \in \mathbb{R}^{km}$, semipassive output $z := col(z_1, ..., z_k) \in \mathbb{R}^{km}$, and internal state $\zeta := col(\zeta_1, ..., \zeta_k) \in \mathbb{R}^{k(n-m)}$. Note that $\epsilon \in \ker\left(I_{km} - \Pi \otimes I_m\right)$, $z \in \ker\left(I_{km} - \Pi \otimes I_m\right)$, and $\zeta \in \ker\left(I_{k(n-m)} - \Pi \otimes I_{n-m}\right)$ define equations of the form

$$\epsilon_i - \epsilon_j = 0,\ z_i - z_j = 0,\ \zeta_i - \zeta_j = 0, \qquad \text{(A.1)}$$

for some $i, j \in \mathcal{I}$. Let $\mathcal{I}_\Pi$ be the set of pairs (i, j) for which (A.1) holds. We want to show that $z_i - z_j = 0$, $\zeta_i - \zeta_j = 0$, and $\epsilon_i = 0$ restricted to $\mathcal{I}_\Pi$ are globally asymptotically stable under the conditions supplied in the *Theorem 1.* Note that (H6.2) implies that the internal dynamics (17) are *convergent.* Assumptions (H4.1) and (H6.2), smoothness of the vectorfields, and boundedness of the closed loop system imply the existence of positive definite functions $V_1(\epsilon)$ and $V_2(\zeta)$ such that

$$\dot{V}_1(\epsilon, x) \le -\kappa c_0 \sum_{(i,j)\in\mathcal{I}_\Pi} |\epsilon_i - \epsilon_j|^2 \ \ \forall\ \ \epsilon, x,$$

$$\dot{V}_2(\zeta, z) \le \sum_{(i,j)\in\Pi} \left(-\alpha|\zeta_i - \zeta_j|^2 + c_1|\zeta_i - \zeta_j||z_i - z_j|\right),$$

for some constants $c_0, c_1, \alpha \in \mathbb{R}_{>0}$ and κ as in (10). Let

$$V_3(z) = \frac{1}{2}\left|\left(I_{km} - \Pi \otimes I_m\right) z\right|^2 = \frac{1}{2}\sum_{(i,j)\in\mathcal{I}_\Pi} |z_i - z_j|^2.$$

Then, using the assumption that there exists X such that $(I - \Pi)L = X(I - \Pi)$, the time derivative of V_3 along the trajectories of the closed loop system is given as

$$\begin{aligned}\dot{V}_3(\zeta, z, \epsilon) = &\sum_{(i,j)\in\mathcal{I}_\Pi} |z_i - z_j|^T\left(a(\zeta_i, z_i) - a(\zeta_j, z_j)\right)\\ &- \frac{\gamma}{2} z^T (I_{km} - \Xi)^T ((X + X^T) \otimes I_m)(I_{km} - \Xi) z\\ &- \frac{\gamma}{2} z^T (I_{km} - \Xi)^T ((X + X^T) \otimes I_m)(I_{km} - \Xi)\epsilon,\end{aligned}$$

with $\Xi := \Pi \otimes I_m$. Again, using smoothness of the vectorfields, ultimate boundedness of the solutions, and assumption (H6.1) it follows that

$$\begin{aligned}\dot{V}_3 \le\ &(c_2 - \gamma\lambda') \sum_{(i,j)\in\mathcal{I}_\Pi} |z_i - z_j|^2\\ &+ \sum_{(i,j)\in\mathcal{I}_\Pi} \left(c_3|z_i - z_j||\zeta_i - \zeta_j| + \gamma\bar{\lambda}|z_i - z_j||\epsilon_i - \epsilon_j|\right),\end{aligned}$$

for some constants $c_2, c_3 \in \mathbb{R}_{>0}$, $\bar{\lambda} > 0$ being the largest eigenvalue of the symmetric matrix $\frac{1}{2}(X + X^T)$, and λ' the largest number such that (H6.1) holds. Let $\mathcal{V} = V_1 + V_2 + V_3$, then from the previous results, it follows that

$$\begin{aligned}\dot{\mathcal{V}} \le &\sum_{(i,j)\in\mathcal{I}_\Pi} ((c_2 - \gamma\lambda')|z_i - z_j|^2 - \kappa c_0|\epsilon_i - \epsilon_j|^2 \qquad \text{(A.2)}\\ &- \alpha|\zeta_i - \zeta_j|^2 + (c_1 + c_{3)})|\zeta_i - \zeta_j||z_i - z_j|\\ &+ \gamma\bar{\lambda}|z_i - z_j||\epsilon_i - \epsilon_j|).\end{aligned}$$

straightforward computations show that

$$\gamma > \gamma' := \frac{1}{\lambda'}\left(\frac{(c_1 + c_3)^2}{4\alpha} + c_2\right), \qquad \text{(A.3)}$$

$$\kappa > \kappa' := \frac{\bar{\lambda}^2}{4c_0\lambda'}\left(\frac{\gamma^2}{\gamma - \gamma'}\right), \qquad \text{(A.4)}$$

implies $\dot{\mathcal{V}} \le -\sigma \sum_{(i,j)\in\mathcal{I}_\Pi}(|z_i - z_j|^2 + |\zeta_i - \zeta_j|^2 + |\epsilon_i - \epsilon_j|^2)$, for some positive constant σ. Hence, it can be concluded that $\dot{\mathcal{V}}$ is negative definite for $\gamma > \gamma'$ and $\kappa > \kappa'$, and therefore the set $\ker\left(I_{k(n+m)} - \Pi \otimes I_{n+m}\right)$ contains a globally asymptotically stable subset. ■