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Robust Performance Analysis of Affine Single Parameter-dependent Systems with Polynomially Parameter-dependent Lyapunov Matrices ⋆

Mahdieh Sadat Sadabadi and Alireza Karimi

Automatic Control Laboratory, Ecole Polytechnique Fédérale de Lausanne (EPFL), Lausanne, Switzerland
(e-mails: mahdieh.sadabadi@epfl.ch and alireza.karimi@epfl.ch).

Abstract: This paper addresses the problem of H_∞ and H_2 performance analysis of continuous-time affine single parameter-dependent systems with polynomially parameter-dependent Lyapunov matrices. First, some necessary and sufficient conditions in terms of linear matrix inequalities (LMIs) are provided using (D, G) scaling approach. Then, the results are used to deal with the problem of fixed-order H_∞ and H_2 controller design via bilinear matrix inequalities (BMIs) in a nonconservative way. Simulation results demonstrate the effectiveness of the proposed approach.

1. INTRODUCTION

Robust performance analysis of uncertain linear time invariant (LTI) systems using linear matrix inequalities (LMIs) is a challenging issue in the robust control theory. The starting point for dealing with this problem is based on the concept of quadratic stability which considerably suffers from the conservatism imposed by the use of fixed Lyapunov matrix (e.g. Palhares et al. [1997]). To improve the results and reduce the conservatism, special structures for the Lyapunov matrices have been considered, such as affine linear parameter-dependent (Feron et al. [1996], Peaucelle et al. [2000], de Oliveira et al. [2004a,b], Ebihara and Hagiwara [2006]) and more recently, (homogeneous) polynomially parameter-dependent Lyapunov matrices (Chesi et al. [2005a,b], Oliveira and Peres [2005, 2007], Ebihara et al. [2009], Zhang et al. [2010]).

In Ebihara and Hagiwara [2006] and Ebihara et al. [2009], necessary and sufficient conditions in terms of LMIs for robust Hurwitz stability analysis of affine single parameter-dependent matrices with polynomially parameter-dependent Lyapunov matrices have been proposed. Then, their results have been extended to deal with the affine multiple parameter-dependent systems and some sufficient conditions for the existence of the linear parameter-dependent Lyapunov matrices have been developed. The proposed results of Ebihara et al. [2009] are applicable to H_∞ performance analysis problem of systems with an affine single parameter-dependent state matrix A and parameter-independent matrices (B, C).

In this paper, the problem of H_∞ and H_2 performance analysis of continuous-time LTI affine single parameter-dependent systems is formulated as a set of LMIs by means of polynomially parameter-dependent Lyapunov matrices. The proposed approach is based on (D, G) scaling (Meinsma and Fu [1997], Ebihara [2008], Zhang et al. [2010]) which can convert inequality conditions depending on an uncertain parameter, being hence a feasibility problem of infinite dimension, to finite-dimensional LMIs. The obtained results can be used to provide some necessary and sufficient conditions in terms of Bilinear Matrix Inequalities (BMIs) for fixed-order controller design of affine single parameter-dependent systems. To the best of our knowledge, this is the first time that such necessary and sufficient conditions for fixed-order control synthesis have been developed. The contributions of this work with respect to Ebihara et al. [2009] are the extension of the results to

- H_∞ performance analysis in the case of the affine single parameter-dependent uncertainty in all matrices A, B, and C;
- H_2 performance analysis;
- fixed-order controller design: a BMI-based approach

The organization of the paper is as follows: Section 2 presents some preliminaries and the problem statement. Sections 3 and 4 are devoted to the main results. The problem of fixed-order controller synthesis is presented in Section 5. Simulation examples are given in Section 6. The paper ends with concluding remarks in Section 7.

The notation used in this paper is standard. In particular, I_n and $0_{n\times 1}$ are the $n \times n$ identity matrix and the zero vector of dimension n, respectively. The symbols $trace(A)$, $\otimes$, and $A(i : j, :)$ denote the trace of matrix A, the Kronecker product, and the extraction of the i^{th} through the j^{th} row of matrix A, respectively. The symbol $\mathrm{He}\{A\}$ is a notation for $A + A^T$. For the simplicity of presentation, symbol $\star$ indicates the symmetric blocks.

2. PRELIMINARIES

Consider a continuous-time LTI affine single parameter-dependent system as follows:

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$$\begin{aligned} \dot{x} &= A(\theta)x + B(\theta)u \\ y &= C(\theta)x \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$. The real matrices A, B, and C are of appropriate dimensions. It is assumed that the triplet (A, B, C) belongs to the following uncertainty domain:

$$\begin{aligned} A(\theta) &= A_0 + \theta A_1 \\ B(\theta) &= B_0 + \theta B_1 \\ C(\theta) &= C_0 + \theta C_1 \end{aligned} \tag{2}$$

where $\theta \in [-1, 1]$, without loss of generality. The assumptions about the system are as follows:

- Matrix $A(\theta)$ is robustly stable, its eigenvalues lie in the open left-half plane for all $\theta \in [-1, 1]$.
- The system is strictly proper which is a reasonable assumption since all physical systems are strictly proper. Moreover, it is a necessary assumption to have a bounded H_2 norm.

The transfer matrix from the input u to the output y is given by:

$$H(s, \theta) = C(\theta)(sI - A(\theta))^{-1} B(\theta) \tag{3}$$

Lemmas 1 and 2 provide some necessary and sufficient conditions for the satisfaction of H_∞ and H_2 norm bounds of transfer matrix $H(s, \theta)$ given in (3).

Lemma 1. (Bounded Real Lemma for continuous-time systems) The inequality $\|H(s, \theta)\|_\infty < \gamma$ holds for all $\theta \in [-1, 1]$ if and only if there exists a symmetric matrix $P(\theta) > 0$ such that (Boyd et al. [1994]):

$$\begin{aligned} A(\theta)^T P(\theta) + P(\theta)A(\theta) + \gamma^{-1} C(\theta)^T C(\theta) \\ + \gamma^{-1} P(\theta)B(\theta)B(\theta)^T P(\theta) < 0 \end{aligned} \tag{4}$$

Lemma 2. The inequality $\|H(s, \theta)\|_2^2 < \upsilon$ holds for all $\theta \in [-1, 1]$ if and only if there exist symmetric matrices $P(\theta) > 0$ and $Q(\theta) > 0$ such that (Boyd et al. [1994]):

$$A(\theta)^T P(\theta) + P(\theta)A(\theta) + P(\theta)B(\theta)B(\theta)^T P(\theta) < 0 \tag{5}$$

$$\begin{aligned} \begin{bmatrix} P(\theta) & C(\theta)^T \\ C(\theta) & Q(\theta) \end{bmatrix} > 0 \\ trace(Q(\theta)) - \upsilon < 0 \end{aligned} \tag{6}$$

It should be noted that $P(\theta)$ and $Q(\theta)$ in Lemma 1 and Lemma 2 do not have special structures.

Lemma 3. ((D, G) Scaling) Let $\Phi \in \mathbb{R}^{n(k+1) \times n(k+1)}$. Then, the following matrix inequality (Zhang et al. [2010]):

$$(\theta^{[k]} \otimes I_n)^T \Phi (\theta^{[k]} \otimes I_n) < 0 \tag{7}$$

where $\theta^{[k]} = \begin{bmatrix} 1 & \theta & \theta^2 & \cdots & \theta^k \end{bmatrix}^T$, holds for all $\theta \in [-1, 1]$ if and only if there exist a positive-definite matrix $D \in \mathbb{R}^{nk \times nk}$ and a real skew-symmetric matrix $G \in \mathbb{R}^{nk \times nk}$ such that:

$$\Phi + \Delta_k^n(D, G) < 0 \tag{8}$$

where $\Delta_k^n(D, G) \in \mathbb{R}^{n(k+1) \times n(k+1)}$ is defined as follows:

$$\Delta_k^n(D, G) = \begin{bmatrix} \bar{I}_k^n \\ \tilde{I}_k^n \end{bmatrix}^T \begin{bmatrix} D & G \\ G^T & -D \end{bmatrix} \begin{bmatrix} \bar{I}_k^n \\ \tilde{I}_k^n \end{bmatrix}; \quad k \neq 0 \tag{9}$$

where

$$\begin{aligned} \bar{I}_k^n &= [\, I_k \quad 0_{k \times 1} \,] \otimes I_n \\ \tilde{I}_k^n &= [\, 0_{k \times 1} \quad I_k \,] \otimes I_n \end{aligned} \tag{10}$$

and $\Delta_0^n(D, G) = 0$. The inequality in (8) is an LMI with respect to the matrices Φ, D, and G.

3. MAIN RESULTS

In this section, some necessary and sufficient conditions in terms of LMIs for the existence of a linearly parameter-dependent Lyapunov matrix $P(\theta)$ and $Q(\theta)$ given by:

$$P(\theta) = P_0 + \theta P_1 \tag{11}$$

$$Q(\theta) = Q_0 + \theta Q_1 \tag{12}$$

which satisfy (4), (5), and (6), are provided. The main idea behind of these conditions is (D, G) scaling approach which can be used to convert the positivity (negativity) of a polynomial matrix in the following form into the feasibility problem of some LMIs independent on θ.

$$J(\theta) = \sum_{k=0}^{N} \theta^k J_k, \quad \theta \in [-1, 1] \tag{13}$$

where J_k $(k = 0, 1, .., N)$ are Hermitian matrices.

3.1 H_∞ Performance Analysis by Means of Linearly Parameter-dependent Lyapunov Matrices

In this part, the problem of H_∞ performance analysis via linearly parameter-dependent Lyapunov matrices is proposed and the results are given in the following theorem.

Theorem 1. For an affine single parameter-dependent system, the inequality given in (4) holds with $P(\theta)$ in (11) if and only if there exist symmetric matrices P_0 and P_1, $D > 0$, and a real skew-symmetric matrix G of appropriate dimensions such that:

$$\begin{aligned} P_0 + P_1 > 0 \\ P_0 - P_1 > 0 \end{aligned} \tag{14}$$

and

$$\begin{bmatrix} W_1 + \Delta_2^n(D, G) & \star \\ Y_1 & -\gamma I_{m+p} \end{bmatrix} < 0 \tag{15}$$

where

$$W_1 = \mathrm{He} \left\{ \begin{bmatrix} P_0 \\ P_1 \\ 0 \end{bmatrix} [\, A_0 \quad A_1 \quad 0 \,] \right\} \tag{16}$$

$$Y_1 = \begin{bmatrix} B_0^T P_0 & B_1^T P_0 + B_0^T P_1 & B_1^T P_1 \\ C_0 & C_1 & 0 \end{bmatrix} \tag{17}$$

and $\Delta_2^n(D, G)$ is given in (9) with $k = 2$.

The inequality (15) is an LMI with respect to γ and the matrices P_0, P_1, D, and G.

Proof: First, the inequalities in (14) imply that Lyapunov matrix $P(\theta)$ in (11) is positive definite. Then, the inequality given in (4) can be rewritten as follows:

$$\begin{bmatrix} I_n \\ \theta I_n \\ \theta^2 I_n \end{bmatrix}^T \Phi_\infty \begin{bmatrix} I_n \\ \theta I_n \\ \theta^2 I_n \end{bmatrix} < 0 \tag{18}$$

where

$$\begin{aligned} \Phi_\infty =& \mathrm{He} \left\{ \begin{bmatrix} P_0 \\ P_1 \\ 0 \end{bmatrix} [\, A_0 \ A_1 \ 0 \,] \right\} + \gamma^{-1} \begin{bmatrix} C_0^T \\ C_1^T \\ 0 \end{bmatrix} [\, C_0 \ C_1 \ 0 \,] \\ &+ \gamma^{-1} \begin{bmatrix} P_0 B_0 \\ P_0 B_1 + P_1 B_0 \\ P_1 B_1 \end{bmatrix} \begin{bmatrix} P_0 B_0 \\ P_0 B_1 + P_1 B_0 \\ P_1 B_1 \end{bmatrix}^T \end{aligned} \tag{19}$$

Based on (D, G) scaling, (18) is equivalent to the following inequality:

$$\Phi_\infty + \Delta_2^n(D, G) < 0 \tag{20}$$

The last term in the inequality (19) contains the product of unknown Lyapunov matrices; therefore, the inequality (20) is not an LMI. Schur Complement lemma (A. Albert [1969]) can be applied on (19) to convert it to LMI in (15). □

3.2 H_2 Performance Analysis by Means of Linearly Parameter-dependent Lyapunov Matrices

The results of H_2 performance analysis with linearly parameter-dependent Lyapunov matrices are presented in Theorem 2.

Theorem 2. For an affine single parameter-dependent system, the inequalities in (5) and (6) hold with $P(\theta)$ and $Q(\theta)$ given in (11) and (12), respectively if and only if there exist symmetric matrices P_0, P_1, Q_0 and Q_1, a positive definite matrix D, and a real skew-symmetric matrix G of appropriate dimensions such that:

$$\begin{bmatrix} W_1 + \Delta_2^n(D, G) & \star \\ Y_1(1:m,:) & -I_m \end{bmatrix} < 0 \tag{21}$$

$$\begin{bmatrix} P_0 + P_1 & C_0^T + C_1^T \\ C_0 + C_1 & Q_0 + Q_1 \end{bmatrix} > 0 \tag{22}$$

$$\begin{bmatrix} P_0 - P_1 & C_0^T - C_1^T \\ C_0 - C_1 & Q_0 - Q_1 \end{bmatrix} > 0 \tag{23}$$

$$\begin{aligned} trace(Q_0 + Q_1) < \upsilon \\ trace(Q_0 - Q_1) < \upsilon \end{aligned} \tag{24}$$

where W_1 and Y_1 are defined in (16) and (17), respectively.

The above inequalities are LMIs with respect to υ and the matrices P_0, P_1, Q_0, Q_1, D, and G.

Proof: The inequality given in (5) can be rewritten as follows:

$$\begin{bmatrix} I_n \\ \theta I_n \\ \theta^2 I_n \end{bmatrix}^T \Phi_2 \begin{bmatrix} I_n \\ \theta I_n \\ \theta^2 I_n \end{bmatrix} < 0 \tag{25}$$

where

$$\begin{aligned} \Phi_2 =& \text{He}\left\{ \begin{bmatrix} P_0 \\ P_1 \\ 0 \end{bmatrix} [\, A_0 \quad A_1 \quad 0 \,] \right\} \\ &+ \begin{bmatrix} P_0 B_0 \\ P_0 B_1 + P_1 B_0 \\ P_1 B_1 \end{bmatrix} \begin{bmatrix} P_0 B_0 \\ P_0 B_1 + P_1 B_0 \\ P_1 B_1 \end{bmatrix}^T \end{aligned} \tag{26}$$

By applying (D, G) scaling approach and then the Schur Complement lemma on (25), the inequality given in (21) is obtained.

Inequalities in (6) are linear with respect to θ. Therefore, they hold for all $\theta \in [-1, 1]$ if and only if they are satisfied just for $\theta = 1$ and $\theta = -1$. In this way, the other inequalities in (22)-(24) result. □

In the sequel, the problem of H_∞ and H_2 analysis based on polynomially parameter-dependent Lyapunov matrices is proposed.

4. EXTENSION TO POLYNOMIALLY PARAMETER-DEPENDENT LYAPUNOV MATRICES

It has been shown in Bliman [2004] that the inequalities in (4)-(6) depending continuously on the scalar parameter θ have polynomial type solutions with respect to θ of the following form:

$$P_N(\theta) = \sum_{i=0}^{N} \theta^i P_i \tag{27}$$

$$Q_N(\theta) = \sum_{i=0}^{N} \theta^i Q_i \tag{28}$$

with sufficiently high degree N. Therefore, in this section, we are interested in the extension of the previous results to polynomially parameter-dependent Lyapunov matrix $P_N(\theta)$ and $Q_N(\theta)$. The results are summerized in the following subsections. As the degree of the polynomials $P(\theta)$ and $Q(\theta)$ increases, the results converge to the optimal ones. Finally, necessary and sufficient conditions for the robust performance analysis of affine single parameter-dependent systems are derived.

4.1 H_∞ Performance Analysis by Means of Polynomially Parameter-dependent Lyapunov Matrices

Theorem 3. For an affine single parameter-dependent system, the inequality in (4) holds with $P_N(\theta)$ given in (27) if and only if there exist symmetric matrices P_i for $i = 0, 1, \cdots, N$, positive definite matrices D and L, and real skew-symmetric matrices G and K such that:

$$\begin{bmatrix} W_N + \Delta_{N+1}^n(D, G) & \star \\ Y_N & -\gamma I_{m+p} \end{bmatrix} < 0 \tag{29}$$

$$-\frac{1}{2}\begin{bmatrix} 2P_0 & P_1 & \cdots & P_{j-1} & P_j \\ P_1 & 0 & \cdots & 0 & P_{j+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ P_{j-1} & 0 & \cdots & 0 & P_{2j-1} \\ P_j & P_{j+1} & \cdots & P_{2j-1} & 2P_{2j} \end{bmatrix} + \Delta_j^n(L, K) < 0 \tag{30}$$

where

$$j = \begin{cases} \frac{N}{2} & \text{if } N \text{ is even} \\ \frac{N+1}{2} & \text{if } N \text{ is odd} \end{cases} \tag{31}$$

$$W_N = \text{He}\left\{ \begin{bmatrix} P_0 \\ P_1 \\ \vdots \\ P_N \\ 0 \end{bmatrix} [\, A_0 \quad A_1 \quad 0 \quad \cdots \quad 0 \,] \right\} \tag{32}$$

$$Y_N = \begin{bmatrix} P_0 B_0 & C_0^T \\ P_0 B_1 + P_1 B_0 & C_1^T \\ P_1 B_1 + P_2 B_0 & 0 \\ \vdots & \vdots \\ P_{N-1} B_1 + P_N B_0 & 0 \\ P_N B_1 & 0 \end{bmatrix}^T \tag{33}$$

and $\Delta_{N+1}^n(D, G)$ is defined in (9) with $k = N + 1$. Note that if N is an odd number, $2j = N+1$; therefore, $P_{2j} = 0$ is considered.

Proof: The inequality given in (30) directly expresses the positivity of matrix $P_N(\theta)$ based on (D, G) scaling approach in two cases where N is either even or odd. The remaining of the poof is similar to that of Theorem 1. □

4.2 H_2 Performance Analysis by Means of Polynomially Parameter-dependent Lyapunov Matrices

Theorem 4. For an affine single parameter-dependent system, the inequalities in (5) and (6) hold with $P_N(\theta)$ and $Q_N(\theta)$ given in (27) and (28), respectively if and only if there exist symmetric matrices P_i and Q_i for $i = 0, 1, \cdots, N$, positive definite matrices D, L, and H and skew-symmetric matrices G, K, and T such that:

$$\begin{bmatrix} W_N + \Delta^n_{N+1}(D, G) & \star \\ Y_N(1:m,:) & -I_m \end{bmatrix} < 0 \tag{34}$$

$$-\frac{1}{2}\begin{bmatrix} 2Z_0 & Z_1 & \cdots & Z_{j-1} & Z_j \\ Z_1 & 0 & \cdots & 0 & Z_{j+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Z_{j-1} & 0 & \cdots & 0 & Z_{2j-1} \\ Z_j & Z_{j+1} & \cdots & Z_{2j-1} & 2Z_{2j} \end{bmatrix} + \Delta^{n+p}_j(L, K) < 0 \tag{35}$$

$$\frac{1}{2}\begin{bmatrix} 2(q_0 - \upsilon) & q_1 & \cdots & q_{j-1} & q_j \\ q_1 & 0 & \cdots & 0 & q_{j+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{j-1} & 0 & \cdots & 0 & q_{2j-1} \\ q_j & q_{j+1} & \cdots & q_{2j-1} & 2q_{2j} \end{bmatrix} + \Delta^1_j(H, T) < 0 \tag{36}$$

where j, W_N, and Y_N are given in (31), (32), and (33), respectively.

$$Z_i = \begin{cases} \begin{bmatrix} P_i & {C_i}^T \\ C_i & Q_i \end{bmatrix} & \text{for } i = 0, 1 \\[2ex] \begin{bmatrix} P_i & 0 \\ 0 & Q_i \end{bmatrix} & \text{for } i = 2, \ldots, N \end{cases} \tag{37}$$

$$q_i = trace(Q_i); \quad i = 0, 1, \ldots, N \tag{38}$$

Note that $Z_{2j} = 0$ and $q_{2j} = 0$ if N is an odd number.
Proof: The inequalities given in (5) and (6) can be rewritten in the form of (13) as follows:

$$\begin{bmatrix} I_n \\ \theta I_n \\ \vdots \\ \theta^{N+1} I_n \end{bmatrix}^T \Phi_{2_N} \begin{bmatrix} I_n \\ \theta I_n \\ \vdots \\ \theta^{N+1} I_n \end{bmatrix} < 0 \tag{39}$$

$$-(Z_0 + Z_1\theta + \cdots + Z_N\theta^N) < 0 \tag{40}$$

$$q_0 - \upsilon + q_1\theta + \cdots + q_N\theta^N < 0 \tag{41}$$

where

$$\Phi_{2_N} = W_N + Y_N^T(1:m,:)Y_N(1:m,:) \tag{42}$$

Z_i and q_i for $i = 1, \cdots, N$ are defined in (37) and (38). Then, by applying (D, G) scaling approach to the inequalities in (39)-(41), the linear matrix inequalities in (34)-(36) are obtained. □

5. FIXED-ORDER H_∞ AND H_2 CONTROLLER DESIGN

Results of Theorem 3 and 4 can be utilized to design fixed-order dynamic output feedback controllers for affine single parameter-dependent systems represented by the following state space realization:

$$\begin{aligned} \dot{x}_g(t) &= A_g(\theta)x_g(t) + B_g(\theta)u(t) + B_w(\theta)w(t) \\ z(t) &= C_z(\theta)x_g(t) + D_{zu}(\theta)u(t) \\ y(t) &= C_g x_g(t) \end{aligned} \tag{43}$$

where the matrices $A_g(\theta)$, $B_g(\theta)$, $B_w(\theta)$, $C_z(\theta)$, and $D_{zu}(\theta)$ belong to the following uncertainty area:

$$\begin{aligned} A_g(\theta) &= A_{g_0} + \theta A_{g_1} \\ B_g(\theta) &= B_{g_0} + \theta B_{g_1} \\ B_w(\theta) &= B_{w_0} + \theta B_{w_1} \\ C_z(\theta) &= C_{z_0} + \theta C_{z_1} \\ D_{zu}(\theta) &= D_{zu_0} + \theta D_{zu_1} \end{aligned} \tag{44}$$

where $\theta \in [-1, 1]$. The signals $x_g \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_i}$, $w \in \mathbb{R}^r$, $y \in \mathbb{R}^{n_o}$, and $z \in \mathbb{R}^s$ are the state, the control input, the exogenous input, the measured output, and the controlled output, respectively.

The objective is to design a stabilizing dynamic output-feedback controller satisfying some H_∞ and H_2 norm bounds on the transfer function between w and z, $H_{zw}(s, \theta)$. The fixed-order controller $K_c(s)$ is presented as follows:

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c y(t) \\ u(t) &= C_c x_c(t) + D_c y(t) \end{aligned} \tag{45}$$

where $A_c \in \mathbb{R}^{m \times m}$ and B_c, C_c, and D_c are of appropriate dimensions. Then, the closed-loop system $H_{zw}(s, \theta)$ has the following state space realization:

$$\begin{aligned} \dot{x}(t) &= A(\theta)x(t) + B(\theta)w(t) \\ z(t) &= C(\theta)x(t) \end{aligned} \tag{46}$$

where $x(t) = [x_g(t) \quad x_c(t)]^T$ and

$$\begin{aligned} A(\theta) &= \begin{bmatrix} A_g(\theta) + B_g(\theta)D_cC_g & B_g(\theta)C_c \\ B_cC_g & A_c \end{bmatrix} \\ B(\theta) &= \begin{bmatrix} B_w(\theta) \\ 0 \end{bmatrix} \\ C(\theta) &= [C_z(\theta) + D_{zu}(\theta)D_cC_g \quad D_{zu}(\theta)C_c] \end{aligned} \tag{47}$$

The closed-loop matrices $A(\theta)$, $B(\theta)$, and $C(\theta)$ belong to (2); where

$$\begin{aligned} A_0 &= \begin{bmatrix} A_{g_0} + B_{g_0}D_cC_g & B_{g_0}C_c \\ B_cC_g & A_c \end{bmatrix}; \\ A_1 &= \begin{bmatrix} A_{g_1} + B_{g_1}D_cC_g & B_{g_1}C_c \\ 0 & 0 \end{bmatrix}; \\ B_0 &= \begin{bmatrix} B_{w_0} \\ 0 \end{bmatrix}; \quad B_1 = \begin{bmatrix} B_{w_1} \\ 0 \end{bmatrix}; \\ C_0 &= [C_{z_0} + D_{zu_0}D_cC_g \quad D_{zu_0}C_c]; \\ C_1 &= [C_{z_1} + D_{zu_1}D_cC_g \quad D_{zu_1}C_c] \end{aligned} \tag{48}$$

Remark: If the matrix C_g belongs to (2) and B_g is fixed, the same results are obtained; however, the case of uncertainty in both matrices (B_g, C_g) has not been considered in this contribution. Inequalities given in (29) and (30) are necessary and sufficient conditions for fixed-order H_∞ controller design. In this case, H_∞ performance constraints are expressed by a set of BMIs by means of polynomially parameter-dependent Lyapunov matrices.

Similar to the H_∞ control problem, the problem of fixed-order H_2 dynamic output-feedback design of the affine single parameter-dependent systems described by (43) can be reformulated as an optimization problem subject to a set of BMI and LMI constraints in (34)-(36).

To solve optimisation problems involving BMI constraints, several local and global approaches have been developed in the literature (e.g. Safonov et al. [1994], Kocvara and Stingl [2006], Kanev et al. [2004], Dinh et al. [2012]). These methods can be conveniently employed to solve the BMI problem of fixed-order controller design proposed in this paper.

6. SIMULATION EXAMPLES

In this section, three illustrative examples are provided. It should be noted that LMI problems in the examples are solved using YALMIP (Löfberg [2004]) as the interface and SDPT3 (Toh et al. [1999]) as the solver.

Example 1: In this example, the problem of H_∞ analysis of the following stable affine single parameter-dependent system is considered.

$$\begin{aligned} A_0 &= \begin{bmatrix} -4 & 2 & -2 \\ 5 & -6 & 1 \\ -2 & 2 & -7 \end{bmatrix}; \quad A_1 = \begin{bmatrix} -5 & -3 & -13 \\ -5 & 0 & 0 \\ 10 & 13 & 16 \end{bmatrix} \\ B_0 &= \frac{1}{10}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad B_1 = \frac{1}{10}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ C_0 &= \frac{1}{10}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}; \quad C_1 = \frac{1}{10}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \end{aligned} \tag{49}$$

The state matrices A_0 and A_1 are borrowed from Ebihara and Hagiwara [2006]. They have shown that there does not exist a linearly parameter-dependent Lyapunov matrix for the stability analysis of $A(\theta) = A_0 + \theta A_1$ for all $\theta \in [-1, 1]$.

By applying a fine griding on θ, the worst case H_∞ norm of the system can be computed, $\|H(s,\theta)\|_{\infty_{w.c.}} = 1.5336$. The results of Sections 3 and 4 are applied to determine the upper bound of H_∞ norm of the system $(A(\theta), B(\theta), C(\theta))$. The LMI conditions in (14) and (15) become infeasible by linearly parameter-dependent Lyapunov matrices. However, the second-degree polynomially parameter-dependent Lyapunov matrices reach to the worst case H_∞ norm.

Example 2: As the second example, consider the following system borrowed from de Oliveira et al. [2004a]:

$$\begin{aligned} A_0 &= \begin{bmatrix} -0.535 & 0.455 & 0.115 \\ -0.085 & -0.67 & -0.325 \\ 0.45 & -0.21 & -0.17 \end{bmatrix} \\ A_1 &= \begin{bmatrix} -0.095 & -0.355 & 0.785 \\ -0.805 & -0.03 & -0.145 \\ -0.47 & 0.52 & -0.04 \end{bmatrix} \\ B_0 &= B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad C_0 = C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{50}$$

The worst case H_2 norm of the system is $\|H(s,\theta)\|_{2_{w.c.}} = 1.4921$ (a fine girding has been used). Table 1 shows the upper bound of H_2 norm of the system $(A(\theta), B, C)$ and the dual system $(A^T(\theta), C^T, B^T)$ with polynomially parameter-dependent Lyapunov matrices of order N. In Table 1, the symbol $-$ indicates that the LMIs in (34)-(36) are infeasible with the polynomially parameter-dependent Lyapunov matrices $P_N(\theta)$ and $Q_N(\theta)$ of the related order N.

Table 1. Upper bound of $\|H(s,\theta)\|_2$ in Example 2 with polynomially parameter-dependent Lyapunov matrices of order N

N	0	1	2	3	4
$\|H(s,\theta)\|_2$	—	1.7247	1.5052	1.4921	1.4921
$\|H^T(s,\theta)\|_2$	—	2.3161	1.646	1.4923	1.4921

By increasing the order of polynomially parameter-dependent matrices $P_N(\theta)$ and $Q_N(\theta)$ in (27) and (28), better results are obtained and finally the worst case H_2 norm can be reached with $P_N(\theta)$ and $Q_N(\theta)$ of order $N = 4$.

Example 3: Consider the continuous-time polytopic system with two vertices in Dong and Yang [2013]. The system can be easily converted to an affine single parameter-dependent system in (43); where,

$$\begin{aligned} A_g(\theta) &= \begin{bmatrix} -1.346 & 34.065 & 179.82 \\ 0.2424 & -1.135 & -21.69 \\ 0 & 0 & -30 \end{bmatrix} + \theta \begin{bmatrix} 0.356 & -16.65 & -83.67 \\ 0.0223 & 0.2834 & 10.3 \\ 0 & 0 & 0 \end{bmatrix} \\ B_g &= \begin{bmatrix} -91.435 \\ 0 \\ 30 \end{bmatrix} + \theta \begin{bmatrix} -6.345 \\ 0 \\ 0 \end{bmatrix}; \quad B_w = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ C_g &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}; \quad C_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ D_{zu} &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{51}$$

where $\theta \in [-1, 1]$. The objective here is to design a static output feedback H_∞ controller with polynomially parameter-dependent Lyapunov matrices. To this end, an optimization problem, which is the minimization of γ subject to the LMI and BMI constraints in (14) and (15), should be solved. The BMI constraints are solved by using PENBMI, version 2.1, (Kocvara and Stingl [2006]) with initial controller $K_{c_0} = [0 \quad 0]$. The problem is solved after 30 iterations (CPU time $= 9.823 sec$). Resulting static output feedback is given in Table 2.

Table 2. Upper bound of $\|H_{zw}(s,\theta)\|_\infty$ in Example 3

Method	γ	K_c
Crusius *et al*, 1999	9.7315	$[0.5558 \quad 5.0823]$
Shaked, 2003	6.8028	$[0.0536 \quad 0.6384]$
Dong *et al*, 2013	2.3267	$[0.4474 \quad 4.1860]$
Sadabadi *et al*, 2013	1.7947	$[77.1587 \quad 608.8698]$
Proposed method with $N = 1$	1.6602	$[130.3463 \quad 939.3718]$
Proposed method with $N = 2$	1.6446	$[0.1702e3 \quad 1.2234e3]$

The results are compared with the LMI-based methods in Crusius and Trofino [1999], Shaked [2003], Dong and Yang [2013], Sadabadi and Karimi [2013]. It can be observed from Table 2 that the proposed BMI-based method in this paper with polynomially parameter-dependent Lyapunov matrices of order two leads to the best results among the others.

7. CONCLUSION

In this paper, the necessary and sufficient LMI conditions for H_∞ and H_2 performance analysis of continuous-time affine single parameter-dependent systems with polynomially parameter-dependent Lyapunov matrices have been proposed. The fundamental idea of the proposed approach is based on the use of (D, G) scaling which can convert inequality conditions depending on an uncertain parameter to a set of parameter-independent LMIs. The results have been employed to develop the necessary and sufficient conditions for fixed-order H_∞ and H_2 dynamic output feedback controller design for affine single parameter-dependent systems in terms of BMIs. The extension of the results to the robust performance analysis/controller synthesis of affine multi parameter-dependent systems by linearly parameter-dependent Lyapunov matrices leads to only sufficient conditions.

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