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Distributed parameter estimation for adaptive event-triggered control ⋆

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Abstract: The asymptotic stability of distributed-event triggered control for interconnected linear systems can be achieved with time-dependent trigger functions, whose parameters relies on some propoerties of the overall system, which are not known a priori. This paper presents an approach to the distributed estimation of such parameters, which yields to adaptive event-triggered control. We provide proofs of convergence in finite time while the asymptotic stability and the existence of a lower bound for the inter-event times are preserved.

1. INTRODUCTION

The centralized control of large-scale systems in a networked environment require an accurate knowledge of the interaction between subsystems and the consumption of a lot of computation and network resources. Decentralized approaches are not the optimal solutions because the subsystems do not communicate between them even if they significantly interact. By contrast, in distributed control, the local controllers exchange information to compensate the interconnections, and so, the communication network turns to be part of the design problem.

Recent contributions have shown the interest in applying event-triggering to distributed Networked Control Systems (NCS) [Dimarogonas et al., 2012, De Persis et al., 2011, Donkers and Heemels, 2012, Guinaldo et al., 2013, Mazo and Tabuada, 2011, Wang and Lemmon, 2011]. The basic idea in all these contributions is that each subsystem (also called agent or node) decides when to transmit the measurements based only on local information.

The stability of the system is related to algebraic properties, and many control methods use this information into the design. The problem is when we require the knowledge of some properties of the overall system in distributed control systems but we only have access to nearby nodes. For instance, analysis on eigenvalues can be used to get better performance by setting the communication links in distributed control for interconnected linear systems [Gusrialdi and Hirche, 2010], to estimate the worst case of convergence rate in consensus algorithms [Olfati-Saber and Murray, 2004], or to design trigger rules such that asymptotic stability and a lower bound on the inter-event times are guaranteed [Guinaldo et al., 2013, Seyboth et al., 2013].

There are several algorithms to estimate the matrix eigenvalues in a distributed fashion [Kempe and McSherry, 2008, Trefethen and Bau III, 1997]. Most of them apply to Hermitian square matrices, and they have been applied, for example, to estimate the algebraic connectivity of a network [Yang et al., 2010, Aragues et al., 2012].

In this paper, we propose a distributed method to estimate parameters in interconnected linear systems. Since interconnections are characterized by uncertainties in these setups, the computation of the eigenvalues of the closed loop system can be a tough problem. We solve this problem by estimating the largest eigenvalue of the closed loop system without the interconnections, and computing an upper bound on the interconnections for the overall system. The solution lies on a completely distributed approach. The computation required by the proposed method is very low and basically consists of running the `max`-consensus algorithm [Tahbaz-Salehi and Jadbabaie, 2006].

The most recent estimates of the parameters are used to update the trigger functions presented in Guinaldo et al. [2013]. Adaptive event-triggering was suggested in Aragues et al. [2012] for average consensus and multi-agent systems. In our approach, the exchange of information (broadcasted states and parameters estimates) occurs only at event times. We further provide proofs of convergence in finite time, asymptotic stability and exclusion of Zeno behavior.

The rest of the paper is organized as follows: In Section 2 some background and the problem statement are given. Section 3 presents the distributed parameter estimation algorithms. Section 4 discusses adaptive event-triggering and provides the main stability results of this paper. Our method is evaluated in simulations in Section 5. Finally, conclusions and future work are presented in Section 6.

2. BACKGROUND AND PROBLEM STATEMENT

2.1 Matrix and perturbation analysis

Let $A \in \mathbb{C}^{n\times n}$ be a complex matrix, and let us denote $A^* = (\bar{a}_{ji})$, $\lambda(A) = \{\lambda : \det(A - \lambda I) = 0\}$, and

$$\kappa(A) = \|A\|\|A^{-1}\| \quad (0 \notin \lambda(A)), \tag{1}$$

$$\lambda_{max}(A) = \max\{\Re e(\lambda) : \lambda \in \lambda(A)\}, \tag{2}$$

where $\|\cdot\|$ denotes the induced 2-norm or spectral norm.

The exponential of A is defined as $e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$.

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Let Y be an invertible matrix such that $A = YBY^{-1}$ and $\mu_{max}(A) = \max\{\mu : \mu \in \lambda((A + A^*)/2)\}$. It follows that $\|e^{At}\| = \|Ye^{Bt}Y^{-1}\| \leq \kappa(Y)e^{\mu_{max}(B)t}$ [Van Loan, 1977], being $\kappa(Y)$ is defined according to (1). Thus, assume that A is *diagonalizable*, i.e., there exists a matrix D, where $D = \text{diag}(\lambda(A))$, and a matrix V of eigenvectors, such that $A = VDV^{-1}$. Then it holds

$$\|e^{At}\| \leq \kappa(V)e^{\mu_{max}(D)t} = \kappa(V)e^{\lambda_{max}(D)t} = \kappa(V)e^{\lambda_{max}(A)t}, \tag{3}$$

where $\lambda_{max}(A)$ is defined according to (2).

Let the matrix A be perturbed by some matrix E. A result from semigroup theory (see Kato [1966]) states that if $\|e^{At}\| \leq ce^{\beta t}$ for some constants $c > 0$ and β, then

$$\|e^{(A+E)t}\| \leq ce^{(\beta + c\|E\|)t}. \tag{4}$$

2.2 Maximum and minimum consensus algorithms

Consider a group of $N \in \mathbb{N}$ agents. Let us denote the internal state of the agent i at time $t > 0$ by $y_i(t) \in \mathbb{R}$, and $y_i(0)$ are the initial state value. Assume that the agents exchange information with nearby agents. This information is represented by an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, \ldots, N\}$ is the set of agents, and the pair (i, j) is in $\mathcal{E}$ if the nodes i and j exchange data.

The minimum consensus algorithm results in the convergence of all the states to the minimum of the initial state values. At each step, the node i updates its state to the minimum state value of the neighboring nodes and its own. In Tahbaz-Salehi and Jadbabaie [2006], it is proved that the agents reach the consensus in a finite number of steps and $\lim_{t\to\infty} y(t) = \min_{i=1,\ldots,N} y_i(0)\mathbf{1}$, where $\mathbf{1}$ represents the vector with all entries equal to one. Equivalent results can be obtained for the maximum consensus.

For fixed topologies, connected undirected graphs or strongly connected digraphs, the `max` (`min`) consensus is reached in a finite number of iterations $l_{max} < N - 1$ [Olfati-Saber and Murray, 2004], where l_{max} is the diameter of the graph, i.e., the length of the *longest shortest path* between any two nodes [Godsil and Royle, 2001].

2.3 Problem statement

Consider a system of N linear time-invariant subsystems. The dynamics of each subsystem is given by

$$\dot{x}_i(t) = A_i x_i(t) + B_i u_i(t) + \sum_{j \in N_i} H_{ij} x_j(t), \quad \forall i = 1, ..., N \tag{5}$$

where N_i is the set of "neighbors" of the subsystem i, i.e., the set of subsystems that directly drive agent i's dynamics, and H_{ij} is the interaction term between agent i and agent j.

In Guinaldo et al. [2013], an event-triggered control law is proposed so that the system (5) can achieve asymptotic stability while decreasing the average frequency of communication between nodes, and excluding the Zeno behavior, i.e., the occurrence of two consecutive transmission events at the same instance of time.

The control law is computed based on the last transmited or broadcasted state $x_{b,i}$, rather than on the continuous measurements, as

$$u_i(t) = K_i x_{b,i}(t) + \sum_{j \in N_i} L_{ij} x_{b,j}(t), \quad \forall i = 1, ..., N \tag{6}$$

where K_i is the feedback gain for the nominal subsystem i, L_{ij} is a set of decoupling gains, and $A_{K,i} = A_i + B_i K_i$ is assumed to be Hurwitz.

Error functions are defined as

$$e_i(t) = x_{b,i}(t) - x_i(t). \tag{7}$$

Each agent i decides when to transmit its state to the neighborhood based on some function of the local error (7). More specifically, the proposed trigger functions for the system (5) are

$$f_i(t, e_i(t)) = \|e_i(t)\| - c_1 e^{-\alpha t}, \alpha, c_1 > 0. \tag{8}$$

Event times $\{t_k^i\}_{k=0}^{\infty}$ for each subsystem i are determined recursively by the event trigger function as $t_{k+1}^i = \inf\{t : t > t_k^i, f_i(t, e_i(t)) > 0\}$.

To guarantee the stability of the interconnected system (5) with control law (6) and update mechanism defined by trigger functions (8), the parameter α is constrained by some information of the overall system. Next section discusses this more in detail and presents a proposed distributed parameter estimation approach such that the system reaches the equilibrium asymptotically.

3. DISTRIBUTED PARAMETERS ESTIMATION

3.1 Motivation

The system (5) with control law (6) can be rewriten in terms of the error (7) as

$$\dot{x}_i(t) = A_{K,i} x_i(t) + B_i K_i e_i(t) + \sum_{j \in N_i} \left(\Delta_{ij} x_j(t) + B_i L_{ij} e_j(t)\right), \tag{9}$$

where $\Delta_{ij} = B_i L_{ij} + H_{ij}$ are the coupling terms. In general, $\Delta_{ij} \neq \mathbf{0}$.

Note that if $\Delta_{ij} \neq \mathbf{0}$, the dynamics of $\dot{x}_i(t)$ explicitly depends on $x_j(t), \forall j \in N_i$. To solve this coupling problem, we can study the stability for the overall system.

Let us define $x = (x_1^T, x_2^T, ..., x_N^T)^T$, $e = (e_1^T, e_2^T, ..., e_N^T)^T$, and the following block matrices

$$A_K = \text{diag}(A_{K,1}, A_{K,2}, ..., A_{K,N}) \tag{10}$$

$$\Delta = \{\Delta_{ij}\}, \quad \Delta_{ij} = \mathbf{0} \quad \text{if } i = j \text{ or } j \notin N_i, \tag{11}$$

$B = \text{diag}(B_1, B_2, ..., B_N)$, and $K = \{K_{ij}\}$ where $K_{ij} = K_i$ if $i = j$ and $K_{ij} = L_{ij}$ otherwise.

Thus, the dynamics of the overall system is given by

$$\dot{x}(t) = (A_K + \Delta)x(t) + BKe(t). \tag{12}$$

Under the assumptions that $A_{K,i}$, $\forall i = 1, \ldots, N$ are diagonalizable, and that the coupling terms are such that $\kappa(V)\|\Delta\| < |\lambda_{max}(A_K)|$ holds, the system is asymptotically stable and the state is upper bounded by

$$\|x(t)\| \leq \kappa(V)\Big(e^{-(|\lambda_{max}(A_K)| - \kappa(V)\|\Delta\|)t}\big(\|x(0)\| - \frac{c_1}{|\lambda_{max}(A_K)| - \kappa(V)\|\Delta\| - \alpha}\big)\big) + e^{-\alpha t}\frac{\|BK\|\sqrt{N}c_1}{|\lambda_{max}(A_K)| - \kappa(V)\|\Delta\| - \alpha}\Big),$$

if the parameter α of the trigger functions (8) is constrained as $\alpha < |\lambda_{max}(A_K)| - \kappa(V)\|\Delta\|$ [Guinaldo et al., 2013]. $\lambda_{max}(A_K)$ is defined according to (2) and $\kappa(V)$

is the condition number (1), where V is the matrix of eigenvectors of A_K. $\|\Delta\|$ is the induced 2-norm of Δ.

Hence, the parameter α is restricted by the values of three parameters of the global system, and this information is not accessible by the individual nodes.

The next section presents distributed algorithms to estimate $\lambda_{max}(A_K)$ and $\kappa(V)$, and an upper bound for $\|\Delta\|$.

3.2 Distributed computation of $\lambda_{max}(A_K)$ and $\kappa(V)$

The definition of A_K (10) corresponds to the direct sum of $A_{K,i}$. The following properties are used in the sequel:

$$(A_1 \oplus A_2) \cdot (B_1 \oplus B_2) = A_1B_1 \oplus A_2B_2 \quad (13)$$

$$(A \oplus B)^{-1} = A^{-1} \oplus B^{-1}, \quad (14)$$

if A^{-1} and B^{-1} exist.

Proposition 1. If A_K is defined according to (10) and $A_{K,i} = V_iD_iV_i^{-1}$ holds $\forall i = 1, \ldots, N$, it follows that

$$\lambda_{max}(A_K) = \max_{i=1,\ldots,N}\{\lambda_{max}(A_{K,i})\} \quad (15)$$

$$\kappa(V) = \max_{i=1,\ldots,N}\{\|V_i\|\} \cdot \max_{i=1,\ldots,N}\{\|V_i^{-1}\|\}. \quad (16)$$

Proof. Using the property (13), it follows that $A_K = \oplus_i A_{K,i} = \oplus_i V_iD_iV_i^{-1} = \oplus_i V_i \oplus_i D_i \oplus_i V_i^{-1}$. Let us denote $D = \oplus_i D_i$. Since D is a diagonal matrix that contains the eigenvalues of all the matrices $A_{K,i}$, (15) is proven.

Let $V = \oplus_i V_i$. Since V is a block diagonal matrix, it holds that $\|V\| = \max\{\|V_i\|_{i=1,\ldots,N}\}$. From (14), it follows that $V^{-1} = \oplus_i V_i^{-1}$ and so, $\|V^{-1}\| = \max\{\|V_i^{-1}\|_{i=1,\ldots,N}\}$, which yields (16), from the definition (1).

From the results of Proposition 1, the algorithm for the distributed computation of $\lambda_{max}(A_K)$ and $\kappa(V)$ basically consists of running three `max`-consensus algorithms:

Algorithm 1. Each node i mantains estimates $\hat{\lambda}^i_{max}(A_K)$, $\hat{p}_i(V)$, $\hat{p}_i(V^{-1})$, and $\hat{\kappa}_i(V) = \hat{p}_i(V) \cdot \hat{p}_i(V^{-1})$, where $\hat{p}_i(X)$ denotes the estimation by the node i of the norm of a matrix X.

(1.1) At $k = 0$, each node initializes $\hat{\lambda}^i_{max}(A_K) = \lambda_{max}(A_{K,i})$, $\hat{p}_i(V) = \|V_i\|$, $\hat{p}_i(V^{-1}) = \|V_i^{-1}\|$, and $\hat{\kappa}_i(V) = \hat{p}_i(V) \cdot \hat{p}_i(V^{-1})$ that are computed offline by each node.

(1.2) At each step $k \geq 1$, each node i broadcasts the estimates $\hat{\lambda}^i_{max}(A_K)$, $\hat{p}_i(V)$, and $\hat{p}_i(V^{-1})$ to all $j \in N_i$.

(1.3) Then, the node i updates all the variables as $\hat{\lambda}^i_{max}(A_K) = \max_{j\in N_i\cup\{i\}}\{\hat{\lambda}^j_{max}(A_K)\}$, $\hat{p}_i(V) = \max_{j\in N_i\cup\{i\}}\{\hat{p}_j(V)\}$, $\hat{p}_i(V^{-1}) = \max_{j\in N_i\cup\{i\}}\{\hat{p}_j(V^{-1})\}$, and $\hat{\kappa}_i(V) = \hat{p}_i(V) \cdot \hat{p}_i(V^{-1})$.

3.3 Distributed computation of $\|\Delta\|$

The definition of Δ is given in (11), and includes the coupling terms, which might not be well known. The algorithm proposed in this section will provide an upper bound for $\|\Delta\|$ under the following assumption.

Assumption 2. Each subsystem i does not have full information about Δ_{ij}, $j \in N_i$, but it has the knowledge of some upper bounds δ_{ij} so that $\|\Delta_{ij}\| \leq \delta_{ij}$.

Proposition 3. If Assumption 2 applies, then $\|\Delta\|$ can be upper bounded as

$$\|\Delta\| \leq \sqrt{\Big(\max_{i=1,\ldots,N}\{\sum_{j\in N_i}\delta_{ij}\}\Big)\Big(\max_{j=1,\ldots,N}\{\sum_{i\in N_j}\delta_{ij}\}\Big)}. \quad (17)$$

Proof. For a block partitioned matrix $A = \{A_{ij}\}$, where $A_{ij} \in \mathbb{R}^{n_i\times n_j}$, it holds that $\|A\| \leq \|\mu(A)\|$ [Smoktunowicz, 2008], where $(\mu(A))_{ij} = \|A_{ij}\|$.

Another useful inequality between matrix norms is $\|A\| \leq \sqrt{\|A\|_1\|A\|_\infty}$, where $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are the induced 1-norm and $\infty-$norm, respectively [Meyer, 2001].

Thus, it follows that

$$\|\Delta\| \leq \|\mu(\Delta)\| \leq \sqrt{\|\mu(\Delta)\|_1\|\mu(\Delta)\|_\infty}. \quad (18)$$

Since Assumption 2 applies, and from the definition of $\|\cdot\|_1$ and $\|\cdot\|_\infty$, it yields (17) straightforwardly.

Remark 4. According to the the definitions of the norms $\|A\|_1 = \max_j \sum_{i=1}^m |a_{ij}|$ and $\|A\|_\infty = \max_i \sum_{j=1}^n |a_{ij}|$ for any $m \times n$ matrix, the main advantage of using (18) is that the sums are computed row/column by row/column, so that the distributed computation is easier.

The following algorithm is used by each node i to compute an upper bound for $\|\Delta\|$ in a distributed manner.

Algorithm 2. Each node i mantains estimates $\hat{p}_i(\Delta_2)$, $\hat{p}_i(\Delta_1)$, and $\hat{p}_i(\Delta_\infty)$ to compute an upper bound for $\|\Delta\|$, $\|\Delta\|_1$, and $\|\Delta\|_\infty$, respectively.

(2.1) At $k = 0$, each node i initializes $\hat{p}_i(\Delta_2) = \sum_{j\in N_i}\delta_{ij}$, and sends to each neighbor $j \in N_i$ the upper bound δ_{ij} for the corresponding coupling term.

(2.2) At $k = 1$, the local estimates $\hat{p}_i(\Delta_2)$, $\hat{p}_i(\Delta_1)$, and $\hat{p}_i(\Delta_\infty)$ are updated to $\hat{p}_i(\Delta_1) = \sum_{j\in N_i}\delta_{ji}$, $\hat{p}_i(\Delta_\infty) = \sum_{j\in N_i}\delta_{ij}$, $\hat{p}_i(\Delta_2) = \sqrt{\hat{p}_i(\Delta_1)\hat{p}_i(\Delta_\infty)}$.

(2.3) At each step $k \geq 2$, it broadcasts the estimates $\hat{p}_i(\Delta_1)$ and $\hat{p}_i(\Delta_\infty)$ to the neighbors $j \in N_i$.

(2.4) Then, the node i updates the variables as $\hat{p}_i(\Delta_1) = \max_{j\in N_i\cup\{i\}}\{\hat{p}_j(\Delta_1)\}$, $\hat{p}_i(\Delta_\infty) = \max_{j\in N_i\cup\{i\}}\{\hat{p}_j(\Delta_\infty)\}$, and $\hat{p}_i(\Delta_2) = \sqrt{\hat{p}_i(\Delta_1)\hat{p}_i(\Delta_\infty)}$.

Algorithms 1 and 2 basically solve three and two `max`-consensus problems, respectively. For periodic and synchronous updates in the nodes, the `max`-consensus to estimate $\lambda_{max}(A_K)$, $\kappa(V)$, and $\|\Delta\|$ are reached in a finite number of iterations $l_{max} < N - 1$, as commented in Section 2.2. However, this does not hold for event-triggering in general. Next section describes how the previous algorithms can be adapted to event-triggering and some other properties regarding the stability of the system.

4. ADAPTIVE EVENT-TRIGGERED CONTROL

Trigger functions (8) depend on two parameters c_1 and α. Both parameters determine the performance, and moreover, α is constrained by $|\lambda_{max}(A_K)| - \kappa(V)\|\Delta\|$ in Guinaldo et al. [2013].

According to the discussion and the results of the previous section, we propose new trigger functions as

$$f_i(t, e_i(t), \alpha_i(t)) = \|e_i(t)\| - c_1e^{-\alpha_i(t)t}, \ \alpha_i(t) > 0, \quad (19)$$

where the parameter $\alpha_i(t)$ is updated at local event instances of time denoted by $\{t_k^i\}_{k=0}^{\infty}$ as follows:

$$\alpha_i(t_k^i) = \gamma(|\hat{\lambda}_{max}^i(A_K)| - \hat{\kappa}_i(V)\hat{p}_i(\Delta_2)),\ 0 < \gamma < 1, \quad (20)$$

and remains constant in the inter-event times: $\alpha_i(t) = \alpha_i(t_k^i), \quad \forall t \in [t_k^i, t_{k+1}^i)$.

Assumption 5. Let us assume that the interconnection terms are such that $\kappa(V)\sqrt{\|\mu(\Delta)\|_1\|\mu(\Delta)\|_\infty} < |\lambda_{max}(A_K)|$ holds.

The previous assumption is just a little more restrictive than the case of study of Guinaldo et al. [2013] already discussed, in which $\kappa(V)\|\Delta\| < |\lambda_{max}(A_K)|$ was required to ensure the stability of the system (12). This way, Assumption 5 guarantees a positive value for α.

The adaptive event-triggered control algorithm can be summarized as follows:

Algorithm 3. Each node mantains the estimates for $\lambda_{max}(A_K)$, $\kappa(V)$, and $\|\Delta\|$ as described in Section 3, computes the control law based on the broadcasted states, and transmit the values of the state and the parameters at event times $\{t_k^i\}_{k=0}^{\infty}$.

(3.1) At $t = 0$, all nodes perform the initialization steps (1.1) and (2.1) of algorithms 1 and 2, respectively. They also set $\alpha_i = \gamma(|\hat{\lambda}_{max}^i(A_K)| - \hat{\kappa}_i(V)\hat{p}_i(\Delta_2))$, $\hat{\lambda}_{max}^j(A_K) = -\infty$, $\hat{p}_j(V) = \hat{p}_j(V^{-1}) = 0$, and $\hat{p}_j(\Delta_1) = \hat{p}_j(\Delta_\infty) = 0\ \forall j \in N_i$.

(3.2) Once (3.1) has concluded, each node performs the update (2.2) of Algorithm 2.

(3.3) In the node i, any time new information (state and parameters estimates) is received from the neighbors $j \in N_i$ at some t_k^j, the control law (6) as well as the parameters are updated according to steps (1.3) and (2.4) of algorithms 1 and 2, respectively.

(3.4) When a local event is detected, the state $x_i(t_k^i)$, and the estimates $\hat{\lambda}_{max}^i(A_K)$, $\hat{p}_i(V)$, $\hat{p}_i(V^{-1})$, $\hat{p}_i(\Delta_1)$, and $\hat{p}_i(\Delta_\infty)$ are broadcasted to the neighbors $j \in N_i$.

(3.5) Then, it updates the values of α_i according to (20) and computes the control law (6).

Remark 6. The value of $\alpha_i(t)$ is only updated at the local event times to avoid discontinuities in the error threshold in the inter-event times that could cause Zeno behavior.

As a result of Algorithm 3, the set of α_i converges to the same value $\alpha_m = \gamma(|\lambda_{max}(A_K)| - \kappa(V)\sqrt{\|\mu(\Delta)\|_1\|\mu(\Delta)\|_\infty})$. Note that α_m corresponds to the minimum value of $\alpha_i(t)$, $\forall i, t$. The value α_m is reached by all agents in finite time. However, due to the asynchronous and aperiodic update of the triggering mechanism, it is not straightforward to determine this instance of time analytically. Next results state this more rigorously.

Proposition 7. Let us assume that the node i_1 has at time $t = 0$ the maximum value of a parameter π, so that $\pi_{i_1} = \pi_{max}$. If $\|x_j(0)\| > 0\ \forall j = 1, \dots, N$, then there exist sequences of broadcasting finite times $t_1^{i_1} < t_{k_2}^{i_2} < \cdots < t_{k_{l_{max}}}^{i_{l_{max}}} < \infty$, so that at time $t_{k_{l_{max}}}^{i_{l_{max}}}$ the $\mathtt{max}$-consensus is reached for the parameter π, i.e. $\pi_j = \pi_{max}, \forall j = 1, \dots, N$.

Proof. The $\mathtt{max}$-consensus is reached when the information is transmited from the agent i_1 to any other node in the network following a path that has at most a l_{max} length, being l_{max} the diameter of the graph.

At the first event instance of time $t_1^{i_1}$ for the node i_1, π_{max} reaches the neighborhood N_{i_1}. When the next event occurs at $t > t_1^{i_1}$ for any of the neighbors $j \in N_{i_1}$, π_{max} reaches the neighborhoods $\{N_j\}$, and so on. The process is repeated no more than l_{max} times. Let us call $t_{k_{l_{max}}}^{i_{l_{max}}}$ the time at which the last node in the network updates $\pi_{i_{l_{max}}} = \pi_{max}$. At this time, the consensus is reached.

The existence of the sequence of broadcasting times is guaranteed by the triggering mechanism for non zero initial conditions $\|x_i(0)\|$, since the equilibrium is reached asymptotically. Therefore, there always exists a sufficiently large t such that $\|e_i(t)\| \geq c_1 e^{-\alpha_i(t)t}$.

Corollary 8. The trigger functions (19) end to

$$f_i(t, e_i(t)) = \|e_i(t)\| - c_1 e^{-\alpha_m t},\ \forall i = 1, \dots, N,$$

in finite time, and the time at which this occurs is when the $\mathtt{max}$-consensus is reached for all the estimated parameters. Moreover, $\alpha_m = \gamma(|\lambda_{max}(A_K)| - \kappa(V)\sqrt{\|\mu(\Delta)\|_1\|\mu(\Delta)\|_\infty})$ is the minimum value of all the sequences of values $\alpha_i(t)$, $\forall i$, $\forall t$.

Proof. The first part follows straightforward from Proposition 7. For the second part, note that $\lambda_{max}(A_{K,i}) < 0, \forall i = 1, \dots, N$. Thus, running a $\mathtt{max}$-consensus to estimate $\lambda_{max}(A_K)$ causes that $|\hat{\lambda}_{max}^i(A_K)|$ is a sequence of decreasing values. Moreover, the $\mathtt{max}$-consensus to estimate $\kappa(V)$ and $\|\Delta\|$ yields sequences of increasing values for the estimates $\hat{\kappa}_i(V)$ and $\hat{p}_i(\Delta_2)$. Thus, $\{\alpha_i(t)\}$ are a piecewise constant decreasing functions.

Theorem 9. Consider the interconnected linear system (5) with the adaptive event-triggered control described in Algorithm 3. Then, for all initial conditions $x(0)$ such that $\|x_i(0)\| > 0,\ \forall i = 1, \dots, N$, the closed-loop system does not exhibit Zeno behavior, and moreover, it converges asymptotically to the equilibrium.

Proof. From (12), it follows that $x(t) = e^{(A_K+\Delta)t}x(0) + \int_0^t (e^{(A_K+\Delta)(t-s)}BKe(s))ds$. Thus, the state $x(t)$ can be upper bounded by $\|x(t)\| \leq \|e^{(A_K+\Delta)t}\|\|x(0)\| + \int_0^t \|(e^{(A_K+\Delta)(t-s)}\|\|BK\|\|e(s)\|)ds$. Because A_K is diagonalizable, (3) can be applied to bound the exponential of A_K such that $\|e^{A_K t}\| \leq \kappa(V)e^{\lambda_{max}(A_K)t}$. Therefore, (4) can be used to compute an upper bound for $\|e^{(A_K+\Delta)t}\|$, giving as a result $\|e^{(A_K+\Delta)t}\| \leq \kappa(V)e^{(\lambda_{max}(A_K)+\kappa(V)\|\Delta\|)t}$. From (18) and noting that $\lambda_{max}(A_K) < 0$, it follows that $\|e^{(A_K+\Delta)t}\| \leq \kappa(V)e^{-\lambda_M t}$, where $\lambda_M = |\lambda_{max}(A_K)| - \kappa(V)\sqrt{\|\mu(\Delta)\|_1\|\mu(\Delta)\|_\infty}$.

Note that from the definition of the error $e(t)$ and Corollary 8: $\|e(s)\| \leq \sqrt{N}\max_i\|e_i(s)\| \leq \sqrt{N}\max_{i=1,\dots,N} c_1 e^{-\alpha_i(s)s} \leq \sqrt{N}c_1 e^{-\alpha_m s}$. Hence, it yields $\|x(t)\| \leq \kappa(V)\Big(e^{-\lambda_M t}\|x(0)\| + \int_0^t (e^{-\lambda_M(t-s)}\|BK\|\sqrt{N}c_1 e^{-\alpha_m s})ds\Big)$. Solving the integral and retriving the expression for α_m, it follows that

$$\|x(t)\| \leq \kappa(V)\Big(e^{-\lambda_M t}\|x(0)\| + \tfrac{\|BK\|\sqrt{N}c_1}{\lambda_M(1-\gamma)}\big(e^{-\gamma\lambda_M t} - e^{-\lambda_M t}\big)\Big), \quad (21)$$

that converges asymptotically to the origin if $0 < \gamma < 1$.

Before proving the exclusion the Zeno behavior, let us first show that there exists a constant c_2 such that $c_2 e^{-\alpha_m t} \le c_1 e^{-\alpha_i(t)t} \le c_1 e^{-\alpha_m t}$, $\forall i = 1, \dots, N$ and $t \ge 0$. Denote $\alpha_{max} = \max_{i=1,\dots,N}\{\alpha_i(0)\}$. According to Corollary 8, $\{\alpha_i(t)\}$ are piecewise constant decreasing functions, so they take the maximum value at $t = 0$, and therefore $c_1 e^{-\alpha_{max} t} \le c_1 e^{-\alpha_i(t)t}$. Moreover, all the nodes reach the value α_m in finite time. Let us denote this instance of time as t_{α_m}. Thus, there exists a constant c_2, $0 < c_2 \le c_1$, such that $c_2 e^{-\alpha_m t_{\alpha_m}} = c_1 e^{-\alpha_{max} t_{\alpha_m}}$. For instance, $c_2 = c_1 e^{-(\alpha_{max}-\alpha_m)t_{\alpha_m}} \le c_1$. Then, it holds that

$$c_2 e^{-\alpha_m t} \le c_1 e^{-\alpha_i(t)t} \le c_1 e^{-\alpha_m t}, \forall i, \forall t. \tag{22}$$

Finally, we show that the inter-event times are lower-bounded by a positive constant T. In the inter-event times, it holds that $\dot{e}_i(t) = -\dot{x}_i(t)$, since the broadcasted state $x_{b,i}$ remains constant. Thus, $\|\dot{e}_i(t)\| \le \|\dot{x}_i(t)\| \le \|\dot{x}(t)\| \le \|A_K + \Delta\|\|x(t)\| + \|BK\|\|e(t)\|$ in the inter-event times.

Assume that the subsystem i triggers at $t^\star \ge 0$. It holds $\|e_i(t)\| \le \int_{t^\star}^{t} (\|A_K + \Delta\|\|x(s)\| + \|BK\|\|e(s)\|)ds$. From (21) and using that $\|x(t)\| \le \|x(t^\star)\|$ for $t \ge t^\star$, it yields $\|x(t)\| \le \kappa(V)\Big(e^{-\lambda_M t^\star}\|x(0)\| + \frac{\|BK\|\sqrt{N}c_1}{\lambda_M(1-\gamma)} e^{-\gamma\lambda_M t^\star}\Big)$. If the following constants are defined: $k_1 = \kappa(V)\|A_K + \Delta\|\|x(0)\|$, $k_2 = \|BK\|\sqrt{N}c_1\big(1 + \frac{\kappa(V)\|A_K+\Delta\|}{\lambda_M(1-\gamma)}\big)$, it follows that $|e_i(t)\| \le \int_{t^\star}^{t}(k_1 e^{-\lambda_M t^\star} + k_2 e^{-\gamma\lambda_M t^\star})ds \le (k_1 e^{-\lambda_M t^\star} + k_2 e^{-\gamma\lambda_M t^\star})(t - t^\star)$. The next event is not triggered before $\|e_i(t)\| = c_2 e^{-\alpha_m t} = c_2 e^{-\gamma\lambda_M t}$, according to (22). If we denote $T = t - t^\star$, the minimum inter-event time is the solution of $(k_1 e^{-\lambda_M t^\star} + k_2 e^{-\gamma\lambda_M t^\star})T = c_2 e^{-\gamma\lambda_M t}$, or equivalently $(\frac{k_1}{c_2} e^{-(1-\gamma)\lambda_M t^\star} + \frac{k_2}{c_2})T = e^{-\gamma\lambda_M T}$, whose right hand side is always positive. This also holds for the left hand side, which is upper bounded by $\frac{k_1+k_2}{c_2}$ and lower bounded by k_2/c_2 for $0 < \gamma < 1$. This yields to a positive value of T for all $t^\star \ge 0$.

5. SIMULATION RESULTS

5.1 System description

The system considered is a collection of $N \times N$ inverted pendulums of mass m and length l coupled by springs with rate k. The topology of the system is depicted in Fig. 1. Each subsystem can be described as

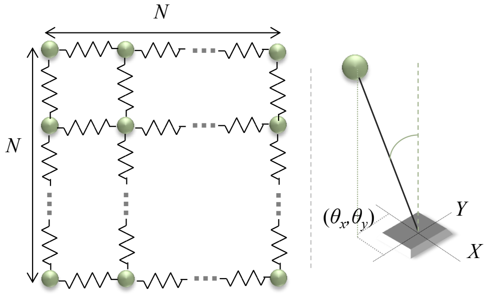

Fig. 1. Scheme of the coupled pendulums mesh.

$$\dot{x}_i = \begin{pmatrix} A_i & \mathbf{0} \\ \mathbf{0} & A_i \end{pmatrix} x_i + \begin{pmatrix} B_i & \mathbf{0} \\ \mathbf{0} & B_i \end{pmatrix} u_i + \sum_{j \in N_i} \begin{pmatrix} H_{ij} & \mathbf{0} \\ \mathbf{0} & H_{ij} \end{pmatrix} x_j,$$

where $A_i = \left(0\ 1;\ \frac{g}{l} - \frac{|N_i|k}{ml^2}\ 0\right)$, $B_i = \left(0;\ \frac{1}{ml^2}\right)$, $H_{ij} = \left(0\ 0;\ \frac{k}{ml^2}\ 0\right)$, $x_i = (x_{i_1}\ x_{i_2}\ x_{i_3}\ x_{i_4})^T$ and $u_i = (u_{i_1}\ u_{i_2})^T$.

The feedback gains K_i are designed to place the poles at $\{-2, -2, -1, -1\}$. The decoupling gains are designed to decouple the system. Let us assume that $\|\Delta_{ij}\| \le \delta_{ij}$, where δ_{ij} are generated randomly in the interval $(0, 0.35|\lambda_{max}(A_K)|/\kappa(V))$ to illustrate the algorithm.

5.2 Distributed parameter estimation

Let us consider a 8×8 pendulum mesh with initial conditions generated in the interval $(-\pi/2, \pi/2)$. With the proposed design, the closed-loop matrices $\{A_{K,i}\}$ are identical for all the subsystems, $A_{K,i} = (0\ 1;\ -2\ -3)$, so that $\lambda_{max}(A_{K,i}) = -1$ and $\kappa(V_i) = 6.1623$. Hence, in this case it follows that $\lambda_{max}(A_K) = \lambda_{max}(A_{K,i})$ and $\kappa(V) = \kappa(V_i)$. The coupling terms Δ_{ij} are generated randomly according to the constraint set before. For the 8×8 mesh and a particular simulation, it results in $\|\Delta\| = 0.1136$.

Since $\lambda_{max}(A_{K,i})$ and $\kappa(V_i)$ are identical for all the nodes, the estimate that affects the proposed adaptive event-triggering mechanism is the upper bound on Δ. The sequence of values of $\{\alpha_i(t)\}$ are depicted in Fig. 2. Note that the final value is the same for all nodes, being $\alpha_m = 0.17$ and $t_{\alpha_m} = 14.4$ s.

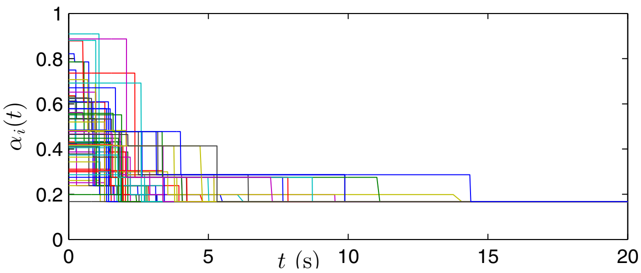

Fig. 2. Sequence of values of $\alpha_i(t)$.

5.3 Performance of the adaptive event-triggered control

Fig. 3 shows the output of the system in a 3D space for the mesh of 8×8 pendulums. The coordinates (θ_x, θ_y) in the XY plane over time are plotted. The parameter c_1 of the trigger functions (19) is 0.8 and the evolution of $\alpha_i(t)$ has been illustrated in the previous section. Note that the system converges asymptotically to the origin.

5.4 Comparative and discussion

Fig. 4 compares the performance for the node (1,1) in three situations:

- Adaptive event-triggering (19) with the same parameters than the previous simulations (solid blue line).
- Event-triggering with constant parameters (8) (dashed red line) with $\alpha = 0.3 < |\lambda_{max}(A_K)| - \kappa(V)\|\Delta\|$.
- Time-driven control (dotted green line) with sampling period $T_s = 0.1$ s.

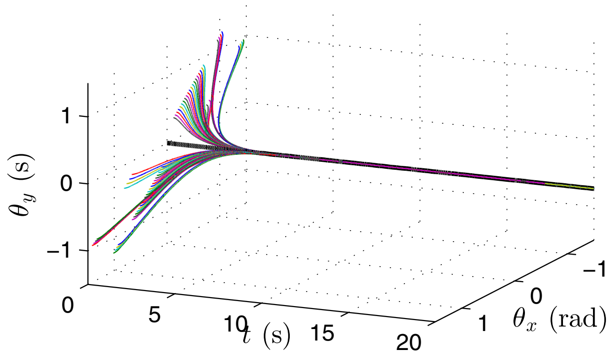

Fig. 3. $x_{i1}(\theta_x)$ and $x_{i3}(\theta_y)$ for a 8×8 mesh of inverted pendulums.

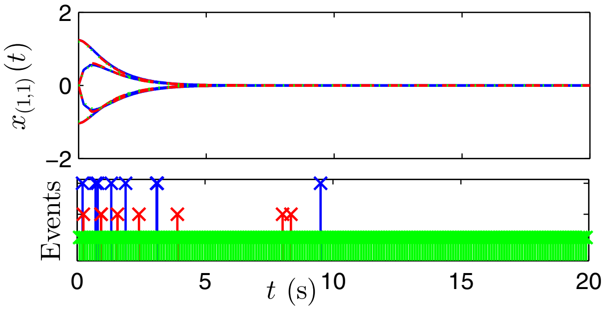

Fig. 4. Adaptive event-triggering, static event-triggering, periodic.

Note that the behavior of the system is very similar in all three situations (see Fig. 4 above) but the need of communication is dramatically reduced with event-based control (see Fig. 4 below). For instance, the number of events generated for the first five seconds of the experiment are 6, 5, and 5 s/$T_s = 50$, respectively.

Let us compare the time of convergence for the parameters estimation. For the proposed approach, this time is $t_{\alpha_m} = 14.4$ s, whereas for the periodic version of the algorithm it would be, at most, $T_s \cdot l_{max} = 0.1 \cdot 14 = 1.4$ s, since the longest path is 14. Hence, periodic and synchronous transmissions would yield a 10 times faster convergence. Nevertheless, it can be argued that the parameter estimation algorithm in this scenario is no more than a means of guaranteeing a *safe* value of α so that the Zeno behavior is excluded, and this is achieved as proven in Theorem 9.

6. CONCLUSION

We have presented a distributed method to compute three parameters for a linear interconnected system: the maximum of the real parts of the eigenvalues of A_K, and the condition number of the eigenvectors matrix $\kappa(V)$, and an upper bound for the norm of the matrix of coupling terms $\|\Delta\|$. Neighboring nodes exchange information and run `max`-consensus for the estimation process following event-based policies. The estimated parameters are used to update the adaptive trigger functions. The convergence of the estimates in finite time, the asymptotic stability of the system and the exclusion of the Zeno behavior have been proved. The results have been illustrated through simulations. Future work will include the study of the impact of model uncertainties and the extension to non-linear systems.

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