Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

A Partial-State Observer for a Class of MIMO Nonlinear Systems ⋆

Lei Wang * **Alberto Isidori** ** **Hongye Su** * **Weihua Xu** *

* *State Key Laboratory of Industrial Control Technology, Institute of Cyber-Systems and Control, Zhejiang University, 310027, Hangzhou, P.R.China (e-mail: lwang,hysu,whxu@iipc.zju.edu.cn).*
** *Department of Computer, Control and Management Engineering, University of Rome "La Sapienza", Rome, Italy (e-mail: albisidori@dis.uniroma1.it)*

Abstract: In this paper, we consider input-affine invertible MIMO nonlinear systems which can be transformed into a special normal form by means of the structure algorithm. The normal form highlights a partial state, a subset of state variables, which plays in this setting a role similar to that of the outputs and its derivatives in a SISO system. It is shown that, if a system in this class can be asymptotically stabilized by means of a static feedback from that partial state, then semiglobal stabilization can be achieved via dynamic feedback driven by the output of the system. The dynamic feedback in question is based a (non–trivial) extension to MIMO systems of the standard high–gain observer.

Keywords: MIMO nonlinear systems, structure algorithm, normal forms, minimum-phase systems, nonlinear observers.

1. INTRODUCTION

A classical problem in nonlinear control theory is the design of feedback laws to the purpose of asymptotically stabilizing a given equilibrium point or to asymptotically track or reject certain signals generated by an independent system. Specific design methods depend, of course, on the kind of information available for feedback. Since full state information is seldom available, design methods based on the availability of partial state information are important.

A relevant class of systems for which systematic design methods are available is that of single–input single–output (SISO) input–affine systems having a well defined relative degree r, in which asymptotic stability can be achieved using *partial state* information, consisting of the output and its first $r-1$ derivatives. Typically, this is the case when a system possesses an asymptotically stable zero dynamics (see e.g. Saberi et al. [1990] and Byrnes and Isidori [1991]). If such partial state is not available, the output can be used – as shown in the pioneering papers Esfandiari and Khalil [1992] and Teel and Praly [1994] – to drive a dynamical system by means of which the partial state in question can be "roughly" approximated by means of a high-gain observer (see, in this respect, also Praly and Jiang [1993], Kaliora et al. [2006], Ahrens and Khalil [2009] and the expository survey Khalil [2008]). This leads to the design of a dynamic control law by means of which asymptotic stability with guaranteed region of attraction can be achieved.

The problem of asymptotic stabilization multi–input multi–output (MIMO) nonlinear systems via output feedback, despite of its obvious relevance, has received little attention in the literature. Of course, there is a class of nonlinear MIMO systems to which the results indicated above can be trivially extended, that is the class of systems having a well-defined vector relative degree $\{r_1, \ldots, r_m\}$, possessing a well-defined global normal form (and globally asymptotically stable zero dynamics), whose so-called "high-frequency gain matrix" can be rendered "positive". But this is indeed a limited class of MIMO systems: even in the case of linear systems, to have a well-defined vector relative degree is a quite restrictive property. A much broader class of MIMO systems is the class of systems that are *invertible* (in the sense of Hirschorn [1979] and Singh [1981]) and in fact, for such systems, it has been shown in Liberzon [2004] that global asymptotic stability can be achieved via *full state* feedback, provided that the system is "minimum-phase" (in the strong sense of Liberzon et al. [2002]). But the problem of achieving stability via *output* feedback is still largely open.

In this paper we consider – as in Liberzon [2004] – invertible nonlinear MIMO systems, and we additionally assume that the so–called *Structure Algorithm* (which, following Hirschorn [1979] and Singh [1981], is used in Liberzon [2004] to characterize the property of invertibility) can be implemented in a simplified manner, by means of state-independent matrix multipliers (see below). The class in question can be considered as an "intermediate" level between the (general) class of invertible systems and the (quite restricted) class of systems possessing a vector relative degree. Systems in this class, if certain vector fields are complete, possess (as shown in Schwartz et al. [1999] and Liu and Lin [2011]) a globally defined *normal form* which can be used as a point of departure for the

⋆ This work is supported by the 111 Project (B07031); National Natural Science Foundation of China (61134007).

design of a *partial state* feedback law yielding, under appropriate "minimum-phase" assumptions, global asymptotic stability or asymptotic stability with guaranteed region of attraction. The partial state in question, though, does not simply consists of the collection of outputs and their derivatives, but rather consists of a set of intermediate variables that are defined at the various stages of the structure algorithm (see again Schwartz et al. [1999]). The purpose of this paper is to show how the components of such partial state can be "roughly" approximated by means of a multivariable (and nontrivial) extension of the high-gain observer of Esfandiari and Khalil [1992], and this provides the basis for the design of a semi-globally stabilizing *dynamic output feedback* law.

This paper is organized as follows. Section 2 presents the basic assumptions and introduces the normal form and the required stabilizability property via partial-state feedback law. In section 3, a special "rough–derivatives" observer of the partial-state is presented, followed by the discussion of semi–global stabilization via dynamic output feedback.

2. SETUP AND ASSUMPTIONS

In this paper, we consider square MIMO nonlinear systems defined by equations of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ y &= h(x). \end{aligned} \tag{1}$$

in which $x \in \mathbf{R}^n$, $u, y \in \mathbf{R}^m$ and the mappings $f(x)$, $g(x)$ and $h(x)$ are smooth mappings, with $f(0) = 0$ and $h(0) = 0$. Throughout the paper we suppose the following assumptions are satisfied.

Assumption 1. There exists a state feedback law $u = \alpha(x) + \beta(x)v$, with invertible $\beta(x)$, such that the resulting system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\alpha(x) + g(x)\beta(x)v, \\ y &= h(x) \end{aligned}$$

has a linear input-output behavior between input v and output y.

Assumption 2. The system is strongly invertible, in the sense of Hirschorn [1979], Singh [1981].

Systems satisfying Assumption 1 have been studied in Isidori [1995], where necessary and sufficient conditions for the existence of such $\alpha(x)$ and $\beta(x)$ are determined. We stress that in the present paper *do not make use* of the feedback law $u = \alpha(x) + \beta(x)v$ indicated in Assumption 1, because this law requires access to the *full* state x and this is not the case in the present setting, where only output feedback is used. We make use of this Assumption only to the purpose of identifying suitable structural properties of the system. In fact, a relevant implication of this assumption is that the so-called *Structure Algorithm* of Silverman Silverman [1969], Hirschorn Hirschorn [1979] and Singh Singh [1981] can be implemented in a simplified manner, because the "row reductions" done at each stage of the algorithm are achieved by means of *x-independent* matrix multipliers.

If Assumptions 1 and 2 hold, it is possible to define – by means of the Structure Algorithm – a local diffeomorphism in the state space by means of which the system is transformed in a system expressed in *normal form* (see Schwartz et al. [1999] and Liu and Lin [2011]). If, in addition, certain vector fields commute (see Schwartz et al. [1999]), the normal form in question – which is globally defined – consists of equations of the form

$$\begin{aligned}
\dot{z} &= f_0(z, (\xi_1^1, \xi_1^2, \cdots, \xi_1^m)) \\
\dot{\xi}_1^1 &= \xi_2^1 \\
&\cdots \\
\dot{\xi}_{n_1-1}^1 &= \xi_{n_1}^1 \\
\dot{\xi}_{n_1}^1 &= a^1(x) + b^1(x)u \\
\dot{\xi}_1^2 &= \xi_2^2 + \delta_{11}^2(a^1(x) + b^1(x)u) \\
&\cdots \\
\dot{\xi}_{n_2-1}^2 &= \xi_{n_2}^2 + \delta_{n_2-1,1}^2(a^1(x) + b^1(x)u) \\
\dot{\xi}_{n_2}^2 &= a^2(x) + b^2(x)u \\
&\cdots \\
\dot{\xi}_1^i &= \xi_2^i + \sum_{j=1}^{i-1} \delta_{1j}^i(a^j(x) + b^j(x)u) \\
&\cdots \\
\dot{\xi}_{n_i-1}^i &= \xi_{n_i}^i + \sum_{j=1}^{i-1} \delta_{n_i-1,j}^i(a^j(x) + b^j(x)u) \\
\dot{\xi}_{n_i}^i &= a^i(x) + b^i(x)u \\
&\cdots
\end{aligned} \tag{2}$$

with

$$y_i = \xi_1^i$$

for $i = 1, \cdots, m$.

In what follows, we assume that the conditions – presented in Schwartz et al. [1999] – for the existence of a globally defined normal form of this kind are fulfilled, and we state it explicitly as follows.

Assumption 3. There exists globally defined diffeomorphism transforming system (1) into a system of the form (2).

The normal form (2) is an obvious point of departure for the design of globally (or semi-globally) stabilizing feedback laws. If, in particular, the upper subsystem of (2), viewed as a system with input $(\xi_1^1, \xi_1^2, \cdots, \xi_1^m)$ and state z, is input-to-state stable, one may guess that global asymptotic stability could be achieved (maybe under appropriate additional assumptions) by means of a feedback law depending only on the *partial-state*

$$\xi = \mathrm{col}(\xi_1^1, \xi_2^1, \ldots, \xi_{n_1}^1, \xi_1^2, \xi_2^2, \ldots, \xi_{n_2}^2, \ldots, \xi_1^m, \xi_2^m, \ldots, \xi_{n_m}^m)\,.$$

In this paper we will not discuss the existence of such partial-state feedback stabilizing law (which will be studied in a paper in preparation Wang et al. [2013]) but rather we assume that such law exists and we concentrate on showing a "separation principle", that is we show how it is possible to design a suitable dynamic feedback – driven by the actual output y of the system – by means of which asymptotic stabilization with guaranteed region of attraction can be obtained.

Assumption 4. There exists a smooth function

$$u = \alpha^*(\xi) \tag{3}$$

such that the equilibrium $(z, \xi) = (0, 0)$ of the closed-loop system (2) – (3) is globally asymptotically and locally exponentially stable.

Examples of systems for which such a feedback law exists can be found in the literature. A trivial example is the case of systems which have well-defined vector relative degree. A substantially more relevant example is the class of system studied in Liberzon [2004]. In fact, as a corollary of Theorem 2 of Liberzon [2004], it is readily seen that a system satisfying Assumptions $1-3$, if $\dim(z) = 0$, can be globally asymptotically stabilized by means of a feedback of the form (3).

In what follows, we assume that a feedback with the properties indicated in Assumption 4 exists and we proceed with the design of a dynamic feedback, driven by the actual output y, that provides suitable asymptotic "proxies" of the ξ^i_j's. The dynamic feedback in question is based on a (non-trivial) extension, to the case of MIMO systems considered in this paper, of the results presented in the seminal works of Esfandiari and Khalil [1992] and Teel and Praly [1994].

3. SEMIGLOBAL STABILIZATION VIA DYNAMIC OUTPUT FEEDBACK

3.1 A recursive observer

In this section we show how the various ξ^i_j's, which are needed to implement the feedback law $u = \alpha^*(\xi)$, can be estimated by means of a multi-output "dirty-derivatives" observer, driven by the output y of the system. To this end, it is convenient to proceed first with defining some "tuning variables".

Set $\mathbf{n_2} = n_1 + n_2$, and define recursively

$$\begin{aligned} \theta_{n_1+1}(x) &= \xi^2_1(x) - \delta^2_{11}\xi^1_{n_1}(x) \\ \theta_{n_1+2}(x) &= \xi^2_2(x) - \delta^2_{21}\xi^1_{n_1}(x) \\ &\cdots \\ \theta_{\mathbf{n_2}-1}(x) &= \xi^2_{n_2-1}(x) - \delta^2_{n_2-1,1}\xi^1_{n_1}(x) \end{aligned} \tag{4}$$

It is easy to check that

$$L_g\theta_{n_1+1}(x) = L_g\theta_{n_1+2}(x) = \cdots = L_g\theta_{\mathbf{n_2}-1}(x) = 0\,.$$

Now set $\mathbf{n_3} = \mathbf{n_2} + n_3$, and define recursively

$$\begin{aligned} \theta_{\mathbf{n_2}+1}(x) &= \xi^3_1(x) - \delta^3_{11}\xi^1_{n_1} - \delta^3_{12}\xi^2_{n_2} \\ \theta_{\mathbf{n_2}+2}(x) &= \xi^3_2(x) - \delta^3_{21}\xi^1_{n_1} - \delta^3_{22}\xi^2_{n_2} \\ &\cdots \\ \theta_{\mathbf{n_3}-1}(x) &= \xi^3_{n_3-1}(x) - \delta^3_{n_3-1,1}\xi^1_{n_1} - \delta^3_{n_3-1,2}\xi^2_{n_2} \end{aligned} \tag{5}$$

It is easy to check that

$$L_g\theta_{\mathbf{n_2}+1}(x) = L_g\theta_{\mathbf{n_2}+2}(x) = \cdots = L_g\theta_{\mathbf{n_3}-1}(x) = 0\,.$$

This procedure can clearly be continued in a similar way. At the end, a set of equations consisting of (4), (5), ... is obtained, which plays a critical role in the following observer design. Note that, $\theta_{\mathbf{n_2}}, \theta_{\mathbf{n_3}}, \cdots$ are not defined, as they will not be needed in the sequel.

We proceed now with the design of a "high-gain" observer. The first n_1 variables $\xi^1_1, \ldots, \xi^1_{n_1}$ can be estimated by a standard system of the form

$$\begin{aligned} \dot{\hat{\xi}}^1_1 &= \hat{\xi}^1_2 + \kappa_0\alpha_1(y_1 - \hat{\xi}^1_1) \\ \dot{\hat{\xi}}^1_2 &= \hat{\xi}^1_3 + \kappa_0^2\alpha_2(y_1 - \hat{\xi}^1_1) \\ &\cdots \\ \dot{\hat{\xi}}^1_{n_1-1} &= \hat{\xi}^1_{n_1} + \kappa_0^{n_1-1}\alpha_{n_1-1}(y_1 - \hat{\xi}^1_1) \\ \dot{\hat{\xi}}^1_{n_1} &= \kappa_0^{n_1}\alpha_{n_1}(y_1 - \hat{\xi}^1_1) \end{aligned} \tag{6}$$

in which the coefficients κ_0 and $\alpha_1, \alpha_2, \ldots, \alpha_{n_1}$ are design parameters. With the associated estimation errors defined as

$$\begin{aligned} e_1 &= \kappa_0^{n_1-1}(y_1 - \hat{\xi}^1_1) \\ e_2 &= \kappa_0^{n_1-2}(\xi^1_2 - \hat{\xi}^1_2) \\ &\cdots \\ e_{n_1} &= (\xi^1_{n_1} - \hat{\xi}^1_{n_1}) \end{aligned} \tag{7}$$

it is readily obtained that

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \cdots \\ \dot{e}_{n_1} \end{pmatrix} = \kappa_0 \begin{pmatrix} -\alpha_1 & 1 & 0 & \cdots & 0 \\ -\alpha_2 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -\alpha_{n_1} & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \cdots \\ e_{n_1} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 1 \end{pmatrix} Z_0$$

in which

$$Z_0(x,u) = a^1(x) + b^1(x)u.$$

Setting $\mathbf{e}_0 = \mathrm{col}(e_1, \ldots, e_{n_1})$, the equation in question can be re-written in compact form as

$$\dot{\mathbf{e}}_0 = \kappa_0 F_0 \mathbf{e}_0 + G_0 Z_0(x,u)$$

in which we assume that F_0 is a Hurwitz matrix.

From this point on, it is convenient to continue by estimating the remaining ξ^j_i's one-by-one. The resulting design might be redundant, but the procedure can be described in a easy recursive fashion. To this end note that, if θ_{n_1+1} were available, ξ^2_2 could be estimated by means of a system of the form

$$\begin{aligned} \dot{\sigma}_{n_1+1} &= \hat{\xi}^2_2 + \kappa_1 c_1(\theta_{n_1+1} - \sigma_{n_1+1}) \\ \dot{\hat{\xi}}^2_2 &= \kappa_1^2 c_2(\theta_{n_1+1} - \sigma_{n_1+1})\,. \end{aligned} \tag{8}$$

In fact, defining estimation errors as

$$\begin{aligned} \varepsilon_{n_1+1} &= \kappa_1(\theta_{n_1+1} - \sigma_{n_1+1}) \\ e_{n_1+1} &= (\xi^2_2 - \hat{\xi}^2_2)\,, \end{aligned} \tag{9}$$

and bearing in mind that

$$\dot{\theta}_{n_1+1} = L_f\theta_{n_1+1}(x) + L_g\theta_{n_1+1}(x)u = L_f\theta_{n_1+1}(x) = \xi^2_2\,,$$

one would obtain

$$\begin{pmatrix} \dot{\varepsilon}_{n_1+1} \\ \dot{e}_{n_1+1} \end{pmatrix} = \kappa_1 \begin{pmatrix} -c_1 & 1 \\ -c_2 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{n_1+1} \\ e_{n_1+1} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} Z_1(x,u)$$

in which

$$Z_1(x,u) = \xi^2_3(x) + \delta^2_{21}(a^1(x) + b^1(x)u)\,.$$

Since θ_{n_1+1} is not available, we drive instead system (8) by

$$\hat{\theta}_{n_1+1} = y_2 - \delta^2_{11}\hat{\xi}^1_{n_1}\,,$$

in which $\hat{\xi}^1_{n_1}$ is the estimate of $\xi^1_{n_1}$, obtained at the previous stage. In summary, the estimation of ξ^2_2 is provided by the system

$$\begin{aligned} \dot{\sigma}_{n_1+1} &= \hat{\xi}^2_2 + \kappa_1 c_1(y_2 - \delta^2_{11}\hat{\xi}^1_{n_1} - \sigma_{n_1+1}) \\ \dot{\hat{\xi}}^2_2 &= \kappa_1^2 c_2(y_2 - \delta^2_{11}\hat{\xi}^1_{n_1} - \sigma_{n_1+1})\,. \end{aligned} \tag{10}$$

with which we associate estimation errors defined as in (9) and the fact

$$\hat{\theta}_{n_1+1} = \theta_{n_1+1} + \delta^2_{11} e_{n_1} \,.$$

. This yields

$$\begin{pmatrix} \dot{\varepsilon}_{n_1+1} \\ \dot{e}_{n_1+1} \end{pmatrix} = \kappa_1 \begin{pmatrix} -c_1 & 1 \\ -c_2 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_{n_1+1} \\ e_{n_1+1} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} Z_1(x,u) \\ -\kappa_1^2 \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \delta^2_{11} e_{n_1} \,.$$

Setting $\mathbf{e}_1 = \mathrm{col}(\varepsilon_{n_1+1}, e_{n_1+1})$, the equation in question can be re-written in compact form as

$$\dot{\mathbf{e}}_1 = \kappa_1 F_1 \mathbf{e}_1 + \kappa_1^2 F_{1,0} \mathbf{e}_0 + G Z_1(x,u)$$

where by construction, we assume that F_1 is a Hurwitz matrix.

To estimate ξ^2_3 we proceed in a similar way. We define

$$\hat{\theta}_{n_1+2} = \hat{\xi}^2_2 - \delta^2_{21} \hat{\xi}^1_{n_1} \,,$$

we observe that

$$\hat{\theta}_{n_1+2} = \theta_{n_1+2} + \delta^2_{21} e_{n_1} - e_{n_1+1} \,,$$

and consider a system

$$\begin{aligned} \dot{\sigma}_{n_1+2} &= \hat{\xi}^2_3 + \kappa_2 c_1 (\hat{\xi}^2_2 - \delta^2_{21} \hat{\xi}^1_{n_1} - \sigma_{n_1+2}) \\ \dot{\hat{\xi}}^2_3 &= \kappa_2^2 c_2 (\hat{\xi}^2_2 - \delta^2_{21} \hat{\xi}^1_{n_1} - \sigma_{n_1+2}) \end{aligned} \tag{11}$$

with which we associate estimation errors defined by

$$\begin{aligned} \varepsilon_{n_1+2} &= \kappa_2 (\theta_{n_1+2} - \sigma_{n_1+2}) \\ e_{n_1+2} &= (\xi^2_3 - \hat{\xi}^2_3) \,. \end{aligned} \tag{12}$$

Setting $\mathbf{e}_2 = \mathrm{col}(\varepsilon_{n_1+2}, e_{n_1+2})$, we obtain an equation of the form

$$\dot{\mathbf{e}}_2 = \kappa_2 F_2 \mathbf{e}_2 + \kappa_2^2 F_{2,0} \mathbf{e}_0 + \kappa_2^2 F_{2,1} \mathbf{e}_1 + G Z_2(x,u)$$

Proceeding in this way, we can estimate all ξ^2_i's. In particular, to estimate $\xi^2_{n_2}$, we use

$$\begin{aligned} \dot{\sigma}_{\mathbf{n_2}-1} &= \hat{\xi}^2_{n_2} + \kappa_{n_2-1} c_1 (\hat{\xi}^2_{n_2-1} - \delta^2_{n_2-1,1} \hat{\xi}^1_{n_1} - \sigma_{\mathbf{n_2}-1}) \\ \dot{\hat{\xi}}^2_{n_2} &= \kappa_{n_2-1}^2 c_2 (\hat{\xi}^2_{n_2-1} - \delta^2_{n_2-1,1} \hat{\xi}^1_{n_1} - \sigma_{\mathbf{n_2}-1}) \,. \end{aligned} \tag{13}$$

With the latter we associate estimation errors defined by

$$\begin{aligned} \varepsilon_{\mathbf{n_2}-1} &= \kappa_{n_2-1} (\theta_{\mathbf{n_2}-1} - \sigma_{\mathbf{n_2}-1}) \\ e_{\mathbf{n_2}-1} &= (\xi^2_{n_2} - \hat{\xi}^2_{n_2}) \,. \end{aligned} \tag{14}$$

for which, setting $\mathbf{e}_{n_2-1} = \mathrm{col}(\varepsilon_{\mathbf{n_2}-1}, e_{\mathbf{n_2}-1})$, we obtain

$$\begin{aligned} \dot{\mathbf{e}}_{n_2-1} = {} & \kappa_{n_2-1} F_{n_2-1} \mathbf{e}_{n_2-1} + \kappa_{n_2-1}^2 F_{n_2-1,0} \mathbf{e}_0 + \cdots + \\ & + \kappa_{n_2-1}^2 F_{n_2-1,n_2-2} \mathbf{e}_{n_2-2} + G Z_{n_2-1}(x,u) \,. \end{aligned}$$

With the same technique we can estimate all ξ^j_i's in (2), until we arrive at the design of a dynamical system that, driven by the measured output y, provides estimates of all the ξ^j_i's. The estimation errors, defined as above, satisfy a set of equations of the form

$$\begin{aligned} \dot{\mathbf{e}}_0 &= \kappa_0 F_0 \mathbf{e}_0 + G_0 Z_0(x,u) \\ & \cdots \\ \dot{\mathbf{e}}_i &= \kappa_i F_i \mathbf{e}_i + \kappa_i^2 F_{i,0} \mathbf{e_0} + \cdots + \kappa_i^2 F_{i,i-1} \mathbf{e}_{i-1} + G Z_i(x,u) \end{aligned}$$

with $i = 1, 2, \ldots, \mathbf{n_m} - n_1 - m + 1$. Setting, for convenience, $\nu = \mathbf{n_m} - n_1 - m + 1$ these equations can be written altogether as

$$\dot{\mathbf{e}} = \mathbf{F}_{\{\kappa_0,\kappa_1,\ldots,\kappa_\nu\}} \mathbf{e} + \mathbf{B} Z(x,u) \tag{15}$$

in which

$$\mathbf{F}_{\{\kappa_0,\kappa_1,\ldots,\kappa_\nu\}} = \begin{pmatrix} \kappa_0 F_0 & 0 & 0 & \cdots & 0 \\ \kappa_1^2 F_{1,0} & \kappa_1 F_1 & 0 & \cdots & 0 \\ \kappa_2^2 F_{2,0} & \kappa_2^2 F_{2,1} & \kappa_2 F_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \\ \kappa_\nu^2 F_{\nu,0} & \kappa_\nu^2 F_{\nu,1} & \kappa_\nu^2 F_{\nu,2} & \cdots & \kappa_\nu F_\nu \end{pmatrix}$$

To the purpose of ensuring convergence when the estimates are used for feedback stabilization, it is of paramount importance to be able to assign an arbitrary high rate of decay to the solutions of the homogeneous equation

$$\dot{\mathbf{e}} = \mathbf{F}_{\{\kappa_0,\kappa_1,\ldots,\kappa_\nu\}} \mathbf{e} \,.$$

To this end, the “gain parameters” $\kappa_0, \kappa_1, \ldots, \kappa_\nu$ must be appropriately scaled. This is actually possible in view of the following result.

Lemma 1. Consider a matrix of the form

$$A(\rho_1, \rho_2) = \begin{pmatrix} A_1(\rho_1) & 0 \\ \rho_2^2 B & \rho_2 A_2 \end{pmatrix}$$

in which the elements of $A_1(\rho_1)$ are polynomial functions of ρ_1. Assume that:

(i) there exist a positive definite matrix P_1 (independent of ρ_1) and a number ρ_1^* such that, if $\rho_1 > \rho_1^*$,

$$P_1 A_1(\rho_1) + A_1^{\mathrm{T}}(\rho_1) P_1 \le -\rho_1 I$$

(ii) the matrix A_2 is Hurwitz.

Then, there is a positive definite matrix P (independent of ρ_2) and a number ρ_2^* such that, if $\rho_2 > \rho_2^*$,

$$P A(\rho_2^3, \rho_2) + A^{\mathrm{T}}(\rho_2^3, \rho_2) P \le -\rho_2 I \,.$$

Proof. Pick a positive definite matrix P_2 satisfying $P_2 A_2 + A_2^{\mathrm{T}} P_2 = -2I$, set $b = |P_2 B|$ and consider the positive definite matrix

$$P = \begin{pmatrix} 2b^2 P_1 & 0 \\ 0 & P_2 \end{pmatrix} \,.$$

Suppose $\rho_1 > \rho_1^*$ and compute the derivative of $V(x) = x^{\mathrm{T}} P x$ along the solutions of $\dot{x} = A(\rho_1, \rho_2) x$, to obtain

$$\dot{V} \le -2b^2 \rho_1 |x_1|^2 - 2\rho_2 |x_2|^2 + 2b\rho_2^2 |x_1||x_2| \,.$$

We seek a choice of ρ_1 that makes

$$\dot{V} \le -\rho_2 (|x_1|^2 + |x_2|^2) \,,$$

which is the case if

$$(2b^2 \rho_1 - \rho_2)|x_1|^2 + \rho_2 |x_2|^2 - 2b\rho_2^2 |x_1||x_2| \ge 0 \,.$$

To this end, pick $\rho_1 = \rho_2^3$ and observe that the desired inequality holds if

$$\begin{pmatrix} 2b^2 \rho_2^2 - 1 & -b\rho_2 \\ -b\rho_2 & 1 \end{pmatrix} > 0 \,,$$

which clearly holds if $b\rho_2 > 1$. Picking

$$\rho_2^* = \max\{1/b, (\rho_1^*)^{1/3}\}$$

completes the proof.

Using this Lemma iteratively, it is easy to arrive at the following conclusion.

Proposition 1. There are positive integers $p_0, p_1, \ldots, p_{\nu^*-1}$, a positive definite matrix $\mathbf{P}$ and a number $\kappa^* > 0$ such that, if

$$\kappa_0 = \kappa^{p_0}, \kappa_1 = \kappa^{p_1}, \ldots, \kappa_{\nu-1} = \kappa^{p_{\nu-1}}, \kappa_\nu = \kappa$$

the inequality

$$\mathbf{P}\mathbf{F}_{\{\kappa_0,\kappa_1,\ldots,\kappa_\nu\}} + \mathbf{F}^{\mathrm{T}}_{\{\kappa_0,\kappa_1,\ldots,\kappa_\nu\}}\mathbf{P} < -\kappa\mathbf{I}$$

holds for all $\kappa > \kappa^*$.

3.2 Convergence analysis

The "dirty-derivatives" observer designed in the previous section is such that the "estimation" error $\mathbf{e}$ satisfies (15) and Proposition 1 holds. Thus, at this point, it should be clear that a "separation principle" for asymptotic stability with guaranteed region of attraction holds. For the sake of completeness and for the readers' convenience, we sketch in what follows the basic arguments needed to arrive at the desired result.

Let $\alpha^*(\xi)$ be a partial-state feedback law with the properties indicated in Assumption 4. Since ξ is not directly and fully available, it is replaced by the corresponding estimate

$$\hat{\xi} = \mathrm{col}\,(y_1, \hat{\xi}_2^1, \cdots, \hat{\xi}_{n_1}^1, y_2, \hat{\xi}_2^2, \cdots, \hat{\xi}_{n_2}^2, \cdots, y_m, \hat{\xi}_2^m, \cdots, \hat{\xi}_{n_m}^m)$$

provided by the observer described in the previous section, which is a linear system, modeled by the equations (6), (10), (11), ... , (13), $\cdots$. Altogether, these can be written in the form

$$\begin{aligned} \dot{\varphi} &= A_\kappa\varphi + B_\kappa y \\ \hat{\xi} &= C\varphi + Dy \end{aligned} \tag{16}$$

with state

$$\varphi = \mathrm{col}(\hat{\xi}_1^1, \hat{\xi}_2^1, \ldots, \hat{\xi}_{n_1}^1, \sigma_1, \hat{\xi}_2^2, \sigma_2, \hat{\xi}_3^2 \ldots, \sigma_\nu, \hat{\xi}_{n_m}^m)\,.$$

According to Proposition 1 the gain parameters $\kappa_0, \ldots, \kappa_\nu$ in (6), (10), (11), ... , (13) have to be powers of a single gain parameter κ and this is reflected in the notation used in (16). Along with this, we consider the equation that relates $\hat{\xi}, \xi$ and estimation error $\mathbf{e}$,

$$\hat{\xi} = \xi - D_\kappa\mathbf{e}$$

in which D_κ is a matrix whose entries are non-positive powers of the parameter κ (see (7), (9), (12), $\cdots$, (14), $\cdots$).

As originally proposed in Esfandiari and Khalil [1992] in a similar setting, the actual control to be implemented on the system is

$$u = \sigma_L(\alpha^*(\hat{\xi})) \tag{17}$$

in which $\sigma_L(r)$ is a saturation function, e.g. any function that coincides with r when $|r| \le L$, is strictly increasing and satisfies $\sigma_L(r) \le 2L$ for all $r \in \mathbf{R}$.

In this way, a closed-loop system is obtained modeled by the equations

$$\begin{aligned} \dot{x} &= \mathbf{f}^*(x) + \Delta_1(x,\mathbf{e}) \\ \dot{\mathbf{e}} &= \mathbf{F}_\kappa\mathbf{e} + \Delta_2(x,\mathbf{e}) \end{aligned} \tag{18}$$

after having set (and invoking, once again, the dependence of the various $\kappa_0, \kappa_1, \ldots, \kappa_{\nu^*}$ on the single parameter κ)

$$\begin{aligned} \mathbf{f}^*(x) &= f(x) + g(x)\alpha^*(\xi) \\ \Delta_1(x,\mathbf{e}) &= g(x)[\sigma_L(\alpha^*(\xi - D_\kappa\mathbf{e})) - \alpha^*(\xi)] \\ \mathbf{F}_\kappa &= \mathbf{F}_{\{\kappa_0,\kappa_1,\ldots,\kappa_\nu\}} \\ \Delta_2(x,\mathbf{e}) &= \mathbf{B}Z(x, \sigma_L(\alpha^*(\xi - D_\kappa\mathbf{e})))\,. \end{aligned}$$

By assumption system (18) has the following properties:

- the equilibrium $x = 0$ of $\dot{x} = \mathbf{f}^*(x)$ is globally asymptotically stable and locally exponentially stable.
- there exist a positive definite matrix $\mathbf{P}$ and a number κ^* such that, if $\kappa > \kappa^*$,

$$\mathbf{P}\mathbf{F}_\kappa + \mathbf{F}_\kappa^{\mathrm{T}}\mathbf{P} < -\kappa\mathbf{I}\,.$$

Using these properties, it is possible to conclude – by means of known methods – the desired semiglobal stabilization result. Details are essentially similar to those used in the context of the problem of semi-global stabilization of a minimum-phase SISO system (see for instance Isidori [1990], Chapter 12) and are not repeated here for the sake of brevity.

Theorem 1. Consider system (1), suppose Assumptions 1 through 4 hold and let the control be provided by (16)–(17). Then, for every choice of a compact set $\mathcal{K}$, there is a choice of the design parameters L, κ such that the equilibrium $(x, \varphi) = (0, 0)$ of the closed-loop is asymptotically stable, with a domain of attraction that contains the set $\mathcal{K}$.

4. CONCLUSION

In this paper, we have considered the class of input–affine MIMO nonlinear systems which are invertible, input–output linearizable and for which a well–defined normal form is obtained by means of the so–called Structure Algorithm. This class includes, as a trivial special case, the class of MIMO systems possessing a well–defined vector relative degree. Then, a "rough–derivatives" observer that extends the principles of Esfandiari and Khalil [1992] and Teel and Praly [1994] in a nontrivial way is designed, so as to arrive at dynamic output feedback that guarantees asymptotic stability with guaranteed region of attraction.

REFERENCES

C. I. Byrnes, A. Isidori. Asymptotic stabilization of minimum phase nonlinear systems. *IEEE Trans. Automatic Control*, vol. 36 , pp. 1122-1137 (1991).

F. Celani and A. Isidori, Output stabilization of strongly minimum–phase systems. *8th IFAC Symposium NOLCOS*, pp. 647-652, Bologna (Italy), 2010.

F. Esfandiari, and H.K. Khalil. Output feedback stabilization of fully linearizable systems. *International Journal of Control*, vol.56, no.5, pp. 1007-1037, 1992.

R. M. Hirschorn. Invertibility of multivariable nonlinear control systems. *IEEE Trans. Automat. Contr.*, vol.24, , pp. 855-865, 1979.

A. K. Imai, R. R. Costa, L. Hsu, G. Tao, P. V. Kokotovic. Multivariable adaptive control using high-frequency gain matrix factorization. *IEEE Trans. Automat. Contr.*, vol.49, no. 7, pp. 1152-1157, 2004.

A. Isidori. *Nonlinear Control Systems.* 3rd ed. New York:Springer,1995.

A. Isidori. *Nonlinear Control Systems II.* New York:Springer,1999.

G. Kaliora, A. Astolfi, L. Praly. Norm estimators and global output feedback stabilization of systems with ISS inverse dynamics. *IEEE Trans. Automat. Contr.*, vol.51, no.3, pp. 493-498, 2006.

D. Liberzon. Output-input stability implies feedback stabilization. *Syst. Control Lett.*, vol.35, no.3-4, pp. 237-248, 2004.

D. Liberzon, S. Morse and D. Sontag. Output–Input Stability and Minimum–phase Nonlinear Systems. *IEEE Trans. Automat. Contr.*, vol.47, no.3, pp. 422-436, 2002.

A. Liu, Z. Lin. On normal forms of nonlinear systems affine in control. *IEEE Trans. Automat. Contr.*, vol.56, no.2, pp. 239-253, 2011.

A. S. Morse. A gain matrix decomposition and some of its applications. *Syst. Control Lett.*, vol.21, pp. 1-10, 1993.

L. Praly, Z. P. Jiang. Stabilization by output feedback for systems with ISS inverse dynamics. *Syst. Control Lett.*, vol.21, pp. 17-22, 1993.

A. Saberi, P. Kokotovic, H. Sussmann. Global stabilization of partially linear composite systems. *SIAM J Control Optimiz*, vol.28, no.6, pp. 1491-1503, 1990.

B. Schwartz, A. Isidori, T.J. Tarn. Global normal forms for MIMO systems, with applications to stabilization and disturbance attenuation. *Math. Control Signals Syst.*, vol.12, pp. 121-142, 1999

S. N. Singh. A modified algorithm for invertibility in nonlinear systems. *IEEE Trans. Automat. Contr.*, vol.26, no.2, pp. 595-598, 1981.

L. M. Silverman. Inversion of multivariable nonlinear systems. *IEEE Trans. Automat. Contr.*, vol.14, no.3, pp. 270-276, 1969.

A. Teel, L. Praly. Tools for semiglobal stabilization by partial state and output feedback. *SIAM Journal on Control and Optimization*, vol.33, no.5, pp. 1443-1488, 1994.

A. H. Ahrens, H. K. Khalil. High-gain observers in the presence of measurement noise: A switched-gain approach. *Automatica*, vol.45, no.4, pp. 936-943, 2009.

H. K. Khalil, High-gain observers in nonlinear feedback control. *Int. Conf. Control, Automation and Systems*,Oct.14-17, Korea, 2008.

L. Wang, A. Isidori, H, Su. Global stabilization of a class of invertible MIMO nonlinear systems. *IEEE Trans. Automat. Contr.*, 2013 (submitted).