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Sensitivity Analysis of Non-linear Performance with Probability Distortion ⋆

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Abstract: Human preference over random outcomes may not be as rational as shown in the expected utility theory. Such an "irrational" (as a matter of fact, closer to reality) behavior can be modeled by distorting the probability of the outcomes. Stochastic control of such a distorted performance is difficult because dynamic programming fails to work due to the time inconsistency. In this paper, we formulate the stochastic control problem with the distorted performance and show that the mono-linearity of the distorted performance, which claims that the derivative of the distorted performance equals the expected value of the sample derivative under a changed probability measure, makes the gradient-based sensitivity analysis suitable for optimization of the distorted performance. We derive the first order optimality conditions (or the differential counterpart of the HJK equation) for the optimal solution. We use the portfolio allocation problem in finance as an example of application.

Keywords: Probability distortion, mono-linearity, sensitivity-based optimization, perturbation analysis, portfolio allocation

1. INTRODUCTION

The goal of a standard control problem in finance is to optimize the expected value of a utility function which represents the investor's different "satisfactions" to different outcomes. The expectation is taken with respect to the natural probability measure $\mathcal{P}$ and is a linear functional in the utility space ($E[f(x)+g(x)] = E[f(x)]+E[g(x)]$, with x representing the outcomes). However, people's "satisfaction" is not always linear (e.g., $E[f(x)+g(x)] \neq E[f(x)]+E[g(x)]$). In other words, the standard expected utility theory cannot model and explain this nonlinearity of people's behavior. Examples include lottery and insurance; in both cases, the expectation of the outcomes (benefit minus cost) are negative and people still buy them; that is, two random outcomes having the same expectation lead to different satisfactions.

This problem has been attracting considerable interests from the research community, especially in the financial sector. A *dual theory* is proposed in Yaari (1987) to deal with this non-linear behavior. The essential point of the dual theory is that instead of distorting the outcomes by a utility function, it distorts the probability of the outcomes. Intuitively, the dual theory captures people's behavior that they usually enlarge the effects of rare events and diminish those of common events. Some excellent progress has been made in portfolio optimization based on this performance model with distorted probabilities, or the "distorted performance" for short Tversky and Kahneman (1992); Karatzas et al. (1991); He and Zhou (2011); Jin and Zhou (2008); Cao and Wan (2013). These works, however, depend on the special structure of a financial market in which the stock prices are not controllable. The formulation of the optimization of the distorted performance and the approach to solve such an optimization problem in a general setting of stochastic control need to be developed.

The main difficulty is that dynamic programming fails in optimization of distorted performance. Because if a policy is optimal for a distorted performance when the process starts from time t, this policy is no longer optimal from time t onward if the process started from time $t' < t$. This property is called time-inconsistent; we need to look for other approaches that do not require the time-consistent property.

In the past decades, the author and his colleagues have been working on the sensitivity-based optimization approach Cao (2007), which is an alternative to dynamic programming. Unlike dynamic programming, which requires time consistency, the sensitivity-based approach is simply based on a direct comparison of the performance of any two policies and therefore is not subject to time consistency. When the two policies under comparison are infinitesimally close to each other, the approach leads to performance derivatives with respect to continuous parameters of policies. The performance and its derivatives can usually be measured on a sample path and therefore it is a sample-path based approach in the same spirit of perturbation analysis (PA) Ho and Cao (1991); Cao (2007). It has been shown that many results by dynamic

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programming can be obtained with the sensitivity based approach Cao (2007).

On one side, the stochastic control with distorted performance, which represents people's preference pattern over outcomes, cannot be solved by dynamic programming; on the other hand, we have the sensitivity-based optimization approach, which does not require the time-consistent property. It comes naturally to try apply this approach to the case with distorted performance. We have successfully applied this approach to the portfolio management problem in Cao and Wan (2013), the results there rely heavily on the simple structure of the market dynamics: the price process of the stocks are not controllable. The goal of this paper is to formulate and study the optimization with distorted performance in a general setting of stochastic control, with the sensitivity-based approach.

In Section 2, we define the performance with probability distortion and formulate the stochastic control problem with distorted performance. In Section 3, we review some fundamental properties of the distorted performance Cao and Wan (2013). It has been shown that the distorted performance enjoys some sort of linearity, called "mono-linearity", which forms the basis of the sensitivity-based approach applied to the problem. In Section 4, we present the general optimality conditions in a differential form (i.e., first order optimality condition); Applying the perturbation analysis principle, and with the mono-linearity, we prove that derivative of the performance potential of an optimal polity forms a martingale with respect to the changed measure. In Sections 5, we discuss dynamic systems described by the standard diffusion processes. We derive the equations for determining the Radon-Nikodym derivatives of the changed measure under any policy w.r.t $\mathcal{P}$, from which we obtain the first order optimality condition in terms of the sample-derivatives of potentials and the system parameters.

In this paper, we show that the mono-linearity of the distorted performance makes the gradient-based sensitivity analysis suitable for optimization of the distorted performance. We prove that at an optimal policy, the sample derivative of the distorted performance is a martingale under the changed measure. First order optimality conditions (or the differential counterpart of the HJK equation) are then obtained for systems with diffusion state processes in terms of system parameters. These results cannot be obtained by dynamic programming.

2. THE BASIC FORMULATION

2.1 The performance with distorted probability

Consider a non-negative, a.s. finite random variable X_θ defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, with θ denoting a parameter. With the expected utility theory, the objective is to maximize the expected utility $U(X_\theta) : \mathcal{R}_+ \to \mathcal{R}_+$:

$$E_{\mathcal{P}}[U(X_\theta)] = \int_{\mathcal{R}_+} \mathcal{P}[U(X_\theta) > x]\}dx, \qquad (1)$$

where $E_{\mathcal{P}}$ denotes the expectation under probability measure $\mathcal{P}$.

However, people's preference cannot always be measured by the expected utility, e.g., they strongly dislike disasters even if they happen rarely. This type of non-linear behavior has been widely studied in the finance and economics community. Widely used is the dual theory proposed in Yaari (1987), which models the fact that people usually subconsciously enlarge the chance of rare events (wining a lottery or encountering a disaster) while diminishing the effect of common evens. Therefore, in the dual theory, we wish to maximize the following performance with a distorted probability (or called the *"distorted performance"*):

$$\eta_{X_\theta} := \widetilde{E}_{\mathcal{P}}[X_\theta] := \int_{\mathcal{R}_+} w\{\mathcal{P}[X_\theta > x]\}dx, \qquad (2)$$

where w is a nonlinear *distortion function*, which is assumed to be strictly increasing and analytical, with $w(0) = 0$ and $w(1) = 1$. Compared with (2), the utility function U in (1) "distorts" the outcome values; this explains the meaning of "dual". In finance, a convex distortion function is used by risk aversion investors, and a concave distortion function for risk taking ones.

2.2 The optimization problem

Consider a stochastic system whose state $S(t)$ follows the standard diffusion process

$$dS(t) = \alpha(t, S(t))dt + \sigma(t, S(t))dW(t), \qquad 0 \le t \le T, \quad (3)$$

with $S(0) = S_0 \in \mathcal{R}^N$, where $S(t) \in \mathcal{R}^N$, and $W(t) \in \mathcal{R}^M$ represents a Brownian motion defined on $(\Omega, \mathcal{F}, \mathcal{P})$, $\alpha(t, S) \in \mathcal{R}^N$, $\sigma(t, S) \in \mathcal{R}^{N\times M}$, and both $\alpha(t, S)$ and $\sigma(t, S)$ are smooth functions on $[0, t] \times \mathcal{R}^N$. Let $\mathcal{F}_t, 0 \le t \le T$, be the filtration generated by $W(t), 0 \le t \le T$; we may take $\omega \in \Omega$ as a sample path of the system states generated by a realization of W, denoted as $\omega := \{W(t), 0 \le t \le T\}$. Sometimes we add ω to indicate the dependence of a random variable on the sample path, e.g., $S(t, \omega)$.

Let $f(t, S)$, $t \in [0, T]$, $S \in \mathcal{R}^N$, denote the time-dependent reward function, and $F(T, S(T))$ denote the terminating reward. The performance concerned is

$$X_\theta(T) = \int_0^T f(r, S(r))dr + F(T, S(T)), \qquad S(0) = S_0, \qquad (4)$$

We wish to maximize the distorted performance

$$\widetilde{E}_{\mathcal{P}}[X_\theta] = \int_{\mathcal{R}_+} w(H_\theta(x))dx, \qquad (5)$$

with $H_\theta(x) = \mathcal{P}[X_\theta(T) > x]$, subject to $\alpha(t, S) \in \mathfrak{A}$, and $\sigma(t, S) \in \mathfrak{B}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are given sets of n-dimensional and $n \times m$ dimensional smooth functions, respectively; and we use θ to denote any parameter of $\alpha(t, S)$ and $\sigma(t, S)$. In a stock market, we may have $\mathfrak{A} = \{all\ \alpha(t, S) :\ \alpha_i(t, S_i) = \mu_i S_i\}$ and $\mathfrak{B} = \{all\ \sigma(t, S) :\ \sigma_{i,j}(t, S) = \delta_{i,j}\sigma_i S_i\}$, with $n = m$ and $\delta_{i,j} = 0$, if $i \ne j$, and $\delta_{i,j} = 1$, if $i = j$; μ_i and σ_i, $i = 1, 2, \cdots, N$, are parameters.

3. FUNDAMENTALS

We first review some fundamental results in Cao and Wan (2013) that motivate the study in this paper.

3.1 A weighted expectation form

Let $H_\theta(x) = \mathcal{P}[X_\theta > x]$ be the decumulative distribution function of a random variable X_θ defined on $(\Omega, \mathcal{F}, \mathcal{P})$, and $G_\theta(\cdot) := H_\theta^{-1}(\cdot)$ be its left-continuous inverse. The distorted performance (2) can be reformulated as

$$\begin{aligned}\widetilde{E}_{\mathcal{P}}[X_\theta] &= \int_{\mathcal{R}_+} w(H_\theta(x))dx \\ &= \int_0^1 G_\theta(w^{-1}(y))dy = \int_0^1 G_\theta(z)\frac{dw(z)}{dz}dz \\ &= E_{\mathcal{P}}\Big[\Big\{G_\theta(z)\frac{dw(z)}{dz}\Big\}\Big|_{z=Z}\Big] \end{aligned} \quad (6)$$

where Z in (6) can be any uniform random variable on $[0,1]$ defined on $(\Omega, \mathcal{F}, \mathcal{P})$.

It is advantageous to take a special form of uniform random variable $Z = H_\theta(X_\theta)$ in (6). Throughout the paper we assume that X_θ has no atom (the case with atoms is technically more involved and will be the topic for further research). In that case, $G_\theta[H_\theta(X_\theta)] = X_\theta$, and (6) and (5) become

$$\widetilde{E}_{\mathcal{P}}[X_\theta] = E_{\mathcal{P}}\Big[X_\theta \frac{dw(z)}{dz}\Big|_{z=H_\theta(X_\theta)}\Big]. \quad (7)$$

3.2 Change of measure

We may use the random variable $\Pi = \frac{dw(z)}{dz}\Big|_{z=H_\theta(X_\theta)}$ in (7) as a Radon-Nikodym derivative to define another measure $\mathcal{Q}_\theta$ on Ω:

$$\frac{d\mathcal{Q}_\theta}{d\mathcal{P}} = \frac{dw(z)}{dz}\Big|_{z=H_\theta(X_\theta)}. \quad (8)$$

Indeed, w is strictly increasing and we have

$$\begin{aligned}E_{\mathcal{P}}[\Pi] &= E_{\mathcal{P}}\Big[\frac{dw(z)}{dz}\Big|_{z=H_\theta(X_\theta)}\Big] \\ &= \int_0^1 \frac{dw(z)}{dz}dz = w(1) - w(0) = 1.\end{aligned}$$

Thus, (7) becomes

$$\widetilde{E}_{\mathcal{P}}[X_\theta] = E_{\mathcal{P}}\Big[X_\theta \frac{d\mathcal{Q}_\theta}{d\mathcal{P}}\Big] = E_{\mathcal{Q}_\theta}[X_\theta]. \quad (9)$$

To define a sample derivative, we use a function-like notation for a random variable: $X_\theta = X_\theta(\omega) := h(\theta, \omega)$. A sample derivative $\frac{\partial}{\partial\theta}\{X_\theta\}$ is defined as a derivative with $\omega = \{W(t), 0 \le t \le T$ fixed for both θ and $\theta + \Delta\theta$:

$$\begin{aligned}\frac{\partial}{\partial\theta}\{X_\theta\} &= \frac{\partial}{\partial\theta}\{X_\theta(\omega)\} = \frac{\partial}{\partial\theta}\{h(\theta,\omega)\} \\ &= \lim_{\Delta\theta\to 0} \frac{X_{\theta+\Delta\theta}(\omega) - X_\theta(\omega)}{\Delta\theta}.\end{aligned}$$

Assumption 1. For any θ, $\frac{\partial}{\partial\theta}\{X_\theta\} = \frac{\partial}{\partial\theta}h(\theta,\omega)$ exist a.s.; $H_\theta(x)$ is continuous differentiallable w.r.t. θ and x; $E_{\mathcal{P}}[\frac{\partial}{\partial\theta}\{X_\theta\}|X_\theta = x]$ is continuous in x; there exists a random variable K_θ with finite expectation, such that,

$$|X_{\theta+\Delta\theta} - X_\theta|\frac{dw(z)}{dz}\Big|_{z=H_\theta(X_\theta)} \le K_\theta|\Delta\theta|, a.s. \quad (10)$$

for $|\Delta\theta|$ small enough.

If $\frac{dw(z)}{dz}$ is bounded in $[0,1]$, then (10) requires X_θ uniformly differentiable at θ. This assumption to needed to ensure the interchangeability of expectation and derivative, a well-known condition in perturbation analysis, and weaker conditions exist Ho and Cao (1991).

Theorem 1. Under Assumption 1, we have

$$\begin{aligned}\frac{d}{d\theta}\widetilde{E}[X_\theta] &= \frac{d}{d\theta}[E_{\mathcal{Q}_\theta}(X_\theta)] \\ &= E_{\mathcal{P}}\Big[\frac{\partial}{\partial\theta}\{X_\theta\}\frac{dw(z)}{dz}\Big|_{z=H_\theta(X_\theta)}\Big] \\ &= E_{\mathcal{Q}_\theta}\Big[\frac{\partial}{\partial\theta}\{X_\theta\}\Big].\end{aligned} \quad (11)$$

This important property is called the *mono-linearity* in Cao and Wan (2013). It shows that when we take derivatives, we may only change the measure at one end of the derivative direction; this property makes it possible to use the sample derivative and is the foundation for our analysis in optimization of distorted performance.

4. OPTIMIZATION OF DISTORTED PERFORMANCE: AN OVERVIEW

4.1 Stochastic control revisited

When $w \equiv 1$, the problem becomes a standard control problem. Let us briefly revisit the stochastic control problem with a sensitivity-based view and then derive the first order optimality condition. On any sample path denoted by ω, we define the sample performance

$$\eta_\theta(\omega) = \int_0^T f_\theta(r, S_\theta(r), \omega)dr + F(T, S_\theta(T), \omega), \quad (12)$$

and the sample potential function

$$g_\theta(t,\omega) = \int_t^T f_\theta(r, S_\theta(r), \omega)dr + F(T, S_\theta(T), \omega). \quad (13)$$

Then $\eta_\theta(\omega) = g_\theta(0,\omega)$ and the system performance is

$$\eta_\theta(S) = E_{\mathcal{P}}\{\eta_\theta(\omega)|S_\theta(0) = S\},$$

and the performance potential function is

$$g_\theta(t,S) = E_{\mathcal{P}}\{g_\theta(t,\omega)|S_\theta(t) = S\}, \qquad 0 \le t \le T,\ \forall S. \quad (14)$$

We can easily prove that $g_\theta(t,S)$ satisfies the Poisson equation

$$\mathbb{A}_\theta g_\theta(t,S) + f_\theta(t,S) = 0, \qquad 0 \le t \le T,\ \forall S, \quad (15)$$

with $g_\theta(T,S) = F(T,S)$, and the infinitesimal generator $\mathbb{A}$ is defined with any smooth function $h(t,S)$ (with subscript θ omitted):

$$\begin{aligned}\mathbb{A}h(t,s) &= \frac{d}{d\tau}E\Big\{h(\tau, S(\tau))|S(t) = S\Big\}_{\tau=t} \\ &= E\Big\{\frac{d}{dt}h(t,S(t))|S(t) = S\Big\}.\end{aligned} \quad (16)$$

We have the performance derivative formula (which can be derived by Dynkin's formula, see Cao et.al (2011))

$$\begin{aligned}\dot{\eta}_\theta(S) &= \dot{g}_\theta(0,S) = E_{\mathcal{P}}\{\dot{g}_\theta(0,\omega)|S_\theta(0) = S\} \\ &= E_{\mathcal{P}}\Big\{\int_0^T [\dot{\mathbb{A}}_\theta g_\theta + \dot{f}_\theta](t, S_\theta(t))dt\Big|S_\theta(0) = S\Big\}.\end{aligned} \quad (17)$$

where $\dot{\eta}_\theta$, etc, denote the partial derivative with respect to θ. By setting $\dot{\eta}_\theta(S) = 0$, we obtain the first order optimality condition as Cao et.al (2011)

$$[\dot{\mathbb{A}}_\theta g_\theta + \dot{f}_\theta](t, S) = 0, \quad 0 \le t \le T, \ \forall\ S. \tag{18}$$

This is the differential version of the HJB equation, and can also be obtained by taking derivative of the standard HJB equation.

On the other hand, taking derivatives with respect to θ on both sides of (15) yields

$$[\dot{\mathbb{A}}_\theta g_\theta + \mathbb{A}_\theta \dot{g}_\theta + \dot{f}_\theta](t, S) = 0, \quad 0 \le t \le T, \ \forall\ S.$$

Thus, the performance derivative formula (17) becomes

$$\dot{\eta}_\theta(S) = -E_{\mathcal{P}}\Big\{ \int_0^T [\mathbb{A}_\theta \dot{g}_\theta(t, S_\theta(t))]dt \Big| S_\theta(0) = S \Big\}, \tag{19}$$

and the first order optimality condition is

$$\mathbb{A}_\theta[\dot{g}_\theta(t, S)] = 0, \qquad 0 \le t \le T, \ \forall\ S. \tag{20}$$

In addition, we need a sample path based condition. From (20), we have

$$\begin{aligned} \mathbb{A}_\theta \dot{g}_\theta(t, S) &= \lim_{\tau \to 0} \frac{1}{\tau} E_{\mathcal{P}}\Big\{ \Big[\dot{g}_\theta(t+\tau, S_\theta(t+\tau)) \\ &\qquad - \dot{g}_\theta(t, S_\theta(t)) \Big] \Big| S_\theta(t) = S \Big\} \\ &= E_{\mathcal{P}}\Big\{ \frac{d}{dt}[\dot{g}_\theta(t, \omega)] \Big| S_\theta(t) = S \Big\}, \qquad 0 \le t \le T, \ \forall\ S. \end{aligned}$$

Thus, the first order optimality condition takes the form

$$\begin{aligned} & E_{\mathcal{P}}\Big\{ d[\dot{g}_\theta(t, \omega)] \Big| S_\theta(t) = S \Big\} \\ &= d\Big\{ E_{\mathcal{P}}\Big[\dot{g}_\theta(\tau, \omega) \Big| S_\theta(t) = S \Big] \Big\}_{\tau = t} = 0, \ 0 \le t \le T, \forall\ S \end{aligned} \tag{21}$$

4.2 The first order optimality condition for distorted performance

Now, we turn to the optimization of distorted performance. Define

$$\xi_T(\omega) = \frac{dw(z)}{dz}\Big|_{z = H_\theta(\eta_\theta(\omega))}, \tag{22}$$

and

$$\begin{aligned} \xi_t &= E_{\mathcal{P}}[\xi_T | \mathcal{F}_t], \qquad 0 \le t \le T, \\ & \textit{with } \xi_0 = E_{\mathcal{P}}[\xi_T | \mathcal{F}_0] = 1. \end{aligned} \tag{23}$$

By definition, ξ_t is a martingale under $\mathcal{P}$. We wish to optimize the distorted performance defined as (cf. (7))

$$\begin{aligned} \eta_\theta(S) &= E_{\mathcal{P}}[g_\theta(0, \omega)\xi_T(\omega) | S_\theta(0) = S] \\ &= E_{\mathcal{Q}_\theta}[\eta_\theta(\omega) | S_\theta(0) = S], \end{aligned} \tag{24}$$

with $\mathcal{Q}$ be a measure defined by the Radon-Nikodym derivative

$$\frac{d\mathcal{Q}_\theta}{d\mathcal{P}}(\omega) = \frac{dw(z)}{dz}\Big|_{z = H_\theta(\eta_\theta(\omega))} = \xi_T(\omega). \tag{25}$$

By mono-linearity (11) in Theorem 1, we have

$$\dot{\eta}_\theta(S) = E_{\mathcal{Q}_\theta}[\dot{\eta}_\theta(\omega) | S_\theta(0) = S].$$

The derivative of the distorted performance (19) is now

$$\dot{\eta}_\theta(S) = -E_{\mathcal{Q}_\theta}\Big\{ \int_0^T [\mathbb{A}_\theta^{\mathcal{Q}_\theta} \dot{g}_\theta^{\mathcal{Q}_\theta}(t, S_\theta(t))]dt \Big| S_\theta(0) = S \Big\}, \tag{26}$$

in which $\mathbb{A}_\theta^{\mathcal{Q}_\theta}$ denotes the infinitesimal generator under measure $\mathcal{Q}_\theta$, and $g_\theta^{\mathcal{Q}_\theta}(t, S_\theta(t))$ is defined in (14) under measure $\mathcal{Q}_\theta$.

Therefore, we obtain the first order optimality condition for the distorted performance: If θ is an optimal policy and $\mathcal{Q}_\theta$ is corresponding distortion measure, then

$$\mathbb{A}_\theta^{\mathcal{Q}_\theta} \dot{g}_\theta^{\mathcal{Q}_\theta}(t, S) = 0, \qquad 0 \le t \le T, \ \forall\ S. \tag{27}$$

Before turning to the sample path based condition, we need to reform $\xi_T(\omega)$. From (12), we have

$$\begin{aligned} \eta_\theta(\omega) &= \int_0^t f(r, S_\theta(r), \omega)dr \\ &\quad + \Big[\int_t^T f(r, S_\theta(r), \omega)dr + F(T, S_\theta(T), \omega) \Big] \\ &= R_\theta(t, \omega) + g_\theta(t, S_\theta(t), \omega), \end{aligned}$$

with $R_\theta(t, \omega)$ and $g_\theta(t, S_\theta(t), \omega)$ denoting the two terms in the sum. $R_\theta(t, \omega)$ satisfies

$$dR(t, \omega) = f(t, S(t), \omega)dt. \tag{28}$$

Set $\dot{w}(z) = \frac{dw(z)}{dz}$, then

$$\begin{aligned} \xi_T &= \dot{w}\{H_\theta[\eta_\theta(\omega)]\} \\ &= \dot{w}\{H_\theta[R_\theta(t, \omega) + g_\theta(t, S_\theta(t), \omega)]\}. \end{aligned}$$

By the Markov property of $S(t)$, given $R_\theta(t, \omega) = R(t)$ and $S_\theta(t, \omega) = S(t)$,

$$\begin{aligned} \xi_t &= E_{\mathcal{P}}[\xi_T | \mathcal{F}_t] \\ &= E_{\mathcal{P}}\Big\{ \dot{w}\{H_\theta[R_\theta(t, \omega) + g_\theta(t, S_\theta(t), \omega)]\} \Big| \mathcal{F}_t \Big\} \end{aligned}$$

is a function of $R(t)$ and $S(t)$. We denote it as

$$\xi_t(\omega) = \xi_t(S(t, \omega), R(t, \omega)). \tag{29}$$

Now, by (21), the sample path based condition is

$$d\Big\{ E_{\mathcal{Q}_\theta}\Big[\dot{g}_\theta(\tau, \omega) \Big| S_\theta(t) = S \Big] \Big\}_{\tau = t} = 0, \qquad 0 \le t \le T, \ \forall\ S. \tag{30}$$

For $\tau \ge t$, we have

$$\begin{aligned} & E_{\mathcal{Q}_\theta}\Big[\dot{g}_\theta(\tau, \omega) \Big| S_\theta(t) = S, R_\theta(t) = R \Big] \\ &= E_{\mathcal{P}}\Big[\dot{g}_\theta(\tau, \omega)\xi_T(\omega) \Big| S_\theta(t) = S, R_\theta(t) = R \Big] \\ &= E_{\mathcal{P}}\Big[\dot{g}_\theta(\tau, \omega)\xi_\tau(S(\tau, \omega), R(\tau, \omega)) \Big| S_\theta(t) = S, R_\theta(t) = R \Big]. \end{aligned}$$

Thus, (30) becomes

$$\begin{aligned} & d\Big(E_{\mathcal{P}}\Big\{ E_{\mathcal{P}}[\dot{g}_\theta(\tau, \omega)\xi_\tau(S(\tau, \omega), R(\tau, \omega)) \\ & \quad | S_\theta(t) = S, R_\theta(t) = R] \Big| S_\theta(t) = S \Big\} \Big)_{\tau = t} = 0, \end{aligned}$$

and we have the optimality equation

$$E_{\mathcal{P}}\Big[d[\dot{g}_\theta(t, \omega)\xi_t(S_\theta(t), R_\theta(t))] \Big| S_\theta(t) = S, R_\theta(t) = R \Big] = 0, \tag{31}$$

or

$$\begin{aligned} & E_{\mathcal{P}}\Big\{ d\Big[\dot{g}_\theta(t, \omega)\xi_t(S_\theta(t), R_\theta(t)) \Big] \Big| S_\theta(0) = S \Big\} = 0, \\ & \qquad 0 \le t \le T, \end{aligned} \tag{32}$$

This equation implies the following theorem:

Theorem 2. For an optimal policy θ, the process $\dot{g}_\theta(t,\omega)\xi_t(S_\theta(t,\omega), R_\theta(t,\omega))$ is a martingale with respect to $\mathcal{F}_t$ and probability measure $\mathcal{P}$.

In particular, when $w \equiv 1$, it is the standard stochastic control problem and for an optimal policy θ, $\dot{g}_\theta(t,\omega)$ is a martingale with respect to $\mathcal{P}$.

Globally, we have

$$\int_0^T E_{\mathcal{P}}\Big\{d\Big[\dot{g}_\theta(t,\omega)\xi_t(S_\theta(t),R_\theta(t))\Big]\Big|S_\theta(0)=S\Big\}=0,$$
$$0 \le t \le T. \qquad (33)$$

In addition, the derivative of the distorted performance (26) becomes

$$\dot{\eta}_\theta(S) = -\int_0^T E_{\mathcal{P}}\Big\{d[\dot{g}_\theta(t,\omega)\xi_t(S_\theta(t),R_\theta(t))]\Big|S_\theta(0)=S\Big\}. \qquad (34)$$

Finally, if $f(t,S) \equiv 0$, then we have $R(t,\omega)=0$ and (29) becomes

$$\xi_t(\omega) = v(t, S(t,\omega)). \qquad (35)$$

In the next two sections, we will derive optimality conditions expressed explicitly in system parameters. The potential-based optimality condition is based on (27); and the sample derivative based conditions are based on (31) or (32).

5. POTENTIAL-BASED OPTIMIZATION CONDITION

5.1 Determining the martingale ξ_t

By the martingale representation theorem (see Karatzas and Shreve (1991), Problem 3.4.16), there exists an $\mathcal{F}_t$-adapted and $\mathcal{R}^n$-valued process $\zeta(t) = (\zeta_1(t),\dots,\zeta_M(t))$ with

$$\sum_{m=1}^M \int_0^T \xi_t^2\zeta_m^2(t)dt < \infty,$$

such that

$$\xi_t = E_{\mathcal{P}}[\xi_T] + \sum_{m=1}^M \int_0^t \xi_r\zeta_m(r)dW_m(r), \quad 0\le t\le T \quad (36)$$

holds a.s. $\mathcal{P}$. Therefore,

$$d\xi_t = \xi_t\zeta(t)dW(t), \qquad (37)$$

and from which, we get

$$\xi_t(\omega) = \exp\Big\{-\frac{1}{2}\int_0^t \zeta^2(s)ds + \int_0^t \zeta(s)dW(s)\Big\}. \qquad (38)$$

We first assume $f(t,s) \equiv 0$ and use the form in (35) $\xi_t(\omega) = v(t,S(t,\omega))$. Since ξ_t is a martingale, we have $\mathbb{A}_t\xi_t = 0$. Thus, the function $v(t,S)$ satisfies the following Poisson equation

$$\frac{\partial}{\partial t}v(t,S) + \sum_{i=1}^N \alpha_i(t,S(t))\frac{\partial}{\partial S_i}v(t,S)$$
$$+\frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N(\sigma\sigma^T)_{ij}\frac{\partial^2}{\partial S_i\partial S_j}v(t,S)=0, \qquad (39)$$

with the boundary condition

$$v(0,S) = E_{\mathcal{P}}[\xi_T|\mathcal{F}_0] = 1, \qquad (40)$$

and

$$v(T,S) = \frac{dw(z)}{dz}\Big|_{z=H_\theta(\eta_\theta(\omega))}, \qquad (41)$$

where $\eta_\theta(\omega) = F(T,S)$.

On the other hand, by the Ito rule, we have

$$\begin{aligned} d\xi_t &= d[v(t,S(t))] \\ &= \frac{\partial}{\partial t}v(t,S)dt + \sum_{i=1}^N \frac{\partial}{\partial S_i}v(t,S)[\alpha_i(t,S(t))dt \\ &\quad + \sum_{j=1}^M \sigma_{ij}(t,S(t))dW_j(t)] \\ &\quad + \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N(\sigma\sigma^T)_{ij}\frac{\partial^2}{\partial S_i\partial S_j}v(t,S)dt. \end{aligned}$$

Comparing this equation with (39), we have

$$\begin{aligned} d\xi_t &= \sum_{i=1}^N \frac{\partial}{\partial S_i}v(t,S)\Big[\sum_{j=1}^M \sigma_{ij}(t,S(t))dW_j(t)\Big] \\ &= \sum_{j=1}^M\Big[\sum_{i=1}^N \sigma_{ij}(t,S(t))\frac{\partial}{\partial S_i}v(t,S)\Big]dW_j(t). \end{aligned}$$

Comparing this with (37), we have

$$\sum_{i=1}^N \sigma_{ij}(t,S(t))\Big[\frac{\partial}{\partial S_i}v(t,S)\Big] = v(t,S(t))\zeta_j(t),$$
$$j = 1,2\cdots,M. \qquad (42)$$

Therefore, $\zeta_j(t)$, $j=1,2,\cdots,M$, can be obtained from (41), with $v(t,S)$ being the solution to (39) and (40).

When $f \neq 0$, we take the $R(t)$ in (28) as a system state and add (28) to the system equations:

$$\begin{cases} dR(t) = f(t,S(t))dt, \\ dS(t) = \alpha(t,S(t))dt + \sigma(t,S(t))dW(t). \end{cases} \qquad (43)$$

Set $S_0(t) = R(t)$ and $\alpha_0(t,S) = f(t,S)$, $\sigma_{0,i}(t,S) = \sigma_{j,0}(t,S) = 0$. This equation looks the same as a regular system with one more dimension. With this notation, we may denote

$$\xi_t(\omega) = v(t,S(t)),$$

which has the same form as (35), with $S(t) = (S_0(t), S_1(t), \cdots, S_N(t))^T$.

With this modification, all the results (36-41) carries over except that in the boundary condition (40), we have $\eta_\theta(\omega) = R + F(T,S)$, $R = S_0(T)$.

Furthermore, with this modification all the other results for $f \equiv 0$ apply to the case $f \neq 0$. Therefore, in what follow we only work on the form (35).

5.2 The optimality condition for (24)

Now, we calculate the sample derivative

$$dE\{\dot{g}_\theta[t, S_\theta(t)]\xi_t[S_\theta(t)]|S_\theta(t) = S\}$$
$$= E\big\{d\{\dot{g}_\theta[t, S_\theta(t)]\xi_t[S_\theta(t)]\}|S_\theta(t) = S\big\}.$$

in the optimality condition (**??**). First, we have

$$\begin{aligned} &d\{\dot{g}_\theta[t, S_\theta(t)]\xi_t[S_\theta(t)]\} \\ &= d\{\dot{g}_\theta[t, S_\theta(t)]\}\xi_t[S_\theta(t)] + \dot{g}_\theta[t, S_\theta(t)]d\{\xi_t[S_\theta(t)]\} \\ &\quad + d\{\dot{g}_\theta[t, S_\theta(t)]\}d\{\xi_t[S_\theta(t)]\}, \end{aligned} \tag{44}$$

where

$$\dot{g}_\theta(t, S_\theta(t)) = \frac{d}{d\theta}\Big\{E_{\mathcal{P}}\Big[\int_t^T f(r, S_\theta(r), \omega)dr + F(T, S_\theta(T), \omega)\Big|S_\theta(t) = S\Big]\Big\}.$$

By Ito rule, we have

$$\begin{aligned} &d\dot{g}_\theta(t, S_\theta(t)) \\ &= \frac{\partial}{\partial t}\dot{g}_\theta(t, S_\theta(t))dt + \sum_{i=1}^N \frac{\partial}{\partial S_i}\Big[\dot{g}_\theta(t, S_\theta(t))\Big] \\ &\quad \Big[\alpha_i(t, S(t))dt + \sum_{j=1}^M \sigma_{i,j}(t, S(t))dW_j(t)\Big] \\ &\quad + \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^M(\sigma\sigma^T)_{i,j}(t, S(t))\frac{\partial^2}{\partial S_i\partial S_j}\dot{g}_\theta(t, S_\theta(t))dt. \end{aligned}$$

Therefore,

$$\begin{aligned} &E_{\mathcal{P}}\big(d\{\dot{g}_\theta[t, S_\theta(t)]\}\xi_t[S_\theta(t)]|S_\theta(t) = S\big) \\ &= \Big\{\frac{\partial}{\partial t}\dot{g}_\theta(t, S)dt + \sum_{i=1}^N \frac{\partial}{\partial S_i}\Big[\dot{g}_\theta(t, S)\Big][\alpha_i(t, S)dt] \\ &\quad + \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^M(\sigma\sigma^T)_{i,j}(t, S)\frac{\partial^2}{\partial S_i\partial S_j}\dot{g}_\theta(t, S)\Big\}\xi_{\theta,t}(S)dt. \end{aligned}$$

From (37), we have

$$E_{\mathcal{P}}\big(\dot{g}_\theta[t, S_\theta(t)]d\{\xi_{\theta,t}[S_\theta(t)]\}|S_\theta(t) = S\big) = 0,$$

and

$$\begin{aligned} &E_{\mathcal{P}}\big(d\{\dot{g}_\theta[t, S_\theta(t)]\}d\{\xi_t[S_\theta(t)]\}|S_\theta(t) = S\big) \\ &= \sum_{i=1}^N\Big\{\frac{\partial}{\partial S_i}\Big[\dot{g}_\theta(t, S)\Big]\Big[\sum_{j=1}^M \sigma_{i,j}(t, S)\zeta_{j,\theta,t}(S)\Big]\Big\}\xi_{\theta,t}(S)dt. \end{aligned}$$

Therefore, the first order optimality condition (31) is

$$\begin{aligned} &\frac{d}{dt}\Big(E_{\mathcal{P}}\Big\{\Big[\dot{g}_\theta(t, S_\theta(t))\xi_t(S_\theta(t))\Big]\Big|S_\theta(t) = S\Big\}\Big) \\ &= \Big\{\frac{\partial}{\partial t}\dot{g}_\theta(t, S) + \sum_{i=1}^N \frac{\partial}{\partial S_i}\Big[\dot{g}_\theta(t, S)\Big][\alpha_i(t, S)] \\ &\quad + \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^M(\sigma\sigma^T)_{i,j}(t, S)\frac{\partial^2}{\partial S_i\partial S_j}\dot{g}_\theta(t, S) \\ &\quad + \sum_{i=1}^N\Big(\frac{\partial}{\partial S_i}\Big[\dot{g}_\theta(t, S)\Big]\Big[\sum_{j=1}^M \sigma_{i,j}(t, S)\zeta_{j,\theta,t}(S)\Big]\Big)\Big\}\xi_{\theta,t}(S). \\ &= 0, \qquad\qquad 0 \le t \le T,\ \forall S. \end{aligned}$$

Because $\xi_{\theta,t}(S) > 0$, so the first order optimality condition (31) becomes

$$\begin{aligned} &\frac{\partial}{\partial t}\dot{g}_\theta(t, S) + \sum_{i=1}^N \frac{\partial}{\partial S_i}\Big[\dot{g}_\theta(t, S)\Big][\alpha_i(t, S)] \\ &\quad + \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^M(\sigma\sigma^T)_{i,j}(t, S)\frac{\partial^2}{\partial S_i\partial S_j}\dot{g}_\theta(t, S) \\ &\quad + \sum_{i=1}^N\Big(\frac{\partial}{\partial S_i}\Big[\dot{g}_\theta(t, S)\Big]\Big[\sum_{j=1}^M \sigma_{i,j}(t, S)\zeta_{j,\theta,t}(S)\Big]\Big) = 0, \\ &\qquad\qquad 0 \le t \le T,\ \forall S. \end{aligned} \tag{45}$$

Condition (44) can also be obtained directly from (27). From the Girsanov theorem, with (38), under measure $\mathcal{Q}$, the drift of the Brownian motion has to change according to $-\zeta(t)$:

$$dW^{\mathcal{Q}}(t) = dW(t) - \zeta(t)dt.$$

Under measure $\mathcal{Q}$, the system equation has to be modified according to the above equation:

$$dS(t) = [\alpha(t, S(t)) + \sigma(t, S(t))\zeta(t)]dt + \sigma(t, S(t))dW^{\mathcal{Q}}(t), \tag{46}$$

with $S(0) = S$. Applying $\mathbb{A}_\theta^{\mathcal{Q}}$ according to the system equation (45) to $\dot{g}_\theta(t, S)$ yields (44).

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