Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Stability Criteria for Cascaded Nonlinear Stochastic Systems Admitting Not Necessarily Unbounded Decay Rate

Hiroshi Ito * **Yuki Nishimura** **

* *Department of Systems Design and Informatics, Kyushu Institute of Technology, Iizuka, Fukuoka 820-8502, Japan (e-mail: hiroshi@ces.kyutech.ac.jp)*
** *Department of Mechanical Engineering, Kagoshima University, 1-21-40, Korimoto, Kagoshima, 890-0065, Japan (e-mail: yunishi@mech.kagoshima-u.ac.jp)*

Abstract: This paper considers global asymptotic stability and robustness of cascaded nonlinear stochastic systems, and presents Lyapunov-based criteria. Constituent systems are characterized in terms of dissipation inequalities almost in the form of input-to-state stability (ISS) which is popular for both stochastic and deterministic systems. As an important unique feature of stochastic systems, this paper first demonstrates that assuming ISS systems having unbounded decay rate is restrictive, which contrasts sharply with the deterministic case. This motivates the second part of the main results focusing on ISS systems with bounded decay rate as well as systems which are not ISS. With these developments, this paper proposes a framework based on integral input-to-state stability (iISS) and demonstrates its usefulness.

Keywords: Stochastic systems; Nonlinear systems; Interconnected systems; Integral input-to-state stability; Lyapunov methods.

1. INTRODUCTION

Dynamical systems often have fluctuant elements and exhibit distributions due to fluctuations. Stochastic differential equations of Itô form is useful for modeling the fluctuations. In control engineering and systems science, one often starts from modules and then connect them for understanding and synthesizing a lager system. How much do the fluctuations affect this module approach? This paper addresses this question in view of integral input-to-state stability (iISS) by solving problems of stability and robustness of cascaded systems. The aim is at spotlighting unique points arising from the random fluctuations.

Lyapunov-type methods has been extensively studied in the literature of robust nonlinear control of stochastic systems, e.g. Krstić and Deng [1998], Wu et al. [2007], Xie and Tian [2009], Yu and Xie [2010], Yu et al. [2010] to name a few. It is now widely known that replacing the derivative of a Lyapunov function along system trajectories by an infinitesimal generator involving a Hessian term is the technical key to dealing with stochastic systems. The Hessian term makes it difficult to construct Lyapunov functions in establishing stability of interconnected systems. Liu et al. [2008] tackled cascaded systems and proposed a stability criterion when subsystems are input-to-state stable (ISS) in probability (Tang and Basar [2001]). Although cascaded systems had no feedback loop, the criterion was referred to a small-gain condition. This terminology might not be intuitive, but the result nicely described how the Hessian term bothers us. Wu et al. [2013] assumed concavity of system gains to get rid of the Hessian.

One of useful facts for deterministic systems is that a cascade of ISS systems is always ISS (Sontag and Teel [1995]). For example, the system $\dot{x}(t) = -x(t)^3 + u(t)$ satisfying the dissipation inequality

$$\dot{V}(x) \leq -\frac{3}{2}V(x)^2 + \frac{3}{2}|u|^{4/3}$$

with $V(x) = x^2$ is ISS since the decay rate $3V(x)^2/2$ on the right-hand side is radially unbounded in V. It is known that a deterministic ISS system can always admit a radially unbounded decay rate (Sontag and Wang [1995]). Following the seminal work for deterministic systems (Sontag and Teel [1995]), Liu et al. [2008] formulated cascades of stochastic ISS systems with radially unbounded decay rates. To develop a stochastic counterpart of the principle of ISS cascade, is it reasonable to assume that the decay rate is unbounded? This paper gives a characterization elucidating the fact that the unboundedness of decay rates is demanding for stochastic systems even if ISS is assumed.

Bounded decay rates were also considered in preceding studies (Yu and Xie [2010], Yu et al. [2010] to name a few). However, their idea to tackle bounded decay rates or non ISS systems is as simple as seeing if summing up (i.e., linear combination) Lyapunov functions of individual subsystems establishes stability of interconnected systems. For deterministic systems, it is known that the effectiveness of linear combination is very limited, and it results in stability criteria which are far more conservative than those utilizing nonlinear combination (Ito [2006], Ito and Jiang [2009], Praly et al. [2010]). Indeed, the fact that cascade of ISS is always ISS cannot be explained by the linear combination. Is it possible to effectively use nonlinear combination for stochastic systems to obtain less conservative criteria as done for deterministic systems? This paper provides an affirmative answer to this question, and proposes an iISS framework for stochastic systems where decay rates are allowed to be merely positive definite. Due to space limitation, all proofs are omitted.

Notation: The symbols $\mathbb{P}$ and $\mathbb{E}$ denote the probability and the expected value, respectively. Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$. For a given vector x of the n-dimensional real vector space $\mathbb{R}^n$, the Euclidean norm is denoted by $|x|$. For a matrix X, $|X|_{\mathcal{F}}$ denotes the Frobenius norm defined by $|X|_{\mathcal{F}} = \sqrt{\mathrm{Tr}\{X^T X\}}$, where the superscript T indicates the transpose of a matrix, and Tr is the trace of a square matrix. A continuous function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{P}$ and one writes $\zeta \in \mathcal{P}$ if $\zeta(s) > 0$ for all $s \in \mathbb{R}_+ \setminus \{0\}$, and $\zeta(0) = 0$. A continuous function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is of class $\mathcal{P}$ and strictly increasing. It is of class $\mathcal{K}_\infty$ if, in addition, $\lim_{s\to\infty} \zeta(s) = \infty$. A continuous function $\eta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if, for each fixed t, the function $\eta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, $\eta(s, \cdot)$ is decreasing and $\lim_{t\to\infty} \eta(s, t) = 0$. For any given $\zeta \in \mathcal{K}$, define the operator $\zeta^{\ominus}$: $[0, \infty] \to [0, \infty]$ as $\zeta^{\ominus}(s) = \sup\{v \in [0, \infty) : s \geq \zeta(v)\}$. By definition, one has $\zeta^{\ominus}(s) = \zeta^{-1}(s)$ for $s < \lim_{\tau\to\infty} \zeta(\tau)$, and $\zeta^{\ominus}(s) = \infty$ elsewhere. Any non-decreasing continuous function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is extended to the operator ζ: $[0, \infty] \to [0, \infty]$ as $\zeta(s) = \sup_{v\in\{w\in[0,\infty)\,:\, w\leq s\}} \zeta(v)$.

2. PRELIMINARIES

2.1 Robustness with respect to Deterministic Disturbance

Consider the stochastic differential equation of Itô form

$$dx = f(x, r)dt + h(x)dw, \tag{1}$$

where $x(t) \in \mathbb{R}^N$ is the state and $r(t) \in \mathbb{R}^M$ is the deterministic disturbance which is measurable, locally essentially bounded function of $t \in \mathbb{R}_+$. The drift field $f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ and the diffusion field $h : \mathbb{R}^N \to \mathbb{R}^{N\times S}$ are locally Lipschitz and satisfy $h(0) = 0$. Components of $w \in \mathbb{R}^S$ are mutually independent standard Wiener processes. The following definition is used (Tang and Basar [2001]).

Definition 1. System (1) is said to be input-to-state stable (ISS) in probability if for each $\epsilon > 0$, there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that

$$\mathbb{P}\left\{|x(t)| < \beta(|x(0)|, t) + \gamma\left(\sup_{\tau\in[0,t]} |r(\tau)|\right)\right\} \geq 1 - \epsilon,$$
$$\forall t \in \mathbb{R}_+,\ x(0) \in \mathbb{R}^N \setminus \{0\}. \tag{2}$$

In this paper, system (1) is said to be 0-GAS in probability if (2) is satisfied for $r = 0$ (Krstić and Deng [1998]).

Definition 2. System (1) is said to be integral-input-to-state stable (iISS) in probability if for each $\epsilon > 0$, there exist a class $\mathcal{KL}$ function β, a class $\mathcal{K}$ function μ and a class $\mathcal{K}_\infty$ function χ such that

$$\mathbb{P}\left\{\chi(|x(t)|) < \beta(|x(0)|, t) + \int_0^t \mu(|r(\tau)|)d\tau,\right\} \geq 1 - \epsilon,$$
$$\forall t \in \mathbb{R}_+,\ x(0) \in \mathbb{R}^N \setminus \{0\}. \tag{3}$$

The above is an exact analog of iISS for deterministic systems (Sontag [1998]). The following is a variant.

Definition 3. System (1) is said to be quasi-integral-input-to-state stable (quasi-iISS) in probability if there exists a constant $R > 0$ satisfying the following: for each $\epsilon > 0$, there exist a class $\mathcal{KL}$ function β, class $\mathcal{K}$ functions $\overline{\beta}$, μ, γ, and a class $\mathcal{K}_\infty$ function χ such that

$$\mathbb{P}\left\{\chi(|x(t)|) < \overline{\beta}(|x(0)|) + \int_0^t \mu(|r(\tau)|)d\tau,\right\} \geq 1 - \epsilon,$$
$$\forall t \in \mathbb{R}_+,\ x(0) \in \mathbb{R}^N \setminus \{0\} \tag{4}$$
$$\|r\| < R \ \Rightarrow\ (2). \tag{5}$$

Property (4) does not guarantee 0-GAS in probability. It is stressed that the functions β, $\overline{\beta}$, γ, μ and χ in (2), (3) and (4) may depend on ϵ. Usually, it is inevitable that the smaller ϵ is, the larger β, $\overline{\beta}$, γ and μ should become.

2.2 Robustness with respect to Stochastic Disturbance

Consider the following system involving $\Theta(t) \in \mathbb{R}^{S\times S}$:

$$dx = f(x)dt + h(x)\Theta(t)dw. \tag{6}$$

For each $t \in \mathbb{R}_+$, the matrix Θ is non-negative definite and its (k, l)-component represents the intensity describing the influence of the l-th component of $w(t)$ on $x(t)$ through the k-th column of $h(x)$. It is emphasized that for (6), we do not assume $h(0) = 0$.

As in Krstić and Deng [1998], we define noise-to-state stability (NSS) for system (6) by replacing $|r(\tau)|$ with $|\Theta(\tau)\Theta^T(\tau)|_{\mathcal{F}}$ in (2) of Definition 1. In the same way, integral noise-to-state stability (iNSS) and quasi-integral noise-to-state stability (quasi-iNSS) are defined by replacing $|r(\tau)|$ with $|\Theta(\tau)\Theta^T(\tau)|_{\mathcal{F}}$ in (3) of Definition 2, and (4)-(5) of Definition 3, respectively.

2.3 Lyapunov-type Characterizations

The definitions in the previous section are stated with functions β, $\overline{\beta}$, γ and μ parametrized by the level of unsureness ϵ although the notions by themselves are independent of ϵ. It is the trick of arbitrary ϵ. In analyzing and designing systems, the parameterized functions are not very convenient. This paper makes use Lyapunov characterizations of these notions to get rid of the dependency on ϵ. As usual, the infinitesimal generator $\mathcal{L}$ associated with the systems (1) and (6) is defined as

$$\mathcal{L}V = \frac{\partial V}{\partial x} f + \frac{1}{2}\mathrm{Tr}\left\{Q^T h^T \frac{\partial^2 V}{\partial x^2} h Q\right\}, \tag{7}$$

for any given $\mathbf{C}^2$ function $V : x \in \mathbb{R}^N \mapsto V(x) \in \mathbb{R}_+$, where

$$\begin{array}{ll} Q = I & \text{for (1)}, \\ Q = \Theta(t) & \text{for (6)}. \end{array} \tag{8}$$

Here, the symbol I is the identity matrix of size $S \times S$. For ISS, we have the following is parallel to the deterministic case presented in Sontag and Wang [1995].

Proposition 4. Consider (1). If there exist a positive definite and radially unbounded $\mathbf{C}^2$ function $V : \mathbb{R}^N \to \mathbb{R}_+$, and $\mathbf{C}^0$ functions $\rho \in \mathcal{K}$, $\eta \in \mathcal{P}$ such that the implication

$$|x| \geq \rho(|r|) \ \Rightarrow\ \mathcal{L}V \leq -\eta(V(x)) \tag{9}$$

holds for all $x \in \mathbb{R}^N$ and $r \in \mathbb{R}^M$, then system (1) is ISS in probability.

This proposition is essentially given in Tang and Basar [2001] which assumes $\eta \in \mathcal{K}$. The relaxation into $\eta \in \mathcal{P}$ can be verified by combining Mao [2002] and Krstić and Deng [1998]. Ito and Nishimura [2014] proved the following. .

Proposition 5. Consider (1). If there exist a positive definite and radially unbounded $\mathbf{C}^2$ function $V : \mathbb{R}^N \to \mathbb{R}_+$, and $\mathbf{C}^0$ functions $\alpha \in \mathcal{K}$, $\sigma \in \mathcal{K}$ such that

$$\mathcal{L}V \leq -\alpha(V(x)) + \sigma(|r|) \tag{10}$$

holds for all $x \in \mathbb{R}^N$ and $r \in \mathbb{R}^M$, then system (1) is quasi-iISS in probability.

Proposition 6. Consider (1). Suppose that there exists $D \geq 0$ such that

$$\left(\frac{\partial V}{\partial x}(x)\right)h(x) = 0, \quad \forall V(x) \geq D \tag{11}$$

holds. If there exist a positive definite and radially unbounded $\mathbf{C}^2$ function $V : \mathbb{R}^N \to \mathbb{R}_+$, and $\mathbf{C}^0$ functions $\alpha \in \mathcal{P}$, $\sigma \in \mathcal{K}$ such that (10) holds for all $x \in \mathbb{R}^N$ and $r \in \mathbb{R}^M$, then system (1) is iISS in probability.

Proposition 6 recovers the deterministic characterization developed in Angeli et al. [2000] by $h = 0$ when the stochastic noise is absent.

Remark 7. In Yu and Xie [2010], Yu et al. [2010], a positive definite and radially unbounded function V satisfying (10) with a pair of $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$ is called a stochastic iISS Lyapunov function. Nevertheless, it is not proved there that the existence of a stochastic iISS Lyapunov function implies any trajectory-based property (See [Yu and Xie, 2010, Remark 4]).

For NSS, the characterization in Krstić and Deng [1998] can be modified slightly using Mao [2002] as follow:

Proposition 8. Consider (6). If there exist a positive definite and radially unbounded $\mathbf{C}^2$ function $V : \mathbb{R}^N \to \mathbb{R}_+$, and $\mathbf{C}^0$ functions $\rho \in \mathcal{K}$, $\eta \in \mathcal{P}$ such that the implication

$$|x| \geq \rho(|\Theta\Theta^T|_{\mathcal{F}}) \;\Rightarrow\; \mathcal{L}V \leq -\eta(V(x)) \tag{12}$$

holds for all $x \in \mathbb{R}^N$, then system (6) is NSS.

The next two are given in Ito and Nishimura [2014].

Proposition 9. Consider (6). If there exist a positive definite and radially unbounded $\mathbf{C}^2$ function $V : \mathbb{R}^N \to \mathbb{R}_+$, $\mathbf{C}^0$ functions $\alpha \in \mathcal{K}$ and $\sigma \in \mathcal{K}$ such that

$$\mathcal{L}V \leq -\alpha(V(x)) + \sigma(|\Theta\Theta^T|_{\mathcal{F}}) \tag{13}$$

holds $\forall x \in \mathbb{R}^N$, $\Theta \in \mathbb{R}^{S\times S}$, then system (6) is quasi-iNSS.

Proposition 10. Consider (6). Suppose that there exists $D \geq 0$ such that (11) holds. If there exist a positive definite and radially unbounded $\mathbf{C}^2$ function $V : \mathbb{R}^N \to \mathbb{R}_+$, and $\mathbf{C}^0$ functions $\alpha \in \mathcal{P}$, $\sigma \in \mathcal{K}$ such that (13) holds for all $x \in \mathbb{R}^N$ and $\Theta \in \mathbb{R}^{S\times S}$, then system (6) is iNSS.

For dealing with interconnection of systems, we will not use (11) directly in order not to exclude stochastic noises.

3. FRAGILITY OF DECAY RATES: A MOTIVATING RESULT

It is a widely known fundamental that in Lyapunov-type analysis, the difference between the stochastic case and the deterministic case boils down to the diffusion term $(1/2)\mathrm{Tr}\left\{Q^T h^T \partial^2 V/\partial x^2 hQ\right\}$ in the infinitesimal generator $\mathcal{L}V$ in (7). The diffusion term which is present only in the stochastic case implies that the dissipation characterization of stochastic systems is fragile compared to that of deterministic systems. This section shows a novel characterization giving an insight into this unavoidable fact, and demonstrates a unique need for tackling systems having bounded decay rates in the stochastic case.

Definition 11. Given $\alpha \in \mathcal{P}$, $\sigma \in \mathcal{K}\cup\{0\}$, a $\mathbf{C}^1$ function $T : \mathbb{R}_+ \to \mathbb{R}_+$, and positive integers N, M and S, define $\mathcal{S}(N, M, S, \alpha, \sigma, T)$ as the set of all pairs (Σ, V) such that the system Σ is described by

$$dx = f(x, r)dt + h(x)dw, \quad x\in\mathbb{R}^N,\ r\in\mathbb{R}^M \tag{14}$$

with the S-dimensional standard Wiener process w, where $f : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^N$ and $h : \mathbb{R}^N \to \mathbb{R}^{N\times S}$ are locally Lipschitz and satisfy $f(0, 0) = h(0) = 0$. Moreover, the function $V : \mathbb{R}^N \to \mathbb{R}_+$ is of class $\mathbf{C}^2$, positive definite and radially unbounded and satisfies

$$\mathcal{L}V \leq -\alpha(V(x)) + \sigma(|r|), \quad \forall x \in \mathbb{R}^N,\ r \in \mathbb{R}^M \tag{15}$$

$$T(V(x)) \geq \mathrm{Tr}\left\{h^T(x)\left(\frac{\partial V}{\partial x}(x)\right)^T\left(\frac{\partial V}{\partial x}(x)\right)h(x)\right\},\ \forall x \in \mathbb{R}^N. \tag{16}$$

The following condition elucidates the effect of nonlinear coordinate transformation of V on the dominance of the decay rate α over disturbance rate σ in (15).

Theorem 12. Suppose that positive integers N, M, S and $\mathbf{C}^1$ functions $\alpha \in \mathcal{K}\setminus\mathcal{K}_\infty$, $\sigma \in \mathcal{K}\cup\{0\}$, $T : \mathbb{R}_+ \to \mathbb{R}_+$ are given and satisfy

$$T(s) > 0, \quad \forall s \in [b, \infty). \tag{17}$$

for a real number $b > 0$. Then there exists a $\mathbf{C}^2$ function $F \in \mathcal{K}_\infty$ such that for all $(\Sigma, V) \in \mathcal{S}(N, M, S, \alpha, \sigma, T)$, the function $\hat{V}(x) = F(V(x))$ satisfies

$$\mathcal{L}\hat{V} \leq -\hat{\alpha}(\hat{V}(x)) + \hat{\sigma}(|r|), \quad \forall x \in \mathbb{R}^N,\ r \in \mathbb{R}^M \tag{18}$$

with some $\hat{\alpha} \in \mathcal{K}_\infty$ and $\hat{\sigma} \in \mathcal{K}\cup\{0\}$ if and only if there exists a continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\mathbf{Id} + \omega \in \mathcal{K}_\infty \tag{19}$$

$$\omega\circ\alpha(s) > 0, \quad \forall s \in (0, \infty) \tag{20}$$

$$\lim_{s\to\infty} \alpha(s) \geq \lim_{s\to\infty} (\mathbf{Id} + \omega)\circ\sigma(s). \tag{21}$$

$$\lim_{\tau\to\infty} \log\{\omega\circ(\mathbf{Id} + \omega)^{-1}\circ\alpha(\tau)\} + \int_b^\infty \frac{2\omega\circ(\mathbf{Id} + \omega)^{-1}\circ\alpha(\tau)}{T(\tau)}d\tau = \infty \tag{22}$$

are satisfied. Furthermore,

$$\sigma = 0 \;\Rightarrow\; \hat{\sigma} = 0 \tag{23}$$

can be achieved in (18) whenever (19)-(22) are satisfied.

It is stressed that there exists a continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (19)-(21) if and only if

$$\lim_{s\to\infty} \alpha(s) \geq \lim_{s\to\infty} \sigma(s). \tag{24}$$

The function ω is introduced explicitly in Theorem 12 for (22) involving ω. Condition (22) can be made independent of ω if the inequality of (24) is strict.

Corollary 13. Suppose that positive integers N, M, S and $\mathbf{C}^1$ functions $\alpha \in \mathcal{K}\setminus\mathcal{K}_\infty$, $\sigma \in \mathcal{K}\cup\{0\}$, $T : \mathbb{R}_+ \to \mathbb{R}_+$ are given and satisfy (17) for a real number $b > 0$. Assume that $\lim_{s\to\infty}\alpha(s) \neq \lim_{s\to\infty}\sigma(s)$. Then there exists a $\mathbf{C}^2$ function $F \in \mathcal{K}_\infty$ such that for all $(\Sigma, V) \in \mathcal{S}(N, M, S, \alpha, \sigma, T)$, the function $\hat{V}(x) = F(V(x))$ satisfies (18) with some $\hat{\alpha} \in \mathcal{K}_\infty$ and $\hat{\sigma} \in \mathcal{K}\cup\{0\}$ satisfying (23) if and only if the two conditions

$$\lim_{s\to\infty} \alpha(s) > \lim_{s\to\infty} \sigma(s) \tag{25}$$

$$\int_b^\infty \frac{\alpha(\tau)}{T(\tau)} d\tau = \infty \tag{26}$$

hold.

The following includes the deterministic case $h = 0$ as a special case.

Theorem 14. Suppose that positive integers N, M, S and $\mathbf{C}^1$ functions $\alpha \in \mathcal{K} \setminus \mathcal{K}_\infty$, $\sigma \in \mathcal{K} \cup \{0\}$, $T : \mathbb{R}_+ \to \mathbb{R}_+$ are given and satisfy $T(s) = 0, \quad \forall s \in [c, \infty)$ for a real number $c \geq 0$. Then there exists a $\mathbf{C}^2$ function $F \in \mathcal{K}_\infty$ such that for all $(\Sigma, V) \in \mathcal{S}(N, M, S, \alpha, \sigma, T)$, the function $\hat{V}(x) = F(V(x))$ satisfies (18) with some $\hat{\alpha} \in \mathcal{K}_\infty$ and $\hat{\sigma} \in \mathcal{K} \cup \{0\}$ satisfying (23) if and only if (24) holds.

This theorem exactly recovers the deterministic counterpart indicated by Sontag and Wang [1995], Angeli et al. [2000], Ito [2006] when $T = 0$. The growth order conditions (22) and (26) in Theorem 12 and Corollary 13 reveal a distinctive influence of stochastic noise on the coordinate transformation F of the Lyapunov function V, which never exhibits in the deterministic case. The stochastic effect varies with the choice of coordinate on which V is built. Recall that by virtue of Proposition 4, the system Σ is ISS if (24) holds. Nevertheless, the growth order conditions (22) and (26) prevent us from taking it for granted that an ISS system admit an unbounded decay rate ($\alpha \in \mathcal{K}_\infty$). Notice that if (22) and (26) were not required, a cascade of ISS systems could be always ISS as it is in the deterministic result replying on $\alpha \in \mathcal{K}_\infty$ (Sontag and Teel [1995]). Thereby, Theorem 12 and Corollary 13 elucidates that

- Stochastic noise makes the dominance of the decay rate α over the disturbance rate σ fragile, and the dominance is not preserved by the nonlinear coordinate change of V.
- Assuming $\alpha \in \mathcal{K}_\infty$ for stochastic systems is restrictive since the dominance of the decay rate (24) does not secure the existence of $\hat{\alpha} \in \mathcal{K}_\infty$ achieving (18).

The study in Liu et al. [2008] provided an important step toward Lyapunov-based analysis taking the degradation of $\alpha \in \mathcal{K}_\infty$ into account in constructing a Lyapunov functions for cascaded systems. However, it assumes that each subsystem admits $\alpha \in \mathcal{K}_\infty$. This paper not only develops another tool, but also places a special emphasis on allowing $\alpha \in \mathcal{K}\setminus\mathcal{K}_\infty$ and $\alpha \in \mathcal{P}\setminus\mathcal{K}$ to address the unique issue for stochastic systems as well as non-ISS systems.

Remark 15. In Theorem 12, (resp., Corollary 13), the continuous differentiability of α, σ and T is assumed for guaranteeing the local Lipschitzness of f and h in establishing the necessity of (19)-(22) (resp., (25)-(26)).

4. ROBUSTNESS WITH RESPECT TO DETERMINISTIC DISTURBANCE

4.1 Cascaded System with An Equilibrium at Origin

This section supposes that the system (1) consists of

$$dx_1 = f_1(x, r_1)dt + h_1(x)dw_1 \tag{27}$$

$$dx_2 = f_2(x_2, r_2)dt + h_2(x_2)dw_2, \tag{28}$$

where $x_i(t) \in \mathbb{R}^{N_i}$, $r_i(t) \in \mathbb{R}^{M_i}$, $w_i(t) \in \mathbb{R}^{S_i}$ for $i = 1, 2$, $x(t) = [x_1(t)^T, x_2(t)^T]^T \in \mathbb{R}^N$ with $N = N_1 + N_2$, $r(t) = [r_1^T(t), r_2^T(t)]^T \in \mathbb{R}^M$ with $M = M_1 + M_2$, and $w(t) = [w_1(t)^T, w_2(t)^T]^T \in \mathbb{R}^S$ with $S = S_1 + S_2$. As assumed for (1), the functions $f_1 : \mathbb{R}^N \times \mathbb{R}^{M_1} \to \mathbb{R}^{N_1}$, $f_2 : \mathbb{R}^{N_2} \times \mathbb{R}^{M_2} \to \mathbb{R}^{N_2}$, $h_1 : \mathbb{R}^N \to \mathbb{R}^{N_1 \times S_1}$ and $h_2 : \mathbb{R}^{N_2} \to \mathbb{R}^{N_2 \times S_2}$ are locally Lipschitz and satisfy $h_i(0) = 0$ for $i = 1, 2$. Again, components of w_i are mutually independent standard Wiener processes. The two subsystems (27) and (28) forms a cascade, where (27) and (28) are referred to as a driven system and a driving system, respectively. This section assumes that the subsystems (27) and (28) of (1) satisfy the following.

Assumption 16. For each $i = 1, 2$, there exist a positive definite and radially unbounded $\mathbf{C}^2$ function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ and $\mathbf{C}^1$ functions $\alpha_i \in \mathcal{P}$, $\sigma_1 \in \mathcal{K}$ and $\mathbf{C}^0$ functions $\kappa_i \in \mathcal{K} \cup \{0\}$, $T_1, T_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\mathcal{L}V_1 \leq -\alpha_1(V_1(x_1)) + \sigma_1(V_2(x_2)) + \kappa_1(|r_1|)) \tag{29}$$

$$\mathcal{L}V_2 \leq -\alpha_2(V_2(x_2)) + \kappa_2(|r_2|)) \tag{30}$$

$$\mathrm{Tr}\left\{ h_1^T(x) \left(\frac{\partial V_1}{\partial x_1}(x_1) \right)^T \left(\frac{\partial V_1}{\partial x_1}(x_1) \right) h_1(x) \right\} \leq T_1(V_1(x_1)) \tag{31}$$

$$\mathrm{Tr}\left\{ h_2^T(x_2) \left(\frac{\partial V_2}{\partial x_2}(x_2) \right)^T \left(\frac{\partial V_2}{\partial x_2}(x_2) \right) h_2(x_2) \right\} \leq T_2(V_2(x_2)) \tag{32}$$

hold for all $x_i \in \mathbb{R}^{N_i}$, $x_{3-i} \in \mathbb{R}^{N_{3-i}}$ and $r_i \in \mathbb{R}^{M_i}$.

This assumption implies that the driven system, i.e., x_1-subsystem is quasi-iISS with respect to the input x_2 when $\alpha_1 \in \mathcal{K}$. When either $\alpha_1 \in \mathcal{K}_\infty$ or $\lim_{s\to\infty} \alpha_1(s) \geq \lim_{s\to\infty} \sigma_1(s)$ holds, the x_1-subsystem is ISS with respect to the input x_2. The x_1-subsystem is ISS with respect to the input (x_2, r_1) if $\alpha_1 \in \mathcal{K}_\infty$. Assumption 16 defines the x_2-subsystem in the same way. This subsection aims at deriving conditions under which the cascaded system (27)-(28) is quasi-iISS, iISS or ISS in probability. To this end, we seek a Lyapunov function $V : \mathbb{R}^N \to \mathbb{R}_+$ in the form of

$$V(x) = F_1(V_1) + F_2(V_2) = \int_0^{V_1(x_1)} \lambda_1(s)ds + \int_0^{V_2(x_2)} \lambda_2(s)ds \tag{33}$$

for continuous functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$, satisfying $\lambda_i(s) > 0$ for all $s \in (0, \infty)$. In contrast to the deterministic case (Sontag and Teel [1995], Ito [2006], Ito and Jiang [2009]), in the presence of stochastic noise, it is inevitable that the nonlinear transformations F_i cause degradation of decay rates of subsystems. This section shows how to deal with this fundamental difficulty arising from interconnections without assuming $\alpha_1, \alpha_1 \in \mathcal{K}_\infty$.

4.2 Radially Non-Vanishing Decay Rates

In this section, we deal with subsystems admitting $\alpha_i \in \mathcal{K}$, $i = 1, 2$, i.e., the decay rates are not radially vanishing. When $T_1(0) = 0$ holds, there exists $b \geq 0$ such that

$$T_1(s) = 0, \quad \forall s \in [0, b]. \tag{34}$$

For given $b \geq 0$, let $\overline{\sigma}_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be defined as

$$\overline{\sigma}_2(s) = \begin{cases} 0 & , 0 \leq s < b \\ e^{\displaystyle\int_1^s \frac{\alpha_1(\tau)}{\overline{T}_1(\tau)} d\tau} & , b \leq s < \infty \end{cases} \tag{35}$$

$$\overline{T}_1(s) = \max_{\tau \in [b, s]} T_1(\tau), \quad b \leq s < \infty. \tag{36}$$

We can prove the following.

Theorem 17. Consider (1) consisting of (27) and (28). Suppose that $\alpha_1 \in \mathcal{K}$ and $T_1(0) = 0$. If there exists $\tilde{\alpha}_2 \in \mathcal{K}$ such that

$$\tilde{\alpha}_2(s) \leq \alpha_2(s) - \frac{1}{2}\frac{\sigma_1'(s)}{\sigma_1(s)}T_2(s), \quad \forall s \in \mathbb{R}_+ \tag{37}$$

$$\limsup_{s\to 0} \frac{\overline{\sigma}_2 \circ \alpha_1^{\ominus} \circ 3\sigma_1(s)}{\tilde{\alpha}_2(s)} < \infty \tag{38}$$

are satisfied, then system (27)-(28) is 0-GAS in probability. Furthermore, if

$$\lim_{s\to\infty} \tilde{\alpha}_2(s) = \infty \ \text{ or } \ \lim_{s\to\infty} \sigma_1(s)\kappa_2(1) < \infty \tag{39}$$

is satisfied in addition, then the following hold true:

(i) If there exist $D_i \geq 0$, $i = 1, 2$, such that

$$\left(\frac{\partial V_1}{\partial x_1}(x_1)\right)h_1(x) = 0, \quad \forall |x_1| \geq D_1 \tag{40}$$

$$\left(\frac{\partial V_2}{\partial x_2}(x_2)\right)h_2(x_2) = 0, \quad \forall |x_2| \geq D_2 \tag{41}$$

$$D_2 < \lim_{s\to\infty} \sigma_1^{\ominus} \circ \alpha_1(s) \tag{42}$$

are satisfied, system (27)-(28) is iISS in probability.

(ii) System (27)-(28) is quasi-iISS in probability.

(iii) If $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_\infty$, system (27)-(28) is ISS in probability.

Note that (38) holds automatically if (34) is satisfied with $b > 0$. It is worth mentioning that the pair (40) and (41) is less restrictive than (11) required for all $|x| \geq D$. A non-trivial modification allows us to employ (40) and (41).

A cascade of two deterministic systems admitting $\alpha_1, \alpha_2 \in \mathcal{K}$ is always 0-GAS (Ito [2010]). The following illustrates how such deterministic results are covered as special cases.

Corollary 18. Suppose that (34) holds for a real number $b > 0$, and $T_2 = 0$. If $\alpha_1, \alpha_2 \in \mathcal{K}$ holds, system (27)-(28) is 0-GAS in probability. Furthermore, if

$$\lim_{s\to\infty} \alpha_2(s) = \infty \ \text{ or } \ \lim_{s\to\infty} \sigma_1(s)\kappa_2(1) < \infty \tag{43}$$

is satisfied in addition, then the following hold true:

(i) If there exists $D_1 \geq 0$ such that (40) is satisfied, system (27)-(28) is iISS in probability.

(ii) System (27)-(28) is quasi-iISS in probability.

(iii) If α_1 and α_2 are of class $\mathcal{K}_\infty$, system (27)-(28) is ISS in probability.

Remark 19. Liu et al. [2008] assumes $\kappa_1 = 0$ in addition to $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. Theorem 17 and Corollary 18 not only give an alternative to the criterion proposed in Liu et al. [2008], but also allow us to deal with subsystems which are not equipped with $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$. The next section further relaxes the assumption on α_1, α_2.

4.3 Radially Vanishing Decay Rates

This subsection removes $\alpha_1, \alpha_2 \in \mathcal{K}$ assumed in Theorem 17 and Corollary 18, and allows $\lim_{s\to\infty} \alpha_i(s) = 0$.

Theorem 20. Consider (1) consisting of (27) and (28). Suppose that

$$\int_0^1 \frac{\sigma_1(\tau)}{\alpha_2(\tau)}d\tau < \infty \tag{44}$$

$$\int_1^\infty \frac{\sigma_1(\tau)}{\alpha_2(\tau)}d\tau = \infty \tag{45}$$

$$\kappa_2 = 0 \ \text{ or } \ \left\{\begin{matrix} \alpha_2 \in \mathcal{K} \\ \text{and} \\ \sigma_1 \notin \mathcal{K}_\infty \end{matrix}\right\} \ \text{ or } \ \alpha_2 \in \mathcal{K}_\infty \tag{46}$$

are satisfied. If there exists $K \in [0, 2)$ such that

$$K\alpha_2(s) \geq T_2(s)\left(\frac{\sigma_1'(s)}{\sigma_1(s)} - \frac{\alpha_2'(s)}{\alpha_2(s)}\right), \quad \forall s \in \mathbb{R}_+ \tag{47}$$

holds, then system (27)-(28) is 0-GAS in probability and the following hold true:

(i) If there exist $D_i \geq 0$, $i = 1, 2$, such that (40), (41) and

$$D_2 \begin{cases} < \lim_{s\to\infty} \sigma_1^{\ominus} \circ \alpha_1(s), & \text{if } \alpha_1 \in \mathcal{K} \\ = 0, & \text{otherwise} \end{cases} \tag{48}$$

are satisfied, system (27)-(28) is iISS in probability.

(ii) If $\alpha_1, \alpha_2 \in \mathcal{K}$, system (27)-(28) is quasi-iISS in probability.

(iii) If $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, system (27)-(28) is ISS in probability.

Property (47) holds whenever $T_2 = 0$. In such a case, assumption (45) can be dropped since σ_1 can always be replaced by another class $\mathcal{K}$ function being larger than the original σ_1 and satisfying (45) and (46). It is interesting that in the case of $T_2 = 0$, the set of conditions required by Theorem 20 is identical to that of a deterministic result in Ito [2010] even when $T_1 \neq 0$.

The next corollary is a is a special case of Theorem 20. It requires a restrictive, but simple constraint.

Corollary 21. Consider (1) consisting of (27) and (28). If there exists $\ell_2 > 0$ such that

$$\sigma_1(s) \leq \ell_2\alpha_2(s), \quad \forall s \in \mathbb{R}_+ \tag{49}$$

holds, then the statements *(i)*, *(ii)* and *(iii)* in Theorem 20 hold true.

Without invoking Theorem 20, an alternative proof can be given to the above corollary easily. Indeed, the matching between σ_1 and α_2 in the form of (49) allows us to use linear F_1 and F_2, i.e., constant λ_1 and λ_2.

Remark 22. Corollary 21 can be considered as a rearranged statement of approaches to $\alpha_1, \alpha_2 \notin \mathcal{K}_\infty$ such as Yu and Xie [2010], Yu et al. [2010] using constant λ_1, λ_2.

5. ROBUSTNESS WITH RESPECT TO STOCHASTIC DISTURBANCE

5.1 Cascaded System Allowing for Noise at Origin

In this section, let system (6) consist of

$$dx_1 = f_1(x)dt + h_1(x)\Theta_1(t)dw_1 \tag{50}$$

$$dx_2 = f_2(x_2)dt + h_2(x_2)\Theta_2(t)dw_2, \tag{51}$$

where the functions $f_1 : \mathbb{R}^N \to \mathbb{R}^{N_1}$, $f_2 : \mathbb{R}^{N_2} \to \mathbb{R}^{N_2}$, $h_1 : \mathbb{R}^N \to \mathbb{R}^{N_1\times S_1}$ and $h_2 : \mathbb{R}^{N_2} \to \mathbb{R}^{N_2\times S_2}$ are locally Lipschitz. Just as $h(0) = 0$ was not assumed for (6), here $h_i(0) = 0$ for $i = 1, 2$ are not assumed either. Vectors and their dimension are $x_i(t) \in \mathbb{R}^{N_i}$, $w_i(t) \in \mathbb{R}^{S_i}$ for $i = 1, 2$, $x(t) = [x_1(t)^T, x_2(t)^T]^T \in \mathbb{R}^N$ with $N = N_1 +$

N_2, and $w(t) = [w_1(t)^T, w_2(t)^T]^T \in \mathbb{R}^S$ with $S = S_1 + S_2$. Again, components of w_i are mutually independent standard Wiener processes. As in (6), the (k,l)-component of the non-negative definite matrix $\Theta_i(t) \in \mathbb{R}^{S_i \times S_i}$ which is bounded and piecewise continuous in $t \in \mathbb{R}_+$ denotes the intensity describing the influence of the l-th component of $w_i(t)$ on $x_i(t)$ through the k-th column of h_i. This section assumes the following.

Assumption 23. For each $i = 1, 2$, there exist a positive definite and radially unbounded $\mathbf{C}^2$ function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ and $\mathbf{C}^1$ functions $\alpha_i \in \mathcal{P}$, $\sigma_1 \in \mathcal{K}$ and $\mathbf{C}^0$ functions $\kappa_i \in \mathcal{K} \cup \{0\}$, $H_1, H_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\mathcal{L}V_1 \leq -\alpha_1(V_1(x_1)) + \sigma_1(V_2(x_2)) + \kappa_1(|\Theta_1\Theta_1^T|_\mathcal{F}) \quad (52)$$

$$\mathcal{L}V_2 \leq -\alpha_2(V_2(x_2)) + \kappa_2(|\Theta_2\Theta_2^T|_\mathcal{F}) \quad (53)$$

$$\left| h_1^T(x)\left(\frac{\partial V_1}{\partial x_1}(x_1)\right)^T\left(\frac{\partial V_1}{\partial x_1}(x_1)\right)h_1(x)\right|_\mathcal{F} \leq H_1(V_1(x_1)) \quad (54)$$

$$\left| h_2^T(x_2)\left(\frac{\partial V_2}{\partial x_2}(x_2)\right)^T\left(\frac{\partial V_2}{\partial x_2}(x_2)\right)h_2(x_2)\right|_\mathcal{F} \leq H_2(V_2(x_2)) \quad (55)$$

hold for all $x_i \in \mathbb{R}^{N_i}$, $x_{3-i} \in \mathbb{R}^{N_{3-i}}$ and all non-negative definite matrices $\Theta_i \in \mathbb{R}^{S_i \times S_i}$, where κ_i is the zero function, i.e., $\kappa_i = 0$ if $h_i = 0$.

As in Section 4, this section uses a Lyapunov function V in the form of (33) to develop criteria for quasi-iNSS, iNSS and NSS of the cascaded system (50)-(51).

5.2 Radially Non-Vanishing Decay Rates

Recall that the notions of NSS, quasi-iNSS and iNSS estimate the deviation of the process $x(t)$ from the origin due to the stochastic noise w_i, while the notions of ISS, quasi-iISS and iISS aim at getting rid of the influence of w_i at $x = 0$. As a result, the following theorem establishing NSS, quasi-iNSS and iNSS imposes less restrictive conditions on subsystems than Theorem 17.

Theorem 24. Consider (6) consisting of (50) and (51). Suppose that $\alpha_1, \alpha_2 \in \mathcal{K}$ and

$$\lim_{s\to\infty} \alpha_2(s) = \infty \ \text{ or } \ \lim_{s\to\infty} \sigma_1(s) < \infty \ \text{ or } \ h_2 = 0 \quad (56)$$

are satisfied. Then the following hold true.

(i) If there exist $D_i \geq 0$, $i = 1, 2$, such that (40)-(42) are satisfied, system (50)-(51) is iNSS.

(ii) If

$$\limsup_{s\to\infty} \frac{\alpha_i'(s)}{\alpha_i(s)} H_i(s) < \infty, \ i = 1, 2 \quad (57)$$

$$\limsup_{s\to\infty} \frac{\sigma_1'(s)}{\sigma_1(s)} H_2(s) < \infty \quad (58)$$

are satisfied, system (50)-(51) is quasi-iNSS.

(iii) If $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, (57) and (58) are satisfied, system (50)-(51) is NSS.

5.3 Radially Vanishing Decay Rates

Theorem 25. Consider (6) consisting of (50) and (51). Suppose that (44), (45) and

$$h_2 = 0 \ \text{ or } \left\{ \begin{array}{c} \alpha_2 \in \mathcal{K} \\ \text{and} \\ \sigma_1 \notin \mathcal{K}_\infty \end{array} \right\} \text{ or } \ \alpha_2 \in \mathcal{K}_\infty \quad (59)$$

are satisfied. Then the following hold true.

(i) If there exist $D_i \geq 0$, $i = 1, 2$, such that (40), (41) and (48) are satisfied, system (50)-(51) is iNSS.

(ii) If $\alpha_1, \alpha_2 \in \mathcal{K}$, (58) and

$$\limsup_{s\to\infty} \frac{\alpha_2'(s)}{\alpha_2(s)} H_2(s) < \infty \quad (60)$$

are satisfied, then system (50)-(51) is quasi-iNSS.

(iii) If $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and (58) and (60) are satisfied, system (50)-(51) is NSS.

Corollary 26. Consider (6) consisting of (50) and (51). If there exists $\ell_2 > 0$ such that (49) holds, then the statements *(i)*, *(ii)* and *(iii)* in Theorem 25 hold true without assuming (58) and (60).

6. EXAMPLES

Consider the following functions in Assumption 16.

$$\alpha_1(s) = \frac{s}{1+s}, \ \sigma_1(s) = s, \ \kappa_1 = s, \ T_1(s) = \frac{s^2}{3(1+s)} \quad (61)$$

$$\alpha_2(s) = \frac{s^2}{1+s^2}, \ \kappa_2 = 0, \ T_2(s) = \frac{s^3}{1+s^2}. \quad (62)$$

A cascaded system admitting the above functions with $V_i(x_i) = x_i^2/2$ and $N_i = M_i = S_i = 1$, $i = 1, 2$, is

$$dx_1 = -\frac{x_1(1 + x_2 - r_1)}{2 + x_1^2}dt + \frac{x_1}{\sqrt{6(2 + x_1^2)}}dw_1$$

$$dx_2 = -\frac{x_2^3}{4 + x_2^4}dt + \frac{x_2^2}{\sqrt{8 + 2x_2^4}}dw_2.$$

The method in Liu et al. [2008] cannot be applied since neither $\alpha_1 \in \mathcal{K}_\infty$ nor $\lim_{s\to\infty} \alpha_1(s) \geq \lim_{s\to\infty} \sigma_1(s)$ is satisfied. This cascade satisfies all requirements (34)-(39) in Theorem 17 with $b = 0$, $\overline{\sigma}_2(s) = s^3$, $\overline{T}_1 = T_1$ and $\tilde{\alpha}_2 = \alpha_2/2, \alpha_1 \in \mathcal{K}$. Indeed, the function V in (33) with

$$\lambda_1(s) = \frac{s^3}{50(1+s^3)}, \quad \lambda_2(s) = s$$

achieves

$$\mathcal{L}V \leq -\frac{V_1^4}{300(1+V_1^3)(1+V_1)} - P(V_2)V_2 + \frac{|r_1|}{50}$$

for $P \in \mathcal{K}$. Thus, Theorem 17 guarantees 0-GAS and quasi-iISS for all cascaded systems satisfying Assumption 16 with (61) and (62). Note that (44) is not fulfilled.

Next, consider the following functions

$$\alpha_1(s) = \frac{s}{1+s^2}, \ \sigma_1(s) = s, \ \kappa_1 = 0, \ T_1(s) = s^4 \quad (63)$$

$$\alpha_2(s) = \frac{s}{1+s^2}, \ \kappa_2 = 0, \ T_2(s) = \frac{1}{2} \quad (64)$$

for Assumption 16. A cascaded system admitting the above functions with $V_i(x_i) = x_i^2/2$ and $N_i = M_i = S_i = 1$, $i = 1, 2$, is

$$dx_1 = -\frac{2x_1 - x_1^3(x_2 + r_1)}{4 + x_1^4}dt + \frac{x_1^3}{4}dw_1$$

$$dx_2 = -\frac{2x_2}{4 + x_2^4}dt + \frac{2x_2^2}{(1 + 2x_2^2)^{3/2}}dw_2.$$

Due to $\alpha_1, \alpha_2 \in \mathcal{P} \setminus \mathcal{K}$, neither Theorem 17 nor Liu et al. [2008] is applicable. Nevertheless, it can be verified that (44)-(47) in Theorem 20 are satisfied by (63)-(64) with $K = 1$. Thus, 0-GAS of all cascaded systems satisfying Assumption 16 with (63) and (64) is guaranteed by Theorem 20.

7. CONCLUDING REMARKS

In this paper, the problem of verifying stability and robustness of cascaded nonlinear stochastic systems has been addressed. Issues arising from stochastic noises have been spotlighted. Each subsystem is assumed to be described by a dissipation inequality using the infinitesimal generator associated with a Lyapunov function of the subsystem as usual. This paper has shown an important catch in formulating cascaded systems. For deterministic ISS systems, it is widely known that without loss of generality one can assume that the decay rate of each subsystem is radially unbounded. This paper has proved that assuming ISS systems having unbounded decay rates is restrictive. Motivated by this unique feature of stochastic systems, this paper has developed criteria for stability and robustness of cascaded systems allowing for bounded decay rates. The development leads to an iISS framework for composing Lyapunov functions. Compared with the existing literature on interconnection of iISS-type stochastic systems, the development of this paper is unique in the utilization of nonlinear transformation in constructing Lyapunov functions. It has not only led to less conservative stability criteria, but also allowed us to present the criteria in the way that they explicitly recover deterministic criteria when stochastic noises are absent. Another feature of this paper is that all robustness properties achieved in terms of dissipation inequalities are linked to characterized in terms of trajectories as in the deterministic case, while trajectory-based characterizations have been available only for ISS systems in the literature. In addition to ISS and iISS, this paper has addressed robustness in the sense of NSS and its generalizations.

In Ferreira et al. [2012], a criterion for NSS of interconnected systems is presented based on stochastic passivity (Florchinger [1999]). As in the deterministic case, the passivity approach is based on linear combination of Lyapunov functions of subsystems. On the one hand, this paper has employed nonlinear transformation of Lyapunov functions so that subsystems are not required to be passive. On the other, this paper has dealt with only cascade connection. For feedback interconnection, results were reported by the authors in Ito and Nishimura [2014]. On the surface, the feedback formulation can include the cascaded system as a special case by considering a small ficticious function at a connecting channel. However, it is known for deterministic systems that exploiting the cascade structure can lead less conservative criteria. In particular, decay rates in dissipation inequalities can be allowed to be radially vanishing although it cannot be allowed in the feedback formulation (Ito [2010]). In the stochastic case, looking directly at cascade structure is more important. Reducing magnitude of one connecting channel in small-gain arguments for feedback is not readily implementable. The change of coordinates of subsystems for establishing stability of the overall system causes degradation of dissipation properties of subsystems in the presence of stochastic noises. Reducing magnitude of a connection may result in the increase of loop gain. In fact, getting rid of this undesirable cycle explains unique terms arising from stochastic noises in stability criteria this paper have developed.

REFERENCES

D. Angeli, E.D. Sontag, and Y. Wang. A characterization of integral input-to-state stability. *IEEE Trans. Automat. Contr.*, 45:1082–1097, 2000.

A.S.R. Ferreira, M. Arcak, and E.D. Sontag. A decomposition-based approach to stability analysis of large-scale stochastic systems. In *2012 American Control Conf.*, pages 6382–6387, 2012.

P. Florchinger. A passive system approach to feedback stabilization of nonlinear control stochastic systems. *SIAM J. Control Optim.*, 37:1848–1864, 1999.

H. Ito. State-dependent scaling problems and stability of interconnected iISS and ISS systems. *IEEE Trans. Automat. Contr.*, 51:1626–1643, 2006.

H. Ito. A Lyapunov approach to cascade interconnection of integral input-to-state stable systems. *IEEE Trans. Automat. Contr.*, 55:702–708, 2010.

H. Ito and Z.P. Jiang. Necessary and sufficient small gain conditions for integral input-to-state stable systems: A Lyapunov perspective. *IEEE Trans. Automat. Contr.*, 54:2389–2404, 2009.

H. Ito and Y. Nishimura. Stochastic robustness of interconnected nonlinear systems in an iISS framework. In *2014 American Contro. Conf.*, 2014. to appear.

M. Krstić and H. Deng. *Stabilization of nonlinear uncertain systems*. Springer, New York, 1998.

S.-J. Liu, J.-F. Zhang, and Z.P. Jiang. A notion of stochastic input-to-state stability and its application to stability of cascaded stochastic nonlinear systems. *Acta Mathematicae Applicatae Sinica*, 24:141–156, 2008.

X. Mao. A note on the lasalle-type theorems for stochastic differential delay equations. *J. Mathe. Analysis and Applications*, 268:125–142, 2002.

L. Praly, D. Carnevale, and A. Astolfi. Dynamic vs static scaling: an existence result. In *8th IFAC Symp. Nonlinear Control Systems*, pages 1075–1080, 2010.

E.D. Sontag. Comments on integral variants of ISS. *Systems & Contr. Letters*, 34:93–100, 1998.

E.D. Sontag and A.R. Teel. Changing supply functions in input/state stable systems. *IEEE Trans. Automat. Contr.*, 40:1476–1478, 1995.

E.D. Sontag and Y. Wang. On characterizations of input-to-state stability property. *Systems & Contr. Letters*, 24:351–359, 1995.

C. Tang and T. Basar. Stochastic stability of singularly perturbed nonlinear systems. In *40th IEEE Conf. Decision Control*, pages 399–404, 2001.

Z. Wu, H.R. Karimi, and P. Shi. Small-gain conditions for stochastic network systems. In *52nd IEEE Conf. Decision Contr*, pages 1223–1228, 2013.

Z.-J. Wu, X.-J. Xie, and S.-Y. Zhang. Adaptive backstepping controller design using stochastic small-gain theorem. *Automatica*, 43:608–620, 2007.

X.-J. Xie and J. Tian. Adaptive state-feedback stabilization of high-order stochastic systems with nonlinear parameterization. *Automatica*, 45:126–133, 2009.

X. Yu and X.-J. Xie. Output feedback regulation of stochastic nonlinear systems with stochastic iISS inverse dynamics. *IEEE Trans. Automat. Contr.*, 55:304–320, 2010.

X. Yu, X.-J. Xie, and N. Duan. Small-gain control method for stochastic nonlinear systems with stochastic iISS inverse dynamics. *Automatica*, 46:1790–1798, 2010.