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Asymptotic stability of time-varying distributed parameter semi-linear systems

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Abstract: The asymptotic behaviour is studied for a class of non-linear distributed parameter time-varying dissipative systems. This is achieved by using time-varying infinite-dimensional Banach state space description. Stability criteria are established, which are based on the dissipativity of the system in addition to another technical condition. The general development is applied to semi-linear systems with time varying nonlinearity. Stability criteria are extracted from the previous conditions. These theoretical results are applied to a class of transport-reaction processes. Different types of nonlinearity are studied by adapting the criteria given in the early portions of the paper.

1. INTRODUCTION

Stability is one of the most important aspects of system theory. The fundamental theory of stability is extensively developed for finite-dimensional systems. Many results on the asymptotic behavior of nonlinear infinite-dimensional systems are known, for which the dissipativity property plays an important role, see e.g. [7], [9], [8], [12] , [16], [17], [18], [19].

In [2] and [3], asymptotic stability was studied for a class of semi-linear infinite-dimensional systems. With respect to the domain of definition of the nonlinearity in the system, two scenarios were treated. The first one deals with semi-linear systems with a nonlinear term defined everywhere on the state space. In this case, some stability criteria were established on the basis of the m-dissipativity concept (see [2, Theorem 12 and Corollary 13]). The second case is when the nonlinear term is not necessarily defined everywhere, but only defined on a closed convex subset of the state space. This case is more important from application point of view due to the fact that some physical limitations are imposed. [2, Theorem 16] proves the asymptotic stability under some technical conditions, which will not be easy to check for practicing control engineers. Motivated by this fact, [4] deals with more investigation on the case when the nonlinearity is defined only on a convex subset by adapting some conditions. The objective of this paper is to extend the results developed in [2] and [4] to the time-varying case. This case has many applications, for example catalytic reactors are most often modelled as plug flow reactors and play an important role in many industrial processes (e.g. methanol, ammonia, sulphuric acid, nitric acid and other petrochemicals). Commonly, the conversion within a catalytic reactor decreases with time, as catalytic deactivation or catalyst decay occurs. As this catalytic deactivation occurs, the chemical kinetics change with time.

The paper is organized as follows. Section 2 contains some basic results on nonlinear evolution system theory. In Section 3, an asymptotic stability criterion in Banach space is proved, which is based on a technical and weaker condition than the m-dissipativity concept. In Section 3, asymptotic stability criteria are established for a class of semi-linear infinite-dimensional systems by applying the result stated in the previous section. Section 4 deals with the asymptotic stability of an infinite-dimensional description of a transport-reaction process model.

2. PRELIMINARIES

Let X be a Banach space and $\|\cdot\|$ is the norm on X. Let D be a closed subset of X and let $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ be a family of non-linear operators from D to X. Let us consider the following abstract Cauchy problem:

$$\begin{cases} \dot{x}(t) \;=\; \mathcal{A}(t)x(t) \\ x(s) \;=\; x_s \end{cases} \tag{1}$$

Let us consider the following conditions:

(C1) For all $(t,x)\in\mathbb{R}\times D$

$$\liminf_{h\to 0^+} d(x+h\mathcal{A}(t)x;D)/h = 0, \tag{2}$$

(C2) For all $x,y\in D$, there exists a bounded linear functional f on X such that $f(x-y)=\|x-y\|^2=\|f\|^2$ and there exists a continuous function $\mu:\mathbb{R}\to\mathbb{R}$ such that for all $t\geq 0$

$$f(\mathcal{A}(t)x-\mathcal{A}(t)y)\leq \mu(t)\|x-y\|. \tag{3}$$

The following theorem is an immediate consequence of [18, Theorem 5.1, p. 238] since all its conditions are satisfied by using (C1) and (C2):

Theorem 1. Assume that D is a closed subset of X and $\mathcal{A}(\cdot)\cdot:\mathbb{R}\times D\to X$ is continuous. Suppose that (C1) and (C2) are satisfied. Then, there is a unique solution $x(t,x_s)$ to the system (1) on the interval $[s,\infty)$. Moreover, if $x'_s\in D$, then for all $t\geq s$:

$$\|x(t,x_s)-x(t,x'_s)\|\leq \|x_s-x'_s\|\exp\left(\int_s^t \mu(r)dr\right) \tag{4}$$

Let us consider the family of operators $\Gamma(t,s):D\to D$ defined as follows

$$\Gamma(t,s)x_s = x(t,x_s) \text{ for all } t\geq s\geq 0 \text{ and } x\in D, \tag{5}$$

where $x(t, x_s)$ is the solution of (1). This family is called *nonlinear evolution system* with generator $\mathcal{A}(t)$ and satisfying the following conditions

(i) $\Gamma(s, s) = I$, for all $s \geq 0$.

(ii) $\Gamma(t, r) = \Gamma(t, s)\Gamma(s, r)$ for all $t \geq s \geq r \geq 0$.

(iii) $\|\Gamma(t, s)x - \Gamma(t, s)x'\| \leq k\|x - x'\|$.

(iv) $\frac{\partial}{\partial t}\Gamma(t, s)x = \mathcal{A}(t)\Gamma(t, s)x$, for all $t \geq s \geq 0$ and $x \in D$.

$\Gamma(t, s)$ is called a *nonlinear contraction evolution system* if the constant $k \leq 1$ in condition (iii).

Let us defined the concept of dissipativity as it will play a very important role along the paper. It is strongly related to the concept of nonlinear contraction evolution system.

Definition 2. Let $\{\mathcal{A}(t)\}_{t\geq 0}$ be a family of nonlinear operators with (possibly non-convex) fixed domain D.
(i) The family $\{\mathcal{A}(t)\}_{t\geq 0}$ is said to be *dissipative* if, $\forall\ t \geq 0$, $x,\ y \in D$ and $\lambda > 0$

$$\|x - y\| \leq \|(x - y) - \lambda(\mathcal{A}(t)x - \mathcal{A}(t)y)\| \qquad (6)$$

or equivalently, $\forall\ t \geq 0$ and $x,\ y \in D$, there exists a bounded linear functional f on X such that

$$f(x - y) = \|x - y\|^2 = \|f\|^2 \text{ and } f(\mathcal{A}(t)x - \mathcal{A}(t)y) \leq 0 \quad (7)$$

(ii) $\mathcal{A}(t)$ is said to be *strictly dissipative* if the conditions above hold with strict inequalities, for all $x,\ y \in D$ such that $x \neq y$.

Remark 3. Note that if X is a Hilbert space with inner product $< \cdot, \cdot >$, then $\{\mathcal{A}(t)\}_{t\geq 0}$ is dissipative if and only if for all $x, y \in D$,

$$Re < \mathcal{A}(t)x - \mathcal{A}(t)y, x - y > \leq 0. \qquad (8)$$

An immediate consequence of Theorem 1 is given in the following Corollary:

Corollary 4. Let $\{\mathcal{A}(t)\}_{t\geq 0}$ be a dissipative family of non-linear operators with closed domain D of X and assume that condition (C1) is satisfied, then $\{\mathcal{A}(t)\}_{t\geq 0}$ generates a unique nonlinear contraction evolution system.

In the following definition, we will define the asymptotic stability property:

Definition 5. Let us consider system (1) and assume that $\{\mathcal{A}\}_{t\geq 0}$ generates a nonlinear contraction evolution system $\Gamma(t, s)$. Let us consider an equilibrium point $\overline{x}$ of (1), i.e $\overline{x} \in D$ and $\mathcal{A}(t)\overline{x} = 0$. $\overline{x}$ is said to be an *asymptotically stable equilibrium point of (1) on D* if

$$\forall x_s \in D \quad \lim_{t\to\infty} x(t, x_s) := \lim_{t\to\infty} \Gamma(t, s)x_s = \overline{x}.$$

Remark 6. Note that the asymptotic stability definition captures only the attractivity part of this property since stability is guaranteed by the fact that this definition is given for a system that generates a nonlinear contraction evolution system. Indeed, $\forall x_s, x'_s \in D$,

$$\|\Gamma(t, s)x_s - \Gamma(t, s)x'_s\| \leq \|x_s - x'_s\|,$$

and by using this inequality for $x'_s = \overline{x}$ and the fact that $\overline{x}$ is a fixed point of the evolution system $\Gamma(t, s)$, we have

$$\|x(t, x_s) - \overline{x}\| \leq \|x_s - \overline{x}\|.$$

3. NON-LINEAR SYSTEMS

In this section, we will present the main results concerning asymptotic stability of non-linear systems. These result are strongly related to the well-known Lasalle's invariance principle. In order to state these results, the following concepts and notation are needed. If $\Gamma(t, s)$ is a nonlinear contraction evolution system on D, for any $x_s \in D$, the *orbit* $\gamma(x_s)$ *through* x_s is defined by

$$\gamma(x_s) := \{\Gamma(t, s)x_s : t \geq s \geq 0\},$$

and the (possibly empty) *ω-limit set,* $\omega(x_s)$ of x_s, is defined (with respect to the strong topology) by

$$\omega(x_s) := \{x \in D : \exists t_n \to \infty \text{ such that } x = \lim_{n\to\infty} \Gamma(t_n, s)x_s\}$$

In what follow, we will assume the following:

(C3) The omega limit set $\omega(x_s)$ is $\Gamma(t, s)$-invariant, i.e. for all $t \geq s \geq 0$, $\Gamma(t, s)\omega(x_s) \subset \omega(x_s)$.

Lemma 7. Let $\{\mathcal{A}(t)\}_{t\geq 0}$be a dissipative family of operators defined on a closed subset D of X and satisfies conditions (C1) and (C3). Let $\Gamma(t, s)$ be the contraction evolution system generated by $\{\mathcal{A}(t)\}_{t\geq 0}$. Then,
(i) For any $x_s \in D$, the ω-limit set $\omega(x_s)$ is closed.
(ii) Assume that $\omega(x_s)$ is nonempty. If $\overline{x}$ is a fixed point of $\Gamma(t, s) : \Gamma(t, s)\overline{x} = \overline{x}$ for all $t \geq s \geq 0$, then

$$\omega(x_s) \subset \{x | \|x - \overline{x}\| = r\}$$

with $r \leq \|x_s - \overline{x}\|$.

Proof: (i) This can be proved by taking a convergent sequence in $\omega(x_s)$ and show that its limit is in $\omega(x_s)$.
(ii) Let us assume that $\overline{x}$ is a fixed point of $\Gamma(t, s)$. First we need to prove that the function $t \to \|\Gamma(t, s)x_s - \overline{x}\|$ is decreasing. Without loss of generality let us assume that $\overline{x} = 0$. Using the dissipativity of $\{\mathcal{A}(t)\}$, it can be shown that

$$\frac{d}{dt}\|\Gamma(t, s)x_s\|^2 \leq 2f(\mathcal{A}(t)\Gamma(t, s)x_s) \leq 0 \text{ a.e. in } (0, \infty)$$

By using the fact that $\Gamma(t, s)$ is a contraction evolution system, we have

$$\|\Gamma(t, s)x_s - \Gamma(t, s)x'_s\| \leq \|x_s - x'_s\|.$$

If we choose, $x'_s = \overline{x}$, then the inequality can be written as

$$\|\Gamma(t, s)x_s - \overline{x}\| \leq \|x_s - \overline{x}\|.$$

Let $r = \lim_{t\to\infty} \|\Gamma(t, s)x_s - \overline{x}\|$, then $r \leq \|x_s - \overline{x}\|$. Let consider $y \in \omega(x_s)$, then there exists a sequence $t_n \to \infty$ such that $y = \lim_{n\to\infty} \Gamma(t_n, s)x_s$. therefore $\|y - \overline{x}\| = \lim_{n\to\infty} \|\Gamma(t_n, s)x_s - \overline{x}\| = r$. Thus,

$$\omega(x_s) \subset \{x | \|x - \overline{x}\| = r\}. \square$$

The main result of this section is given by the following theorem.

Theorem 8. Consider the system (1) with $\{\mathcal{A}(t)\}_{t\geq 0}$ is a dissipative family of nonlinear operators such that (C1) and (C3) hold, and let $\Gamma(t, s)$ be the nonlinear contraction evolution system on D, generated by $\mathcal{A}(t)$. Assume that $\overline{x}$ is the unique equilibrium point of (1) and for all $t \geq 0$, $(I - \lambda\mathcal{A}(t))^{-1}$ is compact for some $\lambda > 0$. Then for any $x_s \in D$, $x(t, x_s) := \Gamma(t, s)x_s$ converges, as $t \to \infty$, to $\omega(x_s)$, i.e.

$$\lim_{t\to\infty} d(x(t, x_s), \omega(x_s)) = 0.$$

Proof: First we need to prove the boundness of $\gamma(x_s)$. Since $\overline{x}$ is an equilibrium point of the system, then $\mathcal{A}(t)\overline{x} = 0, \forall t \geq 0$ and $\Gamma(t,s)\overline{x} = \overline{x}$, for all $t \geq s \geq 0$. We have

$$\begin{aligned}\|\Gamma(t,s)x_s\| &\leq \|\Gamma(t,s)x_s - \Gamma(t,s)\overline{x}\| + \|\Gamma(t,s)\overline{x}\| \\ &\leq \|x_s - \overline{x}\| + \|\overline{x}\|\end{aligned}$$

Then, $\gamma(x_s)$ is bounded. Moreover, it can be proved that $\gamma(x_s)$ is pre-compact and therefore there is a Cauchy sequence $\{\Gamma(t_n,s)x\}$ with $t_n \to \infty$ as $n \to \infty$. The limit of this sequence belongs to the omega limit set, i.e. $\omega(x_s) \neq \emptyset$. Assume that there exists a $\epsilon > 0$ and a sequence $\{\Gamma(t_n,s)x_s,\}, t_n \to \infty$ as $n \to \infty$ such that

$$d(\Gamma(t_n,s)x_s, \omega(x_s)) > \epsilon, \quad n = 1, 2, \ldots$$

Using the precompactness of $\{\Gamma(t_n,s)x_s\}$, there is a convergent subsequence of $\{\Gamma(t_n,s)x_s\}$, its limit belongs to $\omega(x_s)$. Then, a contradiction appears. □

In the following theorem, it is proved that if we add the strict dissipatitvity assumption, then asymptotic stability holds:

Theorem 9. Consider the system (1) as in Theorem 8. If in addition $\mathcal{A}(t)$ is strictly dissipative, then $x(t,x_s) \to \overline{x}$ as $t \to \infty$ i.e. $\overline{x}$ is an asymptotically stable equilibrium point of (1) on D.

Proof: It is enough to prove that the omega limit set reduces to the origin (without loss of generality we assume $\overline{x} = 0$).

4. SEMI-LINEAR SYSTEMS

Let us consider a real reflexive Banach space X equipped with the norm $\|\cdot\|$. Let us consider the following class of semi-linear systems:

$$\begin{cases} \dot{x}(t) \ = A_0 x(t) + N_0(t, x(t)) \\ x(0) \ = x_0 \in D(A_0) \cap F \end{cases}, \tag{9}$$

where A_0 is a linear operator defined on its domain $D(A_0)$ and $N_0 : [0,\infty) \times F \to X$ is a nonlinear operator, where F is a closed convex subset F of X such that the following assumptions hold:

(A1) A_0 generates a C_0-semigroup of contractions on X.

(A2) there exists a positive constant λ such that

$$(I - \lambda A_0)^{-1} \text{ is compact in } X \tag{10}$$

(A3) N_0 is a Lipschitz continuous dissipative nonlinear operator on F, with Lipschitz constant l_0.

(A4) for all $x_0 \in D(A_0) \cap F$, (9) has at least one solution, in the sense of [17, Definition 2.111, p.101], denoted by $x(t, x_0)$.

Remark 10. (a) The existence of the trajectories of this class of models has already been studied by several authors. For instance, [19] contains an investigation of the general abstract model (9), where A_0 is the generator of a strongly continuous semigroup $S_0(t)$ on an abstract Banach space X, and the nonlinear operator N_0 is continuous in t and satisfies Lipschitz conditions in x. It is shown in [19, pp.185-186] that equation (9) has a unique local mild solution on some interval $[0,T])$, $T \in [0,\infty)$ given by

$$x(t) = S_0(t)x_0 + \int_0^t S_0(t-s)N_0(x(s))ds, \quad 0 \leq t \leq T.$$

Moreover, if $T < \infty$ then, $\lim_{t\to T} \|x(t)\| = \infty$.

In order to study the asymptotic stability of system (9), the following lemmas are useful.

Lemma 11. Consider a semi-linear system given by (9) and satisfying (A1)-(A3). Then, for any $t \geq 0$, the operator $(I - \lambda\mathcal{A}(t))^{-1}$ is compact in X, where $\lambda > 0$ is a constant such that (10) holds.

Proof: In order to prove the compactness of the operator $(I - \lambda\mathcal{A}(t))^{-1}$ ($\lambda > 0$ be such that (10) holds), we consider any bounded sequence (v_t^n) in X and prove that the sequence $(u_t^n) := (I - \lambda\mathcal{A}(t))^{-1}v_n$, defined in D, has a converging subsequence. □

Lemma 12. Consider a semi-linear system given by (9) and satisfying conditions (A1), (A3) and (A4). Then condition (C1) holds, i.e. for all $(t,x) \in \mathbb{R} \times D$

$$\liminf_{h\to 0^+} d(x + h\mathcal{A}(t)x; D)/h = 0, \tag{11}$$

Proof: According to [16, Corollary 3.8.1, p.123], it suffices to prove the following equivalent condition:

$$D \subset \cap_{\lambda,t>0}\mathcal{R}(I - \lambda\mathcal{A}(t)).$$

This can be shown by using contraction mapping theorem.□

Remark 13. As a consequence of Lemma 12, if the conditions (A1), (A3) and (A4) hold, then the operator $\mathcal{A}$ generates a unique nonlinear contraction semigroup on D (see Corollary 4).

The following theorem follows directly from Lemmas 11 and 12, and Theorem 8.

Theorem 14. Consider a semi-linear system given by (9) and satisfying conditions (A1)-(A4). Assume that $\overline{x}$ is the unique equilibrium point of (9). Then for any $x_0 \in D$,

$$\lim_{t\to\infty} d(x(t,x_0), \omega(x_0)) = 0. \tag{12}$$

If in addition N_0 is strictly dissipative, then $x(t,x_0) \to \overline{x}$ as $t \to \infty$, i.e. $\overline{x}$ is an asymptotically stable equilibrium point of (9) on D.

Proof: Observe that the compactness of $(I - \lambda\mathcal{A})^{-1}$ follows from Lemma 11. By Lemma 12, condition (11) holds. Then all the conditions of Theorem 8 are satisfied. Finally, the asymptotic behavior follows from Theorem 8 when N_0 is (resp strictly) dissipative. □

In order to treat the case when the nonlinear operator N_0 is not necessarily dissipative, we need these new assumptions:

($A1'$) the linear operator A_0 is the infinitesimal generator of an exponentially stable C_0-semigroup of bounded linear operators $S_0(t)$ on a Banach space X such that

$$\|S_0(t)\| \leq Me^{-\mu t},$$

for all $t \geq 0$, for some $\mu > 0$ and $M \geq 1$.

($A3'$) For all $t \geq 0$, $N_0(t)$ is a Lipschitz continuous nonlinear operator defined on a closed convex subset F of X, with Lipschitz constant $l_0(t)$.

In order to apply Theorem 8 to this case the following lemmas are needed.

Lemma 15. Assume that (A1′) and (A3′) are satisfied. If for all $t \geq 0$, $-\omega \geq Ml_0(t)$ $(-\omega > Ml_0(t))$, then the family of operators $\mathcal{A}(t) := A_0 + N_0(t, \cdot)$ is (strictly) dissipative.

Proof: The proof of this lemma is similar to the one of [2, Lemma10] in the case when $M = 1$. For $M \geq 1$, we can define a new equivalent norm $|\cdot|$ such that $|S_0(t)| \leq e^{\omega t}$ for all $t \geq 0$ (see the proof of [2, Corollary 13]).

The main concept in proving Lemma 11 and 12 is the dissipativity, then the following lemma is a consequence of Lemma 15.

Lemma 16. Consider a a semi-linear system given by (9) and satisfying conditions (A1′), (A2), (A3′), (A4) and (A5). Assume that $-\omega \geq Ml_0(t)$ for all $t \geq 0$, then: (i) for all $t \geq 0$, the operator $(I - \lambda\mathcal{A}(t))^{-1}$ is compact and (ii) the condition (C1) holds.

Now we are in position to state the following result:

Theorem 17. Consider a semi-linear system given by (9) and satisfying conditions (A1′), (A2), (A3′) and (A4). Let $\overline{x}$ be the unique equilibrium point of (9). Assume that for all $t \geq 0$, $l_0(t) \leq \mu/M$. Then for any $x_0 \in F$,

$$\lim_{t\to\infty} d(x(t, x_0), \omega(x_0)) = 0.$$

If in addition for all $t \geq 0$, $l_0(t) < \mu/M$, then $x(t, x_0) \to \overline{x}$ as $t \to \infty$, i.e. $\overline{x}$ is an asymptotically stable equilibrium of (9) on F.

Proof: This result is a direct consequence of Lemmas 16, 15 and Theorem 8. □

Remark 18. Technically, the Lipschitz constant is the greatest lower bound of all constants satisfying the Lipschitz condition. In general, we do not need necessarily to determine the Lipschitz constant. Any constant that satisfies the Lipschitz condition will work. Finding the Lipschitz constant will enlarge the interval of satisfaction of the inequality condition $Ml_0(t) \leq \mu$ in Theorem 17.

5. APPLICATION

Many unit operations in chemical plants include transport processes that can best be described by partial differential equations (PDEs): see [13], [10] and [14]. Industrial catalytic reactors are most often modelled as plug flow reactors and play an important role in many industrial processes (e.g. methanol, ammonia, sulphuric acid, nitric acid and other petrochemicals). Commonly, the conversion within a catalytic reactor decreases with time, as catalytic deactivation or catalyst decay occurs. As this catalytic deactivation occurs, the chemical kinetics change.

Our focus in this section is to apply the results of the previous section to the following semi-linear first-order PDEs system in one spatial dimension:

$$\frac{\partial x}{\partial t} = -\frac{\partial x}{\partial z} + Mx(t) + f(t, x(t)) \quad (13)$$

subject to the boundary and initial conditions given by:

$$x(0, t) = 0, \text{ and } x(z, 0) = x_0(z) \quad (14)$$

where $x(z, t) = [x_1(z, t) \dots x_n(z, t)]^T \in H := L^2(0, l)^n$ denotes the vector of state variables, $z \in [0, l]$ and $t \in [0, \infty)$ denote position and time, respectively. $f = [f_1 \dots f_n]$ is a continuous vector function defined on a closed convex subset $F \subset H$. M is a diagonal matrix with nonpositive entries, i.e. $M = diag(-\alpha_i), i = 1, \dots, n$, where $\alpha_i \geq 0$ and $x_0 \in H$. Without loss of the generality assume that $l = 1$.

Actually, the equivalent state space description of the model (13)-(14) is given by (9), where A_0 is the linear (unbounded) operator defined on its domain

$$D(A_0) := \{x \in H : x \text{ is a.c } \frac{dx}{dz} \in H \text{ and } x(0) = 0\}, \quad (15)$$

by

$$A_0 := -\frac{d.}{dx} + M \cdot I \quad (16)$$

and the nonlinear operator N_0 is defined on F by

$$N_0(t, x) := [f_1(t, x) \dots f_n(t, x)]^T. \quad (17)$$

The following lemma focuses on the properties of the linear operator A_0 (see e.g [21] and [2]).

Lemma 19. Let us consider the linear operator A_0 given by (15)-(16). The following properties hold:
(i) A_0 is m-dissipative and generates an exponentially stable C_0-semigroup $S(t)$. Moreover, for all $t \geq 0$,

$$\|S(t)\| \leq e^{-\alpha t}, \text{ with } \alpha = \min_{1\leq i\leq n} \alpha_i.$$

(ii) There exists a positive constant λ such that the operator $(I - \lambda A_0)^{-1}$ is compact.

In what follow, we need to assume the following condition:

(H) For all $x_0 \in D(A_0) \cap F$, system (13)-(14) has at least one solution denoted by $x(t, x_0)$.

Now we are in a position to state the following corollaries about the asymptotic behavior of the system (9), (15)-(17). Two scenarios are presented. First we are interested in a dissipative nonlinearity (Corollary 20) and second we assume that the nonlinearity is not necessarily dissipative, but only Lipschitz continuous (Corollary 21).

Corollary 20. Consider a semi-linear system given by (9), (15)-(17) and satisfying condition (H). Assume that $f(t, \cdot)$ is Lipschitz continuous dissipative on F. Then for any $x_0 \in F$,

$$\lim_{t\to\infty} d(x(t, x_0), \omega(x_0)) = 0.$$

If in addition the family $f(t, \cdot)$ is strictly dissipative, then the equilibrium point $\overline{x}$ is asymptotically stable on F.

Corollary 21. Consider a semi-linear system given by (9), (15)-(17) and satisfying condition (H). Assume that for all $t \geq 0$, $f(t, \cdot)$ is Lipschitz continuous on F, with a Lipschitz constant $l_0(t)$ such that $l_0(t) \leq \alpha$ for all $t \geq 0$. Then for any $x_0 \in F$,

$$\lim_{t\to\infty} d(x(t, x_0), \omega(x_0)) = 0.$$

If in addition $l_0(t) < \alpha$, then the equilibrium point $\overline{x}$ is asymptotically stable on F.

Note that if there exists $i = 1, \ldots, n$ such that $\alpha_i = 0$, then the constant $\alpha = 0$. In this case, the inequality $l_0(t) < \alpha$ cannot be satisfied because $l_0(t)$ is positive. The following theorem is stated to avoid the above situation.

Theorem 22. Consider a semi-linear system given by (9), (15)-(17) and satisfying condition (H). Assume that f is Lipschitz continuous on F, with a Lipschitz constant l_0 such that $l_0 < e^{-1}$. Then for any $x_0 \in F$,

$$\lim_{t\to\infty} d(x(t, x_0), \overline{x}) = 0,$$

i.e. the equilibrium point $\overline{x}$ is asymptoticall stable on F.

We shall prove Theorem 22 by using the following Lemma (see [11, Theorem 2.1.6 (c)] and its proof for more details).

Lemma 23. Let $S_0(t)$ be a C_0-semigroup and $\omega_0 < 0$ denotes its growth constant. For all $\omega \in (0, -\omega_0)$, there exists a constant M_ω such that

$$\forall t \geq 0, \quad \|S_0(t)\| \leq M_\omega e^{-\omega t},$$

where M_ω is given by the following expression

$$M_\omega = e^{\omega t_\omega} M_{0,\omega}$$

and the constants t_ω and $M_{0,\omega}$ are given, respectively, by

$$t_\omega := \inf\{\tau \geq 0, \quad \|S_0(t)\| \leq e^{-\omega t}, \quad \forall t \geq \tau\}$$
$$M_{0,\omega} = \sup\{\|S_0(t)\|, \quad t \in [0, t_\omega]\}.$$

Proof of Theorem 22: Observe that $S_0(t)$ satisfies the following

$$\|S_0(t)\| = \begin{cases} 1, & 0 \leq t < 1 \\ 0, & t \geq 1 \end{cases}$$

Then, (a) the growth constant is equal $-\infty$, (b) the constant t_ω defined in Lemma 23 is equal 1 and (c) the constant $M_{0,\omega}$ is equal 1. Therefore, the condition $\omega > Ml_0(t)$ in Theorem 17 can be replaced by the weaker condition

$$l_0(t) < \sup_{\omega>0} \left(\omega M_\omega^{-1}\right) = \sup_{\omega>0} \left(\omega e^{-\omega}\right) = e^{-1}. \qquad \square$$

Remark 24. The exponential stability of the linearized model of (13)-(14) is proved in [5, Theorem 4.2] under some regularity conditions while our analysis imposes some dissipativity conditions in order to show the asymptotic stability of the nonlinear system. However, the linearization-based technique can be used only if the initial profile is close enough to the equilibrium profile, our approach guarantees the stability for any initial state on the domain of the system generator.

Example 5.1. Catalytic Plug Flow Reactor

Let us consider a catalytic chemical plug flow reactor with the following two endothermic reactions:

$$A \longrightarrow B \quad \text{and} \quad \text{A} \longrightarrow \text{D}$$

If the kinetics of the first reaction is time-varying, first order with respect to the reactant concentration $C_A(mol/l)$ and has an Arrhenius-type dependance on temperature $T(K)$, and if the kinetics of the second reaction is first order kinetics with respect to the reactant concentration, the dynamics of the process are described by the following energy and mass balance PDEs:

$$\frac{\partial T}{\partial \tau} = -v\frac{\partial T}{\partial z} + k_1(t)\Delta H C_A exp\left(-\frac{\alpha}{T}\right) - \beta(T - T_e) \quad (18)$$

$$\frac{\partial C_A}{\partial \tau} = -v\frac{\partial C_A}{\partial z} - k_1(t)C_A exp\left(-\frac{\alpha}{T}\right) - k_2 C_A \quad (19)$$

with the boundary conditions given for all $\tau \geq 0$, by

$$T(0, \tau) = T_{in} \quad \text{and} \quad \text{C}_\text{A}(0, \tau) = \text{C}_\text{A,in}.$$

In the equations above, k_1 is a continuous function of time, ΔH is the heat of reaction and T_e represents the equilibrium profile of jacket temperature.

Let us consider the following new variables

$$x_1 = \frac{T - T_{in}}{T_{in}} \text{ and } x_2 = \frac{C_A - C_{A,in}}{C_{A,in}}$$

and consider the new time variable $t = \tau v$. Then the plug flow reactor PDE model can be written under the form (13) such that the matrix M is defined by

$$M = \frac{1}{v}\begin{bmatrix} -\beta & 0 \\ 0 & -k_2 \end{bmatrix}$$

and the function f is defined on its domain

$$F = \{(x_1, x_2) \in L^2(0, 1)^2 : x_1 \geq -1 \text{ and } -1 \leq x_2 \leq 0\}$$

by

$$f(t, x_1, x_2) := \frac{1}{v}\begin{bmatrix} a(t)(1 + x_2)\exp\left(\frac{-\lambda}{1 + x_1}\right) + \beta x_e \\ -k_1(t)(1 + x_2)\exp\left(\frac{-\lambda}{1 + x_1}\right) \end{bmatrix}$$

where

$$a(t) = k_1(t)\Delta H\frac{C_{A,in}}{T_{in}}, \; \lambda = \frac{\alpha}{T_{in}} \text{ and } x_e = \frac{T_e - T_{in}}{T_{in}}$$

The Lipschitz constant of f can be computed by using similar calculations to [2], and it is given by

$$l_0(t) = \max\left(1, \frac{-\Delta H C_{A,in}}{T_{in}}\right)\frac{k_1(t)}{v}\left[1 + \frac{4}{\lambda e^2}\right],$$

The stability condition is

$$\max\left(1, \frac{-\Delta H C_{A,in}}{T_{in}}\right)k_1(t)\left[1 + \frac{4}{\lambda e^2}\right] < \min(\beta, k_2)$$

Remark 25. Note that if the jacket temperature is used as a state feedback, then the function f will change and then the Lipschitz condition must be checked for the new function.

6. CONCLUSION

In this paper, we have studied the asymptotic stability property of a class of dissipative time-varying infinite-dimensional systems with nonlinearity not defined everywhere. First the stability property was studied for non-linear dissipative systems by using the concept of nonlinear contraction evolution system. Secondly, we have studied a class of semi-linear systems with time-varying nonlinearity. Finally, the previous results were applied to a class of transport-reaction systems that can be described by first-order time-varying hyperbolic PDEs.

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