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A New Approach for the H_2 control of Markov Jump Linear Systems with Partial Information

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Abstract: The goal of this paper is to study the H_2-control for Markov Jump Linear Systems (MJLS) assuming that the controller doesn't have access to the Markov parameter but, instead, there is a detector that emits signals which provides information on this parameter. The idea is to use the information provided by this detector in order to design a feedback linear control that stochastically stabilizes the closed loop system. A Linear Matrix Inequalities (LMI) formulation is provided in order to achieve this goal. In the sequel we deal with the H_2 control problem and we show that again an LMI optimization problem can be formulated in order to design a stochastically stabilizing feedback control with guaranteed H_2-cost. We also present some conditions under which our results recast the usual results for the H_2 control of MJLS as presented in Costa et al. (2004). The case with convex polytopic uncertainty on the parameters of the system and on the transition probability matrix is also considered.

Keywords: Linear systems, jump process, incomplete data, H_2 control.

1. INTRODUCTION

This paper deals with the H_2 control problem for Markov jump linear systems (MJLS) under partial information on the jump parameter. We assume that the controller doesn't have direct access to the Markov parameter but, instead, there is a detector that emits signals which provides information on this parameter. Regarding the MJLS, the partial observation problem may be associated either with the state variable, the Markov chain, or yet with both variables, which is of course the hardest problem. For the control problem with partial observations of the Markov chain the readers are referred, for instance, to Caines and Zhang (1995); Everdij and Blom (1996); Fragoso (1988). The case with partial information of the state and perfect measurement of the Markov chain (including the H_2 control problem) is treated, for instance, in Costa and Fragoso (2007); Costa and Tuesta (2003); de Farias et al. (2000); Dufour and Elliott (1998); Fragoso and Costa (2010). The case in which both the state variable and the Markov chain are only partially observable was also studied in Dufour and Elliott (1998). The H_2-norm control problem of discrete-time Markov jump linear systems when part of, or the total of the Markov states is not accessible to the controller was addressed in do Val et al. (2002). In this case the non-observed part of the Markov states is grouped in a number of clusters of observations, with the case of a single cluster retrieving the situation when no Markov state is observed.

In this paper we first analyze the stochastic stabilizability problem through a feedback control for the MJLS, using the signal from the detector instead of the unknown Markov parameter. We show that the existence of a solution to a set of LMIs provides a stochastically stabilizing feedback gain for the MJLS. In the sequel it is provided an LMI optimization formulation in order to design a stochastically stabilizing feedback control with guaranteed H_2-cost. Notice that following this approach it is possible to get explicit numerical tools for the H_2 control problem of MJLS with partial information, unlike other approaches as, for instance, in Caines and Zhang (1995); Everdij and Blom (1996); Fragoso (1988, 1990). We also present conditions (one of them always satisfied for the limit case in which the detector provides perfect information on the Markov parameter), under which our results recast the usual results for the H_2 control of MJLS as presented in Costa et al. (2004).

The paper is organized as follows. In section 2 we present the notation adopted in this paper. In section 3 we introduce the problem formulation, assumptions, the definition of stochastic stabilizability, and some auxiliary results. In section 4 we present in Theorem 6 the first main result of the paper, dealing with the stochastic stabilizability problem. In section 5 we present in Theorem 15 the second main result of the paper, dealing with the guaranteed H_2-control problem. Section 6 deals with the case in which there is convex polytopic uncertainty on the the matrices of the system and on the transition probability matrix of the Markov chain. The paper is concluded in Section 7 with some numerical examples.

2. NOTATION

For $\mathbb{X}$ and $\mathbb{Y}$ complex Banach spaces we set $\mathbb{B}(\mathbb{X},\mathbb{Y})$ the Banach space of all bounded linear operators of $\mathbb{X}$ into $\mathbb{Y}$, with the uniform induced norm represented by $\|.\|$. For simplicity we set $\mathbb{B}(\mathbb{X}) \triangleq \mathbb{B}(\mathbb{X},\mathbb{X})$. The spectral radius of an operator $\mathcal{T} \in \mathbb{B}(\mathbb{X})$ is denoted by $r_\sigma(\mathcal{T})$. If $\mathbb{X}$ is a Hilbert space then the inner product is denoted by $\langle .;. \rangle$, and for $\mathcal{T} \in \mathbb{B}(\mathbb{X})$, $\mathcal{T}^*$ denotes the adjoint operator of $\mathcal{T}$. As usual, $\mathcal{T} \geq 0$ ($\mathcal{T} > 0$ respectively) will denote that the operator $\mathcal{T} \in \mathbb{B}(\mathbb{X})$ is positive-semi-definite (positive-definite). In particular, we denote respectively by $\mathbb{R}^n$ and $\mathbb{C}^n$ the n dimensional real and complex Euclidean spaces and $\mathbb{B}(\mathbb{C}^n,\mathbb{C}^m)$ ($\mathbb{B}(\mathbb{R}^n,\mathbb{R}^m)$ respectively) the normed bounded linear space of all $m \times n$ complex (real) matrices, with $\mathbb{B}(\mathbb{C}^n) \triangleq \mathbb{B}(\mathbb{C}^n,\mathbb{C}^n)$ ($\mathbb{B}(\mathbb{R}^n) \triangleq \mathbb{B}(\mathbb{R}^n,\mathbb{R}^n)$). Unless otherwise stated, $\|.\|$ will denote the standard norm in $\mathbb{C}^n$, and for $M \in \mathbb{B}(\mathbb{C}^n,\mathbb{C}^m)$, $\|M\|$ denotes the induced uniform norm in $\mathbb{B}(\mathbb{C}^n,\mathbb{C}^m)$. The superscript * indicates the conjugate transpose of a matrix, while $'$ indicates the transpose. Clearly for real matrices * and $'$ will have the same meaning. The identity matrix will be denoted by I and the trace operator by $\mathrm{tr}(.)$. For N integer set $\mathbb{N} \triangleq \{1,\ldots,N\}$. Define $\mathbb{H}^{n,m}$ as the linear space made up of all N-sequences of complex matrices $V = (V_1,\ldots,V_N)$ with $V_i \in \mathbb{B}(\mathbb{C}^n,\mathbb{C}^m)$, $i \in \mathbb{N}$. For simplicity, we set $\mathbb{H}^n \triangleq \mathbb{H}^{n,n}$ and $\mathbb{H}^{n+}$ such that $V = (V_1,\ldots,V_N) \in \mathbb{H}^{n+}$ if $V \in \mathbb{H}^n$ and $V_i \geq 0$ for each $i \in \mathbb{N}$. It is easy to verify that $\mathbb{H}^{n,m}$ is a Hilbert space when equipped with the inner product $\langle .;. \rangle$ given, for $V = (V_1,\ldots,V_N)$ and $S = (S_1,\ldots,S_N)$ in $\mathbb{H}^{n,m}$, by

$$\langle V; S \rangle \triangleq \sum_{i=1}^{N} \mathrm{tr}(V_i^* S_i). \tag{1}$$

For $M_i \in \mathbb{B}(\mathbb{C}^n,\mathbb{C}^m)$, $i \in \mathbb{N}$, we set $\mathrm{diag}[M_i]$ the $Nm \times Nn$ block diagonal matrix formed with M_i in the diagonal and zero elsewhere, that is,

$$\mathrm{diag}[M_i] \triangleq \begin{bmatrix} M_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_N \end{bmatrix}.$$

We define the operators φ and $\hat{\varphi}$ in the following way: for $V = (V_1,\ldots,V_N) \in \mathbb{H}^{n,m}$, considering $V_i = [v_{i1} \ \cdots \ v_{in}] \in \mathbb{B}(\mathbb{C}^n,\mathbb{C}^m)$, $v_{ij} \in \mathbb{C}^m$

$$\varphi(V_i) \triangleq \begin{bmatrix} v_{i1} \\ \vdots \\ v_{in} \end{bmatrix} \in \mathbb{C}^{mn} \quad \text{and} \quad \hat{\varphi}(V) \triangleq \begin{bmatrix} \varphi(V_1) \\ \vdots \\ \varphi(V_N) \end{bmatrix} \in \mathbb{C}^{Nmn}.$$

For two matrices A and B in $\mathbb{B}(\mathbb{C}^n)$ we consider $A \otimes B \in \mathbb{B}(\mathbb{C}^{n^2})$ as the Kronecker product between these matrices (see Brewer (1978)).

Consider the stochastic basis $(\Omega, \mathcal{P}, \mathcal{F}, \{\mathcal{F}_k\})$ and denote by $E(.)$ the expected value operator, and by $E(.|.)$ the conditional expected value. We denote by L_2^n the Hilbert space of sequences of random vectors $z(k)$, $z(k): \Omega \to \mathbb{R}^n$, with $z(k)$ $\mathcal{F}_k$-measurable, such that

$$\|z\|_2^2 \triangleq \sum_{k=0}^{\infty} E(\|z(k)\|^2) < \infty. \tag{2}$$

For $A \in \mathcal{F}$ we set 1_A as the Dirac measure or equivalently, the indicator function of the event A (thus $1_A(\omega) = 1$ if $\omega \in A$, and 0 otherwise).

3. PROBLEM FORMULATION, ASSUMPTIONS AND DEFINITIONS

We consider in this section the following controlled discrete-time linear system with Markov jumps on a probabilistic space $(\Omega, \mathcal{P}, \mathcal{F})$:

$$x(k+1) = A_{\theta(k)} x(k) + B_{\theta(k)} u(k), \tag{3}$$
$$x(0) = x_0, \ \ \theta(0) = \theta_0. \tag{4}$$

Here the state variable is given by $x(k) \in \mathbb{R}^n$ and the control variable by $u(k) \in \mathbb{R}^m$. We consider that $\theta(k)$ is a Markov chain taking values in the set $\mathbb{N} = \{1,\ldots,N\}$ with transition probability matrix $\mathbb{P} = [p_{ij}]$ satisfying:

Hypothesis 1. For each $j \in \mathbb{N}$,

$$\sum_{i=1}^{N} p_{ij} > 0. \tag{5}$$

We assume that $\theta(k)$ *is not directly observed* but, instead, there is a finite set $\mathbb{M} = \{1,\ldots M\}$ such that a signal $\widehat{\theta}(k) \in \mathbb{M}$ is emitted associated to the Markov chain $\theta(k)$, independently of all previous and present values of the other processes. More precisely, let $\widehat{\mathcal{F}}_0$ be the σ-field generated by $\{x(0), u(0), \theta(0)\}$ and $\widehat{\mathcal{F}}_k$ be the σ-field generated by $\{x(0),\, u(0),\, \theta(0),\, \widehat{\theta}(0),\, \ldots,\, x(k),\, u(k),\, \theta(k)\}$ (therefore excluding $\widehat{\theta}(k)$ at time k). We assume that $\widehat{\theta}(k) \in \{1,\ldots,M\}$ is related to $\theta(k)$ in such a way that

$$P(\widehat{\theta}(k) = \ell \mid \widehat{\mathcal{F}}_k) = P(\widehat{\theta}(k) = \ell \mid \theta(k)) = \alpha_{\theta(k)\ell}, \ \ \ell \in \mathbb{M}, \tag{6}$$

with $\sum_{\ell=1}^{M} \alpha_{i\ell} = 1$ for each $i \in \mathbb{N}$. Therefore we have that at each time k we observe the signal $\widehat{\theta}(k)$. We define for each $i \in \mathbb{N}$,

$$\mathcal{I}_i \triangleq \{\ell \in \mathbb{M}; \alpha_{i\ell} > 0\} = \{k_1^i,\ldots,k_{\tau^i}^i\},$$
$$\mathcal{I} \triangleq \bigcup_{i=1}^{N} \mathcal{I}_i \subset \mathbb{M}.$$

We have 2 extreme situations:

a) $M = N$ and $\alpha_{ii} = 1$, for $i \in \mathbb{N}$, which would correspond to the situation in which $\widehat{\theta}(k) = \theta(k)$, that is, $\theta(k)$ is known. In this case $\mathcal{I}_i = \{i\}$ and $\mathcal{I} = \mathbb{N}$.
b) $M = 1$ and $\alpha_{i1} = 1$ for all $i \in \mathbb{N}$, which corresponds to the situation in which $\widehat{\theta}(k)$ doesn't provide any information about $\theta(k)$, that is, $\theta(k)$ is unknown.

We will consider state-feedback controls using the observed emitted signal $\widehat{\theta}(k)$ instead of the unknown variable $\theta(k)$, that is, $u(k)$ will be of the following form:

$$u(k) = K_{\widehat{\theta}(k)} x(k), \tag{7}$$

for $K_\ell \in \mathbb{B}(\mathbb{R}^n,\mathbb{R}^m)$, $\ell \in \mathcal{I}$. Associated to a control as in (7) set for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$,

$$A_{i\ell} \triangleq A_i + B_i K_\ell. \tag{8}$$

We define for each $i \in \mathbb{N}$ the following operators $\mathcal{E}$, $\mathcal{T}$, $\mathcal{L}$ in $\mathbb{B}(\mathbb{H}^n)$. For $V = (V_1,\ldots,V_N) \in \mathbb{H}^n$, and $i,j \in \mathbb{N}$,

$$\mathcal{E}_i(V) = \sum_{j=1}^{N} p_{ij} V_j, \tag{9}$$

$$\mathcal{T}_j(V) = \sum_{i=1}^{N} \sum_{\ell \in \mathcal{I}_i} p_{ij} \alpha_{i\ell} A_{i\ell} V_i A'_{i\ell}, \tag{10}$$

$$\mathcal{L}_i(V) = \sum_{\ell \in \mathcal{I}_i} \alpha_{i\ell} A'_{i\ell} \mathcal{E}_i(V) A_{i\ell}. \tag{11}$$

For $R = \{R_{i\ell} \in \mathbb{B}(\mathbb{R}^n); i \in \mathbb{N},\ \ell \in \mathcal{I}_i\}$, define for $j \in \mathbb{N}$,

$$\mathcal{D}_j(R) = \sum_{i=1}^{N} \sum_{\ell \in \mathcal{I}_i} p_{ij} \alpha_{i\ell} R_{i\ell}. \tag{12}$$

The following result will be useful in the sequel.

Proposition 2. We have that $\mathcal{T}^* = \mathcal{L}$.

Proof. See Costa et al. (2014). □

We recall the following definition of stochastic stabilizability.

Definition 3. We say that system (3) is stochastically stabilizable if there exists $K_\ell \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, $\ell \in \mathcal{I}$, such that for $u(k)$ as in (7) we have, for every initial condition x_0 with finite second moment and every initial Markov state θ_0, that

$$\|x\|_2^2 = \sum_{k=0}^{\infty} E(\|x(k)\|^2) < \infty. \tag{13}$$

We denote by $\mathcal{K}$ the set of feedback gains $K = \{K_\ell; \ell \in \mathcal{I}\}$, such that stochastically stabilizes system (3).

Consider $u(k)$ as in (7). Define

$$Q_i(k) = E(x(k)x(k)^* 1_{\{\theta(k)=i\}}), i \in \mathbb{N},$$

and $Q(k) = (Q_1(k) \ldots, Q_N(k)) \in \mathbb{H}^n$. Let $\mathcal{F}_k$ stand for the σ-field generated by $\{x(0), \theta(0), \widehat{\theta}(0), \ldots, x(k), \theta(k), \widehat{\theta}(k)\}$. The next proposition gives a time evolution for $Q(k)$.

Proposition 4. We have that

$$Q(k+1) = \mathcal{T}(Q(k)). \tag{14}$$

Proof. See Costa et al. (2014). □

Define the matrices

$$\Phi_{ij} = \begin{bmatrix} p_{ij}\alpha_{ik_1^i} I & \cdots & p_{ij}\alpha_{ik_{\tau^i}^i} I \end{bmatrix}, \quad \Phi = \begin{bmatrix} \Phi_{11} & \cdots & \Phi_{N1} \\ \vdots & \ddots & \vdots \\ \Phi_{1N} & \cdots & \Phi_{NN} \end{bmatrix},$$

$$\Psi_i = \begin{bmatrix} A_{ik_1^i} \otimes A_{ik_1^i} \\ \vdots \\ A_{ik_{\tau^i}^i} \otimes A_{ik_{\tau^i}^i} \end{bmatrix}, \quad \Psi = \operatorname{diag}[\Psi_i], \quad \mathcal{A} = \Phi\Psi$$

$$\begin{aligned} &\mathcal{A} \in \mathbb{B}(\mathbb{R}^{Nn^2});\ \Phi \in \mathbb{B}(\mathbb{R}^{n^2(\tau^1+\ldots+\tau^N)}, \mathbb{R}^{Nn^2}) \\ &\Psi_i \in \mathbb{B}(\mathbb{R}^{n^2}, \mathbb{R}^{n^2\tau^i}),\ \Psi \in \mathbb{B}(\mathbb{R}^{Nn^2}, \mathbb{R}^{n^2(\tau^1+\ldots+\tau^N)}). \end{aligned} \tag{15}$$

We have the following matricial representation for the time evolution of $Q(k)$.

Proposition 5. We have that

$$\hat{\varphi}(Q(k+1)) = \mathcal{A}\hat{\varphi}(Q(k)). \tag{16}$$

Proof. See Costa et al. (2014). □

4. STOCHASTIC STABILIZABILITY

In this section we present conditions for stochastic stabilizability of system (3). In subsection 4.1 we derive conditions (sufficient and necessary and sufficient) for the general Markov chain setting. In this scenario, the LMI formulation gives a sufficient condition for stochastic stabilizability. Subsection 4.2 deals with some special cases (including the Bernoulli jump case, which corresponds to the situation in which $p_{ij} = p_j > 0$ for all i, j) where we derive necessary and sufficient LMI conditions.

4.1 General Markov Chain

The following result presents conditions for stochastic stabilizability of system (3).

Theorem 6. Consider the following assertions:

i) System (3) is stochastically stabilizable.
ii) There exists $K_\ell \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, $\ell \in \mathcal{I}$ such that for $A_{i\ell}$ as in (8), and $\mathcal{A}$ as in (15), we have that $r_\sigma(\mathcal{A}) < 1$ (or equivalently, $r_\sigma(\mathcal{T}) < 1$ or $r_\sigma(\mathcal{L}) < 1$).
iii) There exists $K_\ell \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, $\ell \in \mathcal{I}$ and $V \in \mathbb{H}^n$, $V > 0$, such that for $A_{i\ell}$ as in (8),
$$V - \mathcal{T}(V) > 0. \tag{17}$$
iv) There exists $K_\ell \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, $\ell \in \mathcal{I}$ and $P \in \mathbb{H}^n$, $P > 0$, such that for $A_{i\ell}$ as in (8),
$$P - \mathcal{L}(P) > 0. \tag{18}$$
v) There exists $K_\ell \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, G_ℓ, $\ell \in \mathcal{I}$, $R_{i\ell}$, $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$, such that for $A_{i\ell}$ as in (8),
$$\begin{bmatrix} R_{i\ell} & A_{i\ell} G_\ell \\ \star & G_\ell + G'_\ell - \mathcal{D}_i(R) \end{bmatrix} > 0, \quad i \in \mathbb{N},\ \ell \in \mathcal{I}_i. \tag{19}$$

We have that i) ⇔ ii) ⇔ iii) ⇔ iv) and v) $\Longrightarrow$ iii).

Proof. See Costa et al. (2014). □

4.2 Some Necessary and Sufficient LMI Conditions

Let us provide next a hypothesis so that all assertions in Theorem 6 are equivalent.

Hypothesis 7. Assume that $M \le N$, $\mathbb{M} \subset \mathbb{N}$, and for each $i, k \in \mathbb{N}$ and $\xi \in \mathcal{I}_k$, we have that $p_{i\xi} = p_{ik}$.

Notice that the case $M = N$ and $\alpha_{ii} = 1$ for $i \in \mathbb{N}$, so that $\mathcal{I}_i = \{i\}$ (which corresponds to the situation in which $\widehat{\theta}(k) = \theta(k)$, that is, $\theta(k)$ is known) satisfies Hypothesis 7 since that in this case $\mathcal{I}_k = \{k\}$ and clearly $p_{i\xi} = p_{ik}$ for $\xi \in \mathcal{I}_k$. The case in which $p_{ij} = \frac{1}{N}$ for all $i, j \in \mathbb{N}$ (which corresponds to the situation in which all modes are independent and equally like to occur) also satisfies Hypothesis 7. In what follows set $\mathbf{I} = (\mathbf{I}_1, \ldots, \mathbf{I}_N) \in \mathbb{H}^{n+}$ as (notice that, from Hypothesis 1, $\sum_{i=1}^{N} p_{ij} > 0$)

$$\mathbf{I}_j \triangleq \sum_{i=1}^{N} p_{ij} I > 0. \tag{20}$$

We have the following proposition:

Proposition 8. If Hypothesis 7 is satisfied then in Theorem 6 we have that i) $\Longrightarrow$ v).

Proof. See Costa et al. (2014). □

From Theorem 6 and Proposition 8 we have the following corollary.

Corollary 9. If there exists L_ℓ, G_ℓ, $R_{i\ell}$, $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$, such that

$$\begin{bmatrix} R_{i\ell} & A_i G_\ell + B_i L_\ell \\ \star & G_\ell + G'_\ell - \mathcal{D}_i(R) \end{bmatrix} > 0, \quad i \in \mathbb{N},\ \ell \in \mathcal{I}_i \tag{21}$$

then system (3) is stochastically stabilizable with $K_\ell = L_\ell G_\ell^{-1}$. Moreover if Hypothesis 7 holds then system (3) is stochastically stabilizable if and only if there exists L_ℓ, G_ℓ, $R_{i\ell}$, $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$ satisfying (21).

Proof. See Costa et al. (2014). □

The Bernoulli Jump Case:

We consider now another particular scenario in which again a condition written as an LMI formulation is necessary and sufficient for stochastic stabilizability of system (3). This is achieved by assuming that the transition probabilities of the Markov chain satisfy:

Hypothesis 10. For some $p_1 > 0, \ldots, p_N > 0$ we have that

$$p_{ij} = p_j, \quad \forall i, j \in \mathbb{N}, \tag{22}$$

which is the so-called Bernoulli jump case. Defining now the operator

$$\bar{\mathcal{D}}(R) = \sum_{i=1}^{N} \sum_{\ell \in \mathcal{I}_i} p_i \alpha_{i\ell} R_{i\ell}, \tag{23}$$

we have the following result:

Theorem 11. The following assertions are equivalent.

a) System (3) is stochastically stabilizable.
b) There exists $K_\ell \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, $\ell \in \mathcal{I}$ and $X > 0$, such that for $A_{i\ell}$ as in (8),

$$X - \sum_{i=1}^{N} \sum_{\ell \in \mathcal{I}_i} p_i \alpha_{i\ell} A_{i\ell} X A'_{i\ell} > 0. \tag{24}$$

c) There exists $K_\ell \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, G, $R_{i\ell}$, $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$, such that for $A_{i\ell}$ as in (8),

$$\begin{bmatrix} R_{i\ell} & A_{i\ell} G \\ \star & G + G' - \bar{\mathcal{D}}(R) \end{bmatrix} > 0, \quad i \in \mathbb{N},\ \ell \in \mathcal{I}_i. \tag{25}$$

Proof. see Costa et al. (2014) □

From Theorem 11 the following corollary is immediate.

Corollary 12. System (3) is stochastically stabilizable if and only if there exists L_ℓ, G, $R_{i\ell}$, $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$ such that

$$\begin{bmatrix} R_{i\ell} & A_i G + B_i L_\ell \\ \star & G + G' - \bar{\mathcal{D}}(R) \end{bmatrix} > 0, \quad i \in \mathbb{N},\ \ell \in \mathcal{I}_i. \tag{26}$$

Moreover if (26) holds then system (3) is stochastically stabilizable with $K_\ell = L_\ell G^{-1}$.

Proof. See Costa et al. (2014). □

5. H_2-CONTROL

5.1 General Markov Chain

We consider now the following controlled discrete-time linear system with Markov jumps:

$$x(k+1) = A_{\theta(k)} x(k) + B_{\theta(k)} u(k) + E_{\theta(k)} w(k), \tag{27}$$

$$z(k) = C_{\theta(k)} x(k) + D_{\theta(k)} u(k), \tag{28}$$

$$x(0) = x_0, \ \theta(0) \sim \mu. \tag{29}$$

where the output variable is given by $z(k) \in \mathbb{R}^p$, the input variable by $w(k) \in \mathbb{R}^r$ and the initial probability for θ_0 given by $\mathcal{P}(\theta_0 = i) = \mu_i > 0$, $i \in \mathbb{N}$. In what follows we assume, for the definition of the H_2 norm, that $r_\sigma(\mathcal{A}) < 1$ (see (15)) and that $e_s \in \mathbb{R}^r$ is the vector formed by zeros except for the s^{th} element, which is equal to 1. In addition, consider system (27) with a stochastically stabilizing control $u(k) = K_{\widehat{\theta}(k)} x(k)$ as in (7). Set for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$,

$$C_{i\ell} = C_i + D_i K_\ell.$$

The H_2 norm associated to system (27)-(28) with the feedback control gain K_ℓ, $\ell \in \mathcal{I}$, denoted by $\|\mathcal{G}_K\|_2$, is defined as follows.

Definition 13. Denote by z_s the output (28) with initial condition $x_0 = 0$ and $\mu_i = \mathcal{P}(\theta_0 = i) > 0$, $i \in \mathbb{N}$, in (29) and $w(k) = 0$ for $k \geq 1$ and $w(0) = e_s$. The H_2 norm is defined as

$$\|\mathcal{G}_K\|_2^2 = \sum_{s=1}^{r} \|z_s\|_2^2. \tag{30}$$

Notice that this definition is as in Definition 4.7 of Costa et al. (2004), and when restricted to the so-called deterministic case ($N = 1$ and $p_{11} = \alpha_{11} = 1$), it reduces to the usual H_2-norm. As in Proposition 4.8 of Costa et al. (2004) we have the following result.

Proposition 14. Let $V \in \mathbb{H}^{n+}$ and $P \in \mathbb{H}^{n+}$ be the unique solution of the observability and controllability Grammians respectively:

$$V = \mathcal{T}(V) + \mathbf{E}, \text{ (controllability Grammian)} \tag{31}$$

$$P = \mathcal{L}(P) + \mathbf{C}, \text{ (observability Grammian)} \tag{32}$$

where $\mathbf{E} = (\mathbf{E}_1, \ldots, \mathbf{E}_N) \in \mathbb{H}^n$, $\mathbf{C} = (\mathbf{C}_1, \ldots, \mathbf{C}_N) \in \mathbb{H}^n$ are given by

$$\mathbf{E}_j = \sum_{i=1}^{N} \mu_i p_{ij} E_i E'_i, \quad \mathbf{C}_i = \sum_{\ell \in \mathcal{I}_i} \alpha_{i\ell} C'_{i\ell} C_{i\ell}.$$

Then

$$\|\mathcal{G}_K\|_2^2 = \sum_{i=1}^{N} \sum_{\ell \in \mathcal{I}_i} \alpha_{i\ell} \operatorname{tr}(C_{i\ell} V_i C'_{i\ell}) = \sum_{i=1}^{N} \mu_i \operatorname{tr}(E'_i \mathcal{E}_i(P) E_i).$$

Proof. See Costa et al. (2014). □

Consider now the following LMI optimization problem.

$$\Upsilon = \inf_{W_{i\ell}, R_{i\ell}, G_\ell, L_\ell} \sum_{i=1}^{N} \sum_{\ell \in \mathcal{I}_i} \alpha_{i\ell} \operatorname{tr}(W_{i\ell})$$

subject to: for $i \in \mathbb{N}$, $\ell \in \mathcal{I}_i$,

$$\begin{bmatrix} R_{i\ell} - \mu_i E_i E'_i & A_i G_\ell + B_i L_\ell \\ \star & G_\ell + G'_\ell - \mathcal{D}_i(R) \end{bmatrix} > 0, \tag{33}$$

$$\begin{bmatrix} W_{i\ell} & C_i G_\ell + D_i L_\ell \\ \star & G_\ell + G'_\ell - \mathcal{D}_i(R) \end{bmatrix} \geq 0. \tag{34}$$

From Proposition 14 we get the following result.

Theorem 15. For any feasible solution $W_{i\ell}$, $R_{i\ell}$, G_ℓ, L_ℓ of the LMIs (33), (34), we get that $K = \{K_\ell = L_\ell G_\ell^{-1}, \ell \in \mathcal{I}\} \in \mathcal{K}$, and $\|\mathcal{G}_K\|_2^2 \leq \sum_{i=1}^{N} \sum_{\ell \in \mathcal{I}_i} \alpha_{i\ell} \operatorname{tr}(W_{i\ell})$. Thus, $\inf_{K \in \mathcal{K}} \|\mathcal{G}_K\|_2^2 \leq \Upsilon$.

Proof. See Costa et al. (2014). □

5.2 Some Strengthened Results

As in Proposition 8, for the case in which Hypothesis 7 is satisfied, Theorem 15 can be strengthened as follows.

Proposition 16. If Hypothesis 7 holds then in Theorem 15 we have that $\inf_{K\in\mathcal{K}} \|\mathcal{G}_K\|_2^2 = \Upsilon$.

Proof. See Costa et al. (2014). □

The Bernoulli Jump Case:

We suppose now that Hypothesis 10 holds. The first result on H_2 analysis goes next.

Proposition 17. We have that

$$\|\mathcal{G}_K\|_2^2 = \sum_{i=1}^{N}\sum_{\ell\in\mathcal{I}_i} p_i\alpha_{i\ell}\,\mathrm{tr}\{C_{i\ell}XC'_{i\ell}\}, \tag{35}$$

where $X \in \mathbb{R}^{n\times n}$ is the unique solution to the following generalized Lyapunov equation:

$$X = \sum_{i=1}^{N}\sum_{\ell\in\mathcal{I}_i} p_i\alpha_{i\ell}A_{i\ell}XA'_{i\ell} + \sum_{i=1}^{N}\mu_i E_i E'_i. \tag{36}$$

Proof. See Costa et al. (2014) □

The preceding proposition yields the following result, regarding the *synthesis* of H_2 optimal controllers. Set

$$\zeta(W) = \sum_{i=1}^{N}\sum_{\ell\in\mathcal{I}_i} p_i\alpha_{i\ell}\,\mathrm{tr}\{W_{i\ell}\}. \tag{37}$$

It will be shown next that the design of H_2 optimal controllers may be expressed by the following optimization problem, with decision matrices $W_{i\ell}$, G, $R_{i\ell}$, and L_ℓ of appropriate dimensions for $\ell\in\mathcal{I}$, $i\in\mathbb{N}$:

$$\widehat{\Upsilon} = \inf\zeta(W) \tag{38a}$$

subject to

$$\begin{bmatrix} W_{i\ell} & C_iG + D_iL_\ell \\ \star & G + G' - \bar{\mathcal{D}}(R)\end{bmatrix} > 0, \tag{38b}$$

$$\begin{bmatrix} R_{i\ell} - \frac{\mu_i}{p_i}E_iE'_i & A_iG + B_iL_\ell \\ \star & G + G' - \bar{\mathcal{D}}(R)\end{bmatrix} > 0. \tag{38c}$$

Theorem 18. For any feasible solution of the LMIs (38b), (38c), we get that $K = \{K_\ell = L_\ell G^{-1},\ \ell\in\mathcal{I}\}\in\mathcal{K}$, and $\|\mathcal{G}_K\|_2^2 \le \zeta(W)$. Moreover $\inf_{K\in\mathcal{K}}\|\mathcal{G}_K\|_2^2 = \widehat{\Upsilon}$.

Proof. See Costa et al. (2014) □

6. CONVEX POLYTOPIC UNCERTAINTY

In this section we consider the matrices $[A\ B] \triangleq [A_1\ \ldots\ A_N\ B_1\ \ldots\ B_N]$ in (3) or (27) and the transition probability matrix $\mathbb{P}$ are not precisely known but belong to a polytopic domain, that is, $([A\ B], \mathbb{P}) \in \Gamma$, where

$$\Gamma \triangleq \Big\{\Big([A\ B], \mathbb{P}\Big); [A\ B] = \sum_{s=1}^{\xi}\gamma_s\,[A^s\ B^s],$$
$$\mathbb{P} = \sum_{s=1}^{\xi}\gamma_s\mathbb{P}^s,\ \gamma\in\Lambda\Big\},$$

and $\Lambda \triangleq \{\gamma = (\gamma_1,\ldots,\gamma_\xi); \sum_{s=1}^{\xi}\gamma_s = 1,\ \gamma_s\ge 0,\ s = 1,\ldots,\xi\}$, with $[A^s\ B^s] = [A_1^s\ \ldots\ A_N^s\ B_1^s\ \ldots\ B_N^s]$ and the transition probability matrices $\mathbb{P}^s = [p_{ij}^s]$, $s = 1,\ldots,\xi$, assumed to be known. We present next the definition of robust stochastic stabilizability.

Definition 19. We say that system (3) is robust stochastically stabilizable if there exists $K_\ell \in \mathbb{B}(\mathbb{R}^n, \mathbb{R}^m)$, $\ell\in\mathcal{I}$, such that for $u(k)$ as in (7) we have, for every $\Big([A\ B], \mathbb{P}\Big) \in \Gamma$, every initial condition x_0 with finite second moment and every initial Markov state θ_0, that (13) holds. We denote by $\mathcal{K}_r$ the set of feedback gains $K = \{K_\ell; \ell\in\mathcal{I}\}$, such that robust stochastically stabilizes system (3).

We present next a sufficient condition for the existence of $K\in\mathcal{K}_r$.

Proposition 20. Suppose that there exists L_ℓ, G_ℓ, $R_{i\ell}$, $i\in\mathbb{N}$, $\ell\in\mathcal{I}_i$, such that

$$\begin{bmatrix} R_{i\ell} & A_i^sG_\ell + B_i^sL_\ell \\ \star & G_\ell + G'_\ell - \mathcal{D}_i^s(R)\end{bmatrix} > 0,\quad i\in\mathbb{N},\ \ell\in\mathcal{I}_i,\ s = 1,\ldots,\xi, \tag{39}$$

where $\mathcal{D}_j^s(R) = \sum_{i=1}^{N}\sum_{\ell\in\mathcal{I}_i} p_{ij}^s\alpha_{i\ell}R_{i\ell}$. Then system (3) is robust stochastically stabilizable with $K = \{K_\ell = L_\ell G_\ell^{-1}, \ell\in\mathcal{I}\}\in\mathcal{K}_r$.

Proof. See Costa et al. (2014). □

Consider now the following LMI optimization problem.

$$\Upsilon = \inf_{W_{i\ell},R_{i\ell},G_\ell,L_\ell}\sum_{i=1}^{N}\sum_{\ell\in\mathcal{I}_i}\alpha_{i\ell}\,\mathrm{tr}(W_{i\ell})$$

subject to: for $i\in\mathbb{N},\ \ell\in\mathcal{I}_i,\ s = 1,\ldots,\xi$,

$$\begin{bmatrix} R_{i\ell} - \mu_iE_iE'_i & A_i^sG_\ell + B_i^sL_\ell \\ \star & G_\ell + G'_\ell - \mathcal{D}_i^s(R)\end{bmatrix} > 0, \tag{40}$$

$$\begin{bmatrix} W_{i\ell} & C_iG_\ell + D_iL_\ell \\ \star & G_\ell + G'_\ell - \mathcal{D}_i^s(R)\end{bmatrix} \ge 0. \tag{41}$$

We have now a robust version of Theorem 15.

Proposition 21. For any feasible solution $W_{i\ell}$, $R_{i\ell}$, G_ℓ, L_ℓ of the LMIs (40), (41), we get that $K = \{K_\ell = L_\ell G_\ell^{-1}, \ell\in\mathcal{I}\}\in\mathcal{K}_r$, and $\|\mathcal{G}_K\|_2^2 \le \sum_{i=1}^{N}\sum_{\ell\in\mathcal{I}_i}\alpha_{i\ell}\,\mathrm{tr}(W_{i\ell})$. Thus, $\inf_{K\in\mathcal{K}_r}\|\mathcal{G}_K\|_2^2 \le \Upsilon$.

Proof. See Costa et al. (2014) □

7. NUMERICAL EXAMPLES

Example 1

Consider two operation modes, and a binary detector of the form

$$[\alpha_{i\ell}] = \begin{bmatrix} \gamma & 1-\gamma \\ 1-\gamma & \gamma\end{bmatrix},\quad \gamma\in[0,1], \tag{42}$$

so that $N = M = 2$. The parameter γ determines the quality of the estimator; for instance, detection is perfect for $\gamma = 1$, and of little use when $\gamma = 0.5$. Our objective in this simulation is to investigate how the proposed design methods depend upon the quality of the estimator, as far as the guaranteed H_2 cost is concerned.

We also consider in this example the following parameters, chosen randomly:

$$A_1 = \begin{bmatrix} 0.7017 & -1.2276 & 0.3931 & -0.6368 \\ -0.4876 & -0.6699 & -1.7073 & -1.0026 \\ 1.8625 & 1.3409 & 0.2279 & -0.1856 \\ 1.1069 & 0.3881 & 0.6856 & -1.0540\end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.0715 & -0.5420 & 0.6716 & 0.6250 \\ 0.2792 & 1.6342 & -0.5081 & -1.0473 \\ 1.3733 & 0.8252 & 0.8564 & 1.5357 \\ 0.1798 & 0.2308 & 0.2685 & 0.4344 \end{bmatrix},$$

along with

$$B_1 = B_2 = \begin{bmatrix} I_2 \\ 0_2 \end{bmatrix}, \quad C_1 = C_2 = \begin{bmatrix} I_4 \\ 0_{2\times 4} \end{bmatrix},$$

$$D_1 = D_2 = \begin{bmatrix} 0_{4\times 2} \\ I_2 \end{bmatrix}, \quad E_1 = E_2 = I_4,$$

where I_n, 0_n and $0_{n\times m}$ stand for $n \times n$ identity and zero matrices, and $n \times m$ zero matrices, respectively. The initial distribution of the Markov chain was picked as $\mu_1 = \mu_2 = 0.5$, and the transition matrix of the Markov chain was considered in the experiment as

$$P = \begin{bmatrix} 0.6942 & 0.3058 \\ 0.6942 & 0.3058 \end{bmatrix},$$

which fits within the Bernoulli jump case. We obtained in this case the results shown in Figure 1, where the guaranteed H_2 cost from Theorem 15 is compared with the benchmark cost from Theorem 18. As we can see, the solutions coincide in the "exact" detector scenario that $\gamma \in \{0, 1\}$, but the cost from Theorem 15 degrades for worse detectors. The worst H_2 cost corresponds to the maximal entropy scenario ($\gamma = 0.5$) where, as anticipated, the detector's output is of little use in determining the current operation mode.

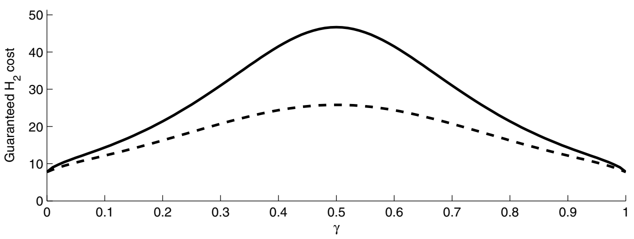

Fig. 1. Example 1: H_2 cost Υ guaranteed by Theorem 15 (solid), and optimal/exact H_2 cost (dashed). The optimal cost coincides with $\widehat{\Upsilon}$ (corresponding to Theorem 18), as expected.

Example 2

In this section we consider Samuelson's macroeconomic model, as in Blair and Sworder (1975). In this case the system parameters are as in (Costa et al., 2004, Section 8.1.2), and we assume that three possible symbols can be detected ($M = 3$). The second and third symbols correspond precisely to the second and third operation modes ($\alpha_{22} = \alpha_{33} = 1$), but we suppose that, whenever the first symbol is detected, there is a probability that the current operation mode is actually "boom" instead of "normal". The probability is governed by a parameter $0 \leq \gamma \leq 0.6$, so that:

$$\alpha_{11} = 1 - \gamma, \qquad \alpha_{12} = \gamma. \tag{43}$$

In this case, the numerical results are shown in Figure 2, where the guaranteed H_2 cost Υ is shown for different values of the probability of erroneous detection γ. As we can see, in this example the proposed control design can be employed in situations where the first operation mode is erroneously detected as frequently as 60% of the time.

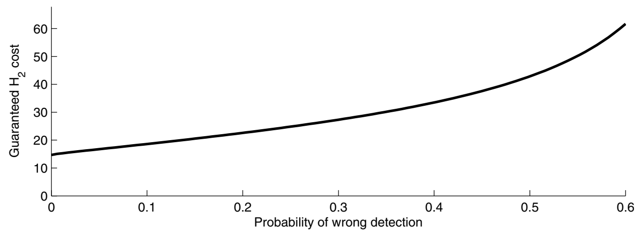

Fig. 2. Example 2: H_2 cost guaranteed by Theorem 15.

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