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On Gaussian filters for continuous-discrete nonlinear systems

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Abstract: This paper considers a continuous-discrete (CD) nonlinear filtering problem in the framework of Gaussian filters. After reviewing the equivalent linearization of static nonlinearities, we derive the continuous-discrete exact Gaussian filter (CD-ExGF) and equivalent linearization Kalman filter (CD-EqKF). It is shown that two CD nonlinear filters have the same time update differential equations for the conditional mean and covariance matrix, and that a difference is in the Kalman gain for the measurement update. Numerical methods of integrating a stochastic differential equation and time update differential equations are developed using the Heun scheme. Results of two simulation studies are included to show the difference and similarity of the CD-EqKF, CD-ExGF and CD extended-Kalman filter (CD-EKF).

Keywords: Nonlinear filtering, Continuous-discrete filtering, Gaussian filters, Exact Gaussian filter, Equivalent linearization Kalman filter, Heun scheme

1. INTRODUCTION

Since it is quite difficult to obtain the Bayesian optimal estimate in nonlinear filtering, many approximate nonlinear filtering methods have been developed in the past [1, 5, 9, 11, 18, 13]. Recent interests in nonlinear filtering include particle filters (PFs) [6, 16, 24], unscented Kalman filters (UKFs) [14], ensemble Kalman filters (EnKFs) [7], Gaussian filters (GF) [12] and quadrature Kalman filters (QKFs) [2], etc.

Though nonlinear filtering problems are usually formulated in discrete-time, real physical systems are continuous in time, so that they are described by stochastic differential equations. In fact, there exist many phenomena that can be modeled as stochastic systems where the measurement of a continuous-time signal is naturally made by using digital devices. Thus, to deal with continuous-discrete (CD) nonlinear filtering problems, the continuous-discrete extended Kalman filter (CD-EKF) has been derived in [13, 9]. Besides, there are many applications of CD nonlinear filtering methods, e.g. GPS/INS [4, 11], target tracking [24, 5], finance [28], hybrid measurements [32].

Also a CD-UKF algorithm with two differential equations for time update of the conditional mean and covariance matrix has been derived from the discrete UKF algorithm by a limiting procedure [26]. Moreover, the continuous-discrete cubature Kalman filter (CD-CKF) algorithm has been developed [3] by using the Ito-Taylor expansion of higher order and the discrete-time CKF [2]. More recently, a CD-CKF with two differential equations for the time update has been developed [27], in which comparison with the Ito-Taylor expansion approach [3] is made to show the relative merits of both approaches from computational point of view. And, by using the divided difference (DD) method [22], we have derived continuous-discrete divided difference filters (CD-DDFs) [30].

In this paper, after reviewing the equivalent linearization method for static nonlinearities [29, 25], we derive the continuous-discrete exact Gaussian filter (CD-ExGF) and equivalent linearization Kalman filter (CD-EqKF), where the latter extends the discrete-time filtering algorithm in [15]. It is shown that the time update algorithm of CD-ExGF is the same as that of CD-EqKF. We also develop a method of implementing the CD nonlinear filters based on the Huen scheme [17].

The paper is organized as follows. In section 2, the CD nonlinear filtering problem is stated, and a general approach of designing CD nonlinear filters is presented. In section 3, we review the method of equivalent linearization of a static nonlinearity, and the CD-ExGF is outlined together with CD-EqKF in section 4. In section 5, numerical procedures for integrating a stochastic differential equation and two time update equations are developed by using the Heun scheme. Section 6 shows results of comparative simulation studies using CD-EKF, CD-EqKF and CD-ExGF. Section 7 concludes the paper.

2. NONLINEAR FILTERING

Consider a nonlinear stochastic system described by

$$dx(t) = f(x(t), t)dt + Ldw(t), \quad 0 \le t \le T \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ is the drift term, $dw(t) \in \mathbb{R}^l$ is the increment of Brownian motion with mean zero and covariance matrix $Qdt \in \mathbb{R}^{l\times l}$, and $L \in \mathbb{R}^{n\times l}$ is the diffusion matrix. We also assume that the drift term satisfies conditions that ensure the existence and uniqueness of the process $x(t), 0 \le t \le T$ [13]. Since L is constant, the system (1) is called a Langevin-type stochastic differential equation.

Suppose that the output observations are taken at discrete times $0 = t_0 < t_1 < \cdots < t_N \le T$. Thus, the observation equation is given by

$$y_k = h_k(x(t_k)) + v_k, \quad k = 0, 1, \cdots, N \tag{2}$$

where $y_k \in \mathbb{R}^p$ is the output vector, $h_k : \mathbb{R}^n \to \mathbb{R}^p$ is the output nonlinear function, and $v_k \in \mathbb{R}^p$ is a Gaussian white noise with mean zero and covariance matrix $R \in \mathbb{R}^{p\times p}$. It is assumed that the observation noise $v_k, k = 0, 1, \cdots, N$ is independent of the initial state $x(0)$ and the Brownian motion $w(t), 0 \le t \le T$.

Let $Y_k = \{y_0, y_1, \cdots, y_k\}$ be the collection of observations up to time t_k. Then, from the Bayesian point of view, the relevant information is in the conditional probability density functions (pdfs) of the state $x(t)$ given Y_k, i.e.

$$p(x(t)|Y_k), \quad t_k \le t \le t_{k+1}$$

where $k = 0, 1, \cdots, N$. We also define the conditional mean estimate and covariance matrix as

$$\hat{x}(t|t_k) = E\{x(t)|Y_k\}$$
$$P(t|t_k) = \mathrm{cov}(x(t)|Y_k), \quad t_k \le t \le t_{k+1}$$

where $k = 0, 1, \cdots$. We see from [13] that the CD nonlinear filtering algorithm consists of two algebraic equations for the measurement update and two forward differential equations for the time update. Thus, the present problem can be solved by iterating the following two steps.

Measurement update At $t = t_k$, update $\hat{x}(t_k|t_{k-1})$ and $P(t_k|t_{k-1})$ based on the observation y_k in order to obtain the filtered estimate $\hat{x}(t_k|t_k)$ and covariance matrix $P(t_k|t_k)$.

Time update For $t_k \le t < t_{k+1}$, integrate two forward differential equations to obtain the predicted estimate $\hat{x}(t_{k+1}|t_k)$ and covariance matrix $P(t_{k+1}|t_k)$, where the initial conditions are the filtered estimate $\hat{x}(t_k|t_k)$ and covariance matrix $P(t_k|t_k)$.

It is well known that the linearization of f and h_k at the filtered and predicted estimates yields the CD-EKF algorithm [13, 9]. The CD-UKF algorithm has been derived from the discrete-time UKF by a limiting procedure [26], and the CD-CKF algorithm has been developed [3], [27]. Also, we have derived CD-DDFs [30], extending DD filters [22]. In this paper, we derive the CD-ExGF by extending ExGF [21], and then the CD-EqKF by using equivalent linearization [15, 20]. We also present numerical methods of simulating the stochastic differential equation and of solving two differential equations in the time update.

3. EQUIVALENT LINEARIZATION OF A STATIC NONLINEARITY

We briefly review the equivalent linearization of a static nonlinearity [25]. Let $x \in \mathbb{R}^n$ be a random vector with mean m_x and covariance matrix $\Sigma_{xx} > 0$. Consider a nonlinear function $y = g(x)$ with $g : \mathbb{R}^n \to \mathbb{R}^p$. Let $m_y := E\{g(x)\}$ and $\Sigma_{yx} := \mathrm{cov}(y, x)$. Then, the linear unbiased minimum variance estimate of y is given by

$$\hat{y} = m_y + \hat{G}(x - m_x)$$

where $\hat{G} = \Sigma_{yx}\Sigma_{xx}^{-1} \in \mathbb{R}^{p\times n}$ is called the equivalent gain matrix in the literature. The covariance matrix Σ_{yy} of y is therefore approximated as

$$\Sigma_{yy} \simeq \mathrm{cov}(\hat{y}) = \hat{G}\Sigma_{xx}\hat{G}^{\mathrm{T}} \tag{3}$$

Moreover, the actual output covariance matrix is bounded below, i.e. $\Sigma_{yy} \ge \hat{G}\Sigma_{xx}\hat{G}^{\mathrm{T}}$.

Proposition 1. Suppose that $x \in \mathbb{R}^n$ is a Gaussian random vector with $N(m_x, \Sigma_{xx})$. Consider a nonlinear function $y = g(x)$ with $g : \mathbb{R}^n \to \mathbb{R}^p$ differentiable, and assume that the following conditions are satisfied:

$$E\{|g_i(x)|\} < \infty, \quad E\{|\partial g_i(x)/\partial x_j|\} < \infty$$

where $i = 1, \cdots, p, j = 1, \cdots, n$. Then, the equivalent gain matrix $\hat{G} \in \mathbb{R}^{p\times n}$ of $g(x)$ is expressed as

$$\hat{G} = E\left\{\frac{\partial}{\partial x}g(x)\right\} = \frac{\partial}{\partial m_x}E\{g(x)\} \tag{4}$$

Proof: See Appendix of [15]. □

The formula (4) implies that the equivalent gain matrix can be obtained by either the expectation of Jacobian matrix of the nonlinearity or the Jacobian of the expectation of it. We should note that this is true under the assumption that x is Gaussian.

We have two linear approximations for $y = g(x)$, i.e. a local approximation:

$$g(x) \simeq g(m_x) + \left[\frac{\partial g(x)}{\partial x}\right]_{x=m_x}(x - m_x) \tag{5}$$

and a "global" approximation:

$$g(x) \simeq E\{g(x)\} + E\left\{\frac{\partial g(x)}{\partial x}\right\}(x - m_x) \tag{6}$$

It is well known that the EKF is derived by using a local approximation of (5), while the EqKF is derived by using a global approximation of (6).

4. CONTINUOUS-DISCRETE EXACT GAUSSIAN FILTER AND EQUIVALENT LINEARIZATION KALMAN FILTER

For simplicity, the Gaussian pdf with $N(m, P)$ is written as [1]

$$\gamma(x - m, P) = \frac{1}{\sqrt{(2\pi)^n|P|}}e^{-\frac{1}{2}(x-m)^{\mathrm{T}}P^{-1}(x-m)}$$

4.1 Measurement update algorithm

The measurement update is performed at $t = t_k$ when a new observation is available. In the following, the a priori conditional pdf is assumed to be Gaussian, i.e.

$$p(x(t_k)|Y_{k-1}) = \gamma(x(t_k) - \hat{x}(t_k|t_{k-1}), P(t_k|t_{k-1})) \tag{7}$$

To derive the measurement update algorithm, we assume that the conditional expectations below are computable by using (7).

Step M1. Compute the one-step prediction

$$\hat{y}_{k|k-1} = E\{h_k(x(t_k))|Y_{k-1}\} \tag{8}$$

where the right-hand side of (8) is a function of $\hat{x}(t_k|t_{k-1})$ and $P(t_k|t_{k-1})$, so that we write it as

$$\hat{y}_{k|k-1} = \psi_k(\hat{x}(t_k|t_{k-1}), P(t_k|t_{k-1})) \tag{9}$$

Step M2. We compute the conditional auto-covariance matrix of y_k and the cross-covariance matrix of $x(t_k)$ and y_k, i.e.

$$V_{k/k-1} = \mathrm{cov}(h_k(x(t_k))|Y_{k-1}) + R \tag{10}$$

and

$$U_{k/k-1} = \mathrm{cov}(x(t_k), h_k(x(t_k))|Y_{k-1}) \tag{11}$$

Thus, by the linear least-squares regression (i.e. the equivalent linearization), the Kalman gain matrix is given by

$$K_k = U_{k/k-1} V_{k/k-1}^{-1} \tag{12}$$

Step M3. By using (9), (10) and (12), we have the measurement update equations

$$\hat{x}(t_k|t_k) = \hat{x}(t_k|t_{k-1}) + K_k[y_k - \hat{y}_{k|k-1}]$$
$$P(t_k|t_k) = P(t_k|t_{k-1}) - K_k V_{k/k-1} K_k^{\mathrm{T}}$$

We note that the form of this measurement update equations is the same as that of the discrete-time nonlinear Kalman filter [1].

To derive the CD-EqKF, we compute the equivalent gain matrix of h_k by

$$\hat{H}_k = \frac{\partial \psi_k(\hat{x}(t_k|t_{k-1}), P(t_k|t_{k-1}))}{\partial \hat{x}(t_k|t_{k-1})} \tag{13}$$

Then, it follows from (3) that $V_{k/k-1}$ of (10) can be approximated as

$$V_{k/k-1}^e = \hat{H}_k P(t_k|t_{k-1}) \hat{H}_k^{\mathrm{T}} + R \tag{14}$$

But, by the first equality of (4), we have exactly

$$U_{k/k-1} = P(t_k|t_{k-1}) \hat{H}_k^{\mathrm{T}} \tag{15}$$

Thus, the measurement update algorithm of CD-EqKF is obtained by replacing (10) and (11) by (14) and (15), respectively.

4.2 Time update algorithm

According to [13], the time update equations satisfied by the conditional mean estimate and the conditional covariance matrix of the prediction error are respectively given by

$$\frac{d}{dt}\hat{x}(t|t_k) = E\{f(x(t), t)|Y_k\} \tag{16}$$

$$\frac{d}{dt}P(t|t_k) = \mathrm{cov}(x(t), f_t|Y_k) + \mathrm{cov}(f_t, x(t)|Y_k) + LQL^{\mathrm{T}} \tag{17}$$

where $f_t := f(x(t), t)$. We now compute the conditional expectations in (16) and (17) with respect to the pdf

$$p(x(t)|Y_k) = \gamma(x(t) - \hat{x}(t|t_k), P(t|t_k)) \tag{18}$$

Step T1. The conditional expectation of the right-hand side of (16) is a function of $\hat{x}(t|t_k)$ and $P(t|t_k)$, so that we write

$$E\{f(x(t), t)|Y_k\} = \varphi_t(\hat{x}(t|t_k), P(t|t_k)) \tag{19}$$

Step T2. It follows from (4) that the conditional covariance matrix is expressed as

$$\begin{aligned}\mathrm{cov}(f_t, x(t)|Y_k) &= E\{f(x(t), t)[x(t) - \hat{x}(t|t_k)]^{\mathrm{T}}|Y_k\} \\ &= \hat{F}_t P(t|t_k)\end{aligned} \tag{20}$$

where the equivalent gain matrix $\hat{F}_t$ is given by

$$\hat{F}_t = \frac{\partial \varphi_t(\hat{x}(t|t_k), P(t|t_k))}{\partial \hat{x}(t|t_k)} \tag{21}$$

Step T3. The time update equations are expressed as

$$\frac{d}{dt}\hat{x}(t|t_k) = \varphi_t(\hat{x}(t|t_k), P(t|t_k)) \tag{22}$$

$$\frac{d}{dt}P(t|t_k) = P(t|t_k)\hat{F}_t^{\mathrm{T}} + \hat{F}_t P(t|t_k) + LQL^{\mathrm{T}} \tag{23}$$

where $t_k \leq t \leq t_{k+1}$.

It follows from (20) that the time update equations of CD-ExGF are the same as those of CD-EqKF, i.e. (22) and (23). Thus, the difference between the two filters is in the evaluation of the conditional auto-covariance matrix $V_{k/k-1} = \mathrm{cov}(y_{t_k}|Y_{k-1})$.

4.3 Relation among CD-ExGF, CD-EqKF and CD-EKF

In the derivation of the algorithm of CD-ExGF and CD-EqKF, it is assumed that the conditional expectations of (8), (10), (11), (19) and (20) are exactly computable under the Gaussian assumptions of (7) and (18). It is well known that these expectations are computable if $h_k(x)$ and $f(x,t)$ are polynomials or monomials and exponential of linear, quadratic and bilinear functions like $e^{\alpha x + \beta x^2}$ and $e^{x_i x_j}$. In fact, in these cases, we can compute ψ_k, $V_{k/k-1}$, $U_{k/k-1}$ and φ_t, $\mathrm{cov}(x(t), f_t|Y_k)$ exactly, so that we have the CD-ExGF algorithm.

- It follows from (20) that both the CD-ExGF and CD-EqKF have the same time update differential equations.
- The difference of the two filters is in the evaluation of the auto-covariance matrix $V_{k/k-1}$ of y_k conditioned on Y_{k-1} in the measurement update.
- Thus, if the observation equation is linear, then the algorithm of CD-ExGF coincides with that of CD-EqKF.
- If we delete $P(t_k|t_{k-1})$ from (9) and (13), and $P(t|t_k)$ from (19) and (21), then the algorithm of CD-EqKF reduces to that of CD-EKF.

If the conditional expectations are not exactly computable, some numerical procedures are needed to approximately evaluate them [12, 31, 2].

5. NUMERICAL PROCEDURE

In this section, we introduce numerical procedures for simulating the stochastic differential equation (1) and for solving the time update equations (22) and (23), by using the Heun scheme [17].

5.1 Solution of stochastic differential equation

Let Δ be a small interval for integration, and let $\bar{x}(t)$ be the approximation of $x(t)$. Define

$$c_1 = f(\bar{x}(t), t)$$
$$c_2 = f(\bar{x}(t) + \Delta c_1 + \sqrt{\Delta} L \bar{w}(t), t + \Delta)$$

Then, the Heun scheme for solving (1) is give by

$$\bar{x}(t + \Delta) = \bar{x}(t) + \frac{\Delta}{2}(c_1 + c_2) + \sqrt{\Delta} L \bar{w}(t)$$

where $t = j\Delta, j = 0, 1, \cdots$, and $\bar{w}(t)$ is the pseudo random number with $N(0, Q)$. It is shown [17] that the the Heun scheme has a strong order of convergence 1.0, while the Euler-Maruyama scheme has a strong order of convergence 0.5. Thus, the Heun scheme with a higher-order accuracy than the Euler-Maruyama scheme is the most appropriate simple procedure for solving our Langevin-type stochastic differential equation of (1).

5.2 Solution of time update differential equations

We consider the integration of (22) and (23) in the interval $t_k \le t \le t_{k+1}$, where the initial conditions are the filtered estimate $\hat{x}(t_k|t_k)$ and covariance matrix $P(t_k|t_k)$. It should be noted that since, the right-hand sides of (22) and (23) are functions of both $\hat{x}(t|t_k)$ and $P(t|t_k)$, they are coupled equations.

Let $m = \hat{x}(t|t_k)$ and $P = P(t|t_k)$. Define $d_1 = \varphi_t(m, P)$. Then the Euler approximation of the solution of (22) at $t + \Delta$ is given by

$$\hat{x}^e(t + \Delta|t_k) = m + \Delta d_1$$

Also, the Euler approximation of the solution of the covariance equation of (23) at $t + \Delta$ is expressed as

$$P^e(t + \Delta|t_k) = P + \Delta\big[\hat{F}_t P + P\hat{F}_t^{\mathrm{T}} + LQL^{\mathrm{T}}\big]$$

where

$$\hat{F}_t = \frac{\partial \varphi_t(m, P)}{\partial m}$$

We write $m^e := \hat{x}^e(t + \Delta|t_k)$, and $P^e := P^e(t + \Delta|t_k)$. In terms of m^e and P^e, we define $d_2 = \varphi_t(m^e, P^e)$. Then, the Heun scheme-based procedure of obtaining the conditional mean at $t + \Delta$ is expressed as

$$\hat{x}(t + \Delta|t_k) = \hat{x}(t|t_k) + \frac{\Delta}{2}(d_1 + d_2) \qquad (24)$$

By using the Euler approximations m^e and P^e, we define

$$\hat{F}_t^e = \frac{\partial \varphi_t(m^e, P^e)}{\partial m^e}$$

Then, the Heun scheme-based time update procedure of the covariance matrix becomes

$$P(t + \Delta|t_k) = P(t|t_k) + \frac{\Delta}{2}\Big[\hat{F}_t P + P\hat{F}_t^{\mathrm{T}} + \hat{F}_t^e P^e + P^e(\hat{F}_t^e)^{\mathrm{T}}\Big] + \Delta LQL^{\mathrm{T}} \qquad (25)$$

The algorithms of (24) and (25) are used in Step T3 to compute the predicted estimate $\hat{x}(t_{k+1}|t_k)$ and covariance matrix $P(t_{k+1}|t_k)$.

6. NUMERICAL EXAMPLES

We show simulation results for two examples; one is a scalar nonlinear system with a square observation, and the other is a parameter estimation problem of a 2nd-order continuous-time AR model.

Example 6.1: Consider a scalar CD system [12]

$$dx(t) = ax(1 - x^2)dt + dw(t) \qquad (26)$$

$$y_k = (x(t_k) - b)^2 + v_k \qquad (27)$$

where $w(t)$ is a Brownian motion with $N(0, qt)$ and v_k is a white noise with $N(0, r)$, and $a > 0$.

It follows that the deterministic system $\dot{x} = ax(1-x^2)$ has three equilibria $\{-1, 0, 1\}$, where $\{-1, 1\}$ are stable and $\{0\}$ is unstable. Also, the stationary pdf of $x(t)$ is given by $p(x) = C\exp\{-\frac{a}{2q}(x^2 - 1)^2\}$ $(C > 0)$, implying that the $x(t)$ process has two operating modes corresponding to two stable equilibria. Also, the observation function of (27) is a square shifted by b as shown in Fig. 1, so that the likelihood function is bimodal. Therefore the present CD nonlinear filtering is a challenging problem.

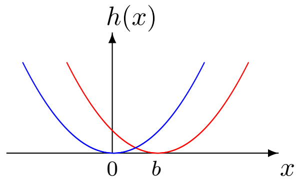

Fig. 1. Output nonlinearity $h(x)$.

The CD-ExGF (CD-EqKF) algorithm for (26) and (27) becomes as follows.

Algorithm of CD-ExGF

1) The measurement update

$$\begin{aligned}
\hat{y}_{k/k-1} &= (\hat{x}(t_k|t_{k-1}) - b)^2 + P(t_k|t_{k-1})\\
\hat{H}_k &= 2(\hat{x}(t_k|t_{k-1}) - b)\\
U_{k/k-1} &= \hat{H}_k P(t_k|t_{k-1})\\
V^e_{k/k-1} &= \hat{H}_k P(t_k|t_{k-1})\hat{H}_k + r\\
V_{k/k-1} &= V^e_{k/k-1} + 2P^2(t_k|t_{k-1})\\
K_k &= U_{k/k-1}V^{-1}_{k/k-1}\\
\hat{x}(t_k|t_k) &= \hat{x}(t_k|t_{k-1}) + K_k[y_k - \hat{y}_{k|k-1}]\\
P(t_k|t_k) &= P(t_k|t_{k-1}) - K_k V_{k/k-1} K_k^{\mathrm{T}}
\end{aligned}$$

2) The time update

$$\begin{aligned}
\frac{d}{dt}\hat{x}(t|t_k) &= \varphi_t(\hat{x}(t|t_k), P(t|t_k))\\
\frac{d}{dt}P(t|t_k) &= 2\hat{F}_t P(t|t_k) + q
\end{aligned}$$

where

$$\begin{aligned}
\varphi_t &= a[\hat{x}(t|t_k) - \hat{x}^3(t|t_k) - 3\hat{x}(t|t_k)P(t|t_k)]\\
\hat{F}_t &= a[1 - 3\hat{x}^2(t|t_k) - 3P(t|t_k)]
\end{aligned}$$

Note that if we employ $V^e_{k/k-1}$ for $V_{k/k-1}$, then we have the CD-EqKF algorithm. Moreover, if we delete $P(t_k|t_{k-1})$ from $\hat{y}_{k/k-1}$, and $P(t|t_k)$ from φ_t and $\hat{F}_t$, then we have the CD-EKF algorithm. □

For simulations, let $q = 0.25$, $r = 0.01$ and $a = 5$, and let τ be the sampling interval, so that we have $t_k = k\tau$ for $k = 0, 1, \cdots, N$, where $N = T/\tau$. Also, we assume that the time interval for simulation is $T = 10$, and the interval for integration is $\Delta = 0.01$.

Figure 2 displays a sample of state and observation processes, and state estimates by CD-EKF, CD-EqKF and CD-ExGF for $b = 0.2$, $\tau = 0.1$, where $x(0) \sim N(0, 1)$ and the initial conditions are set as $\hat{x}(0|-1) = 0, P(0|-1) = 1$. In this particular sample, the CD-EKF made a wrong decision for the operating mode of the state process, while both CD-EqKF and CD-ExGF made a correct decision.

To compare the performance of nonlinear filters, we define the root mean square error (RMSE) as

$$E_N^{(j)} = \sqrt{\frac{1}{N}\sum_{k=0}^{N}(x^{(j)}(t_k) - \hat{x}^{(j)}(t_k|t_k))^2}, \quad j = 1, \cdots, M$$

where M is the number of Monte Carlo runs. Table 1 displays the average RMSE

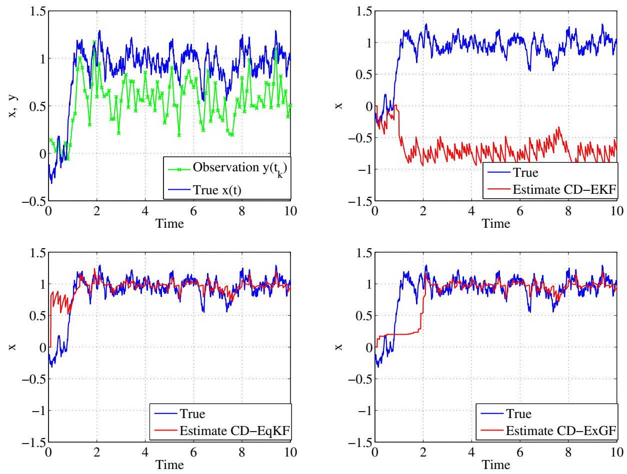

Fig. 2. Sample processes of state, observation and state estimates by CD-EKF, CD-EqKF and CD-ExGF, where $b = 0.2$ and $\tau = 0.1$.

Table 1. Performance of CD nonlinear filters.

	CD-EKF	CD-EqKF	CD-ExGF
$b = 0.1$	–	1.0089	1.0736
	–	(0.8958)	(0.8111)
	36	51	43
$b = 0.2$	0.9441	0.7605	0.8780
	(0.7618)	(0.8480)	(0.7544)
	49	62	50
$b = 0.3$	0.7639	0.5347	0.6445
	(0.6822)	(0.7349)	(0.6619)
	53	72	60
$b = 0.4$	0.5537	0.2884	0.2213
	(0.5473)	(0.5121)	(0.3039)
	65	90	98
$b = 0.5$	0.2449	0.1644	0.1250
	(0.2385)	(0.3764)	(0.0728)
	98	98	100

$$\bar{E} = \frac{1}{M} \sum_{j=1}^{M} E_N^{(j)}, \quad M = 100, \ N = 100$$

where the numbers in parentheses denote the standard deviation of $E_N^{(j)}$, and the numbers in the third row denote the number of success of tracking the state mode. We see from Table 1 that as b increases, the performance of filters and the rate of correct tracking get better in general. Note that for $b = 0$, all the filters with these initial conditions cease to track the state.

Now we consider the case with a poor guess of the initial state treated in [12], where the initial state is given by $x(0) = -0.2$, while the initial conditions for the filters are $\hat{x}(0|-1) = 0.8, P(0|-1) = 2$. Table 2 shows the numbers of success of estimating the mode of operation of three filters for $M = 100$ simulation runs. We see that the increase of b does not help to improve the performance of CD-EKF, and the CD-EqKF shows the best performance among three filters for small values of b. But, the CD-ExGF shows the very good performance for large values of b.

Example 6.2: The CD nonlinear filters are applied to the parameter estimation of a continuous-time AR model of the form [19, 10]

Table 2. Number of success in estimating the operating mode for $M = 100$.

b	CD-EKF	CD-EqKF	CD-ExGF
0	15	53	13
0.1	15	53	13
0.2	15	49	15
0.3	15	37	15
0.4	15	33	15
0.5	15	88	95
0.6	15	94	99

$$\frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_2 x = w(t)$$

where a_1, a_2 are unknown parameters, and w is a white noise with $N(0, q)$. Let $x_1 := x, \ x_2 := \dot{x}, \ x_3 := a_1, \ x_4 := a_2$. Then, the augmented state vector is given by $x = [x_1 \ x_2 \ x_3 \ x_4]^{\mathrm{T}}$, so that the state space model becomes

$$d \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -x_2(t)x_3(t) - x_1(t)x_4(t) \\ 0 \\ 0 \end{bmatrix} dt + \begin{bmatrix} 0 \\ dw(t) \\ 0 \\ 0 \end{bmatrix}$$

$$y_k = x_1(t_k) + v_k$$

where v_k is the white noise with $N(0, r)$. Let $\tau > 0$ be the sampling interval, so that $t_k = k\tau, k = 0, 1, \cdots, N$.

Since the observation function is linear, the algorithm of CD-ExGF is the same as that of CD-EqKF. For the CD-EqKF, the second component of φ_t is given by

$$\varphi_t(2) = -\hat{x}_2(t|t_k)\hat{x}_3(t|t_k) - \hat{x}_1(t|t_k)\hat{x}_4(t|t_k) - P_{23}(t|t_k) - P_{14}(t|t_k) \qquad (28)$$

Thus, if we delete the cross-covariances $P_{23}(t|t_k), P_{14}(t|t_k)$ from the above equation, then we have the CD-EKF algorithm. Hence, in the state and parameter estimation problem for AR models, the difference between two algorithms of CD-EKF and CD-EqKF is very small.

We assume that the initial state is $x(0) = [\xi \ 0 \ a_1 \ a_2]^{\mathrm{T}}$ with $\xi \sim N(0, 0.5^2)$, and $q = 1$, $r = 0.001$, so that the signal to noise ratio of $x_1(t)$ process is about 20dB in magnitude. The initial estimate and covariance matrix are $\hat{x}(0|-1) = [0 \ 0 \ 0 \ 0]^{\mathrm{T}}$ and $P(0|-1) = \rho I_4$, respectively. Also, the time increment for integration is $\Delta = 0.01$ and the simulation interval is $T = 600$ (sec).

Simulation results of the parameter estimation for different values of the sampling interval τ are shown in Tables 3 and 4, where $\hat{a}_1$ and $\hat{a}_2$ respectively denote the averages of $\hat{x}_3(t_N|t_N)$ and $\hat{x}_4(t_N|t_N)$ based on Monte Carlo runs of $M = 100$, and where the numbers in parentheses denote the standard deviation of the estimates. We see that the performance of CD-EqKF (CD-ExGF) is slightly better than that of CD-EKF, especially for large τ. Note that though not shown here, the performance of state estimation by both algorithms was quite close. Thus, it seems that the covariances in (28) much contribute to the parameter estimation than to the state estimation.

It is well known that the performance of nonlinear filters depends on many factors, e.g. the initial state and its a priori estimate, noise variances, the sampling interval, so that stability and convergence of the present parameter estimation algorithm remain to be analyzed [23, 8].

Table 3. Performance of parameter estimation for $a_1 = 3, a_2 = 2$ and $\rho = 10$.

	CD-EKF		CD-EqKF	
τ	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_1$	$\hat{a}_2$
0.1	2.9871	2.0238	2.9991	2.0261
	(0.1321)	(0.1427)	(0.1325)	(0.1428)
0.2	2.9868	2.0197	3.0056	2.0255
	(0.1435)	(0.1446)	(0.1437)	(0.1448)
0.5	2.9630	2.0044	3.0042	2.0217
	(0.2225)	(0.1563)	(0.2197)	(0.1546)
1.0	2.7651	1.8952	2.9351	1.9960
	(0.3935)	(0.2145)	(0.2159)	(0.1635)

Table 4. Performance of parameter estimation for $a_1 = 2, a_2 = 2$ and $\rho = 5$.

	CD-EKF		CD-EqKF	
τ	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_1$	$\hat{a}_2$
0.1	1.9900	2.0163	1.9978	2.0183
	(0.0992)	(0.1193)	(0.0994)	(0.1194)
0.2	1.9896	2.0128	2.0015	2.0175
	(0.1047)	(0.1205)	(0.1049)	(0.1206)
0.5	1.9806	2.0029	2.0027	2.0154
	(0.1422)	(0.1253)	(0.1424)	(0.1250)
1.0	1.9401	1.9658	1.9708	2.0014
	(0.2296)	(0.1673)	(0.2010)	(0.1478)

7. CONCLUSIONS

In this paper, we have derived the continuous-discrete exact Gaussian filter (CD-ExGF) and the equivalent linearization Kalman filter (CD-EqKF). It is shown that two CD nonlinear filters have the same time update differential equations for the conditional mean and covariance matrix, and their difference is in the computation of Kalman gain in the measurement update. Numerical procedures for solving the stochastic differential equation and two time update differential equations are obtained by using the Heun scheme. Simulation results are included to show the difference and similarity of algorithms of CD-EKF, CD-EqKF and CD-ExGF.

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