Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Improved Results on Stability of Time-delay Systems using Wirtinger-based Inequality ⋆

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Abstract: This paper concerns with the problem of delay-dependent stability analysis of time-delay systems. By help of Wirtinger based inequality which gives very close estimating bound of Jensen's inequality, an extended inequality is proposed. Using the new inequality and tuning parameters, a generalized criterion for stability of time-delay systems is established. Two numerical examples are given to describe the less conservatism of the proposed methods.

Keywords: Stability, Time-delay systems, Lyapunov method.

1. INTRODUCTION

During the last decade, many researchers have been devoted much attention to time-delay systems because time-delays are naturally encountered in many dynamic systems such as chemical or process control systems and networked control systems and often sources of poor performance and instability of the system. Existing stability criteria can be classified into two categories, that is, delay-independent ones and delay-dependent ones. It is well known that delay-independent ones are usually more conservative than the delay-dependent ones, so much attention has been paid in recent years to the study of delay-dependent stability conditions. The main issue of delay-dependent stability and stabilization is to reduce the conservatism of stability and stabilization criteria. One of the important index for checking the conservatism is to enlarge the feasible region of the criteria or to get maximum delay bounds for guaranteeing the system stability. Recently, to reduce the conservatism, many methods were adopted based on Lyapunov-Krasovskii approach or Lyapunov-Razumikhin approach. A descriptor model transformation method which is an equivalent model of the original system is introduced in Fridman and Shaked (2003) in which the method significantly reduces the conservatism of the results. By using free-weighting matrices method based on the Newton-Leibniz formula, the further results on time-delay introduced in He et al. (2004).

Very recently, there are a few works (Seuret and Gouaisbaut (2012, 2013a,b)) to reduce Jensen's gap ($g_R(\dot{x}) = \int_a^b \dot{x}(s)R\dot{x}(s)ds - (\int_a^b \dot{x}(s)dsR\int_a^b \dot{x}(s)ds)/(b-a)$ where $R > 0$). Jensen's inequality was used as one of essential techniques to reduce conservatism in dealing with the time delay systems, so it is widely utilized for estimating upper bound of time derivative of constructed Lyapunov functional. Therefore, reducing the Jensen's gap gives more feasible region of stability of time delay systems. In Seuret and Gouaisbaut (2012, 2013a), based on the Wirtinger's inequalities, more close lower bound of Jensen's inequality is proposed as

$$g_R(\dot{x}) \geq \frac{\pi^2}{4}\nu^T(a,b)R\nu(a,b), \tag{1}$$

where $\nu(a,b) = x(b) + x(a) - \frac{2}{b-a}\int_a^b x(s)ds)$, and the choice of a particular signal which satisfies the necessary assumptions to apply the Wirtinger inequalities has been proved. As these result, an integral form of states $\frac{1}{b-a}\int_a^b x(s)ds$ is taken as augmented vectors to include further information of the time-delay systems. Moreover, the improved result which reduces more Jensen's inequality gap as $g_R(\dot{x}) \geq 3\nu^T(a,b)R\nu(a,b)$ instead of (1) has been presented in Seuret and Gouaisbaut (2013b).

In this paper, an extension result, which covers the existing ones as special cases, is introduced to discuss the problem of stability analysis of time-delay systems. In addition, by introducing tuning parameters, the conservatism of stability of the system is reduced without any change of Lyapunov functional and LMI (linear matrix inequality) condition. Numerical examples are given to illustrate that the proposed methods are effective and lead to less conservative results.

Notations: $\mathbb{R}^n$ is the n-dimensional Euclidean space, $X > 0$ (respectively, $X \geq 0$) means that the matrix X is a real symmetric positive definite matrix (respectively, positive semi-definite). $\star$ in a matrix represents the elements below the main diagonal of a symmetric matrix. For $X \in \mathbb{R}^{m\times n}$, $X^\perp$ denotes a basis for the null-space of X.

2. PRELIMINARIES

In this paper, the following time-delay systems are considered:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bx(t-h), \\ x(t) &= \phi(t), \quad \forall t \in [-h,\ 0], \end{aligned} \tag{2}$$

⋆ This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2013R1A1A2A10005201) and BK21+ Program.

where $x(t) \in \mathbb{R}^n$ is the state vector, $h > 0$ is the constant time delay, $\phi(t)$ is a compatible vector valued initial function, and A, B are known real constant matrices with appropriate dimensions.

The following lemma which plays key role in the derivation of the main result is proposed:

Lemma 1. For a given matrix $R = R^T > 0$, scalars $a_1, a_2, \cdots, a_m$ satisfying $a_1 \le a_2 \le \cdots \le a_m$, and all continuously differentiable function x in $[a_1, a_m] \to \mathbb{R}^n$, the following inequality holds:

$$\int_{a_1}^{a_m} \dot{x}^T(s)R\dot{x}(s)ds \ge \sum_{i=1}^{m-1} \frac{1}{a_{i+1}-a_i}\beta^T(a_i, a_{i+1})M\beta(a_i, a_{i+1}), \quad (3)$$

where $\beta(a_1, a_2) = \left[x^T(a_2)\ x^T(a_1)\ \frac{1}{a_2-a_1}\int_{a_1}^{a_2} x(s)ds\right]^T$ and

$$M = \begin{bmatrix} 4R & 2R & -6R \\ \star & 4R & -6R \\ \star & \star & 12R \end{bmatrix}.$$

Proof. It is clear that

$$\int_{a_1}^{a_m} \dot{x}^T(s)R\dot{x}(s)ds = \int_{a_1}^{a_2} \dot{x}^T(s)R\dot{x}(s)ds + \int_{a_2}^{a_3} \dot{x}^T(s)R\dot{x}(s)ds + \cdots + \int_{a_{m-1}}^{a_m} \dot{x}^T(s)R\dot{x}(s)ds.$$

By applying the following relationship proposed in Seuret and Gouaisbaut (2013b) to each subinterval integral terms of above equation:

$$\int_{a_1}^{a_2} \dot{x}^T(s)R\dot{x}(s)ds \ge \frac{1}{a_2-a_1}\Big((x(a_2)-x(a_1))^T R(x(a_2)-x(a_1)) + 3\nu^T(a_1,a_2)R\nu(a_1,a_2)\Big),$$

where $\nu(a_1,a_2) = x(a_2)+x(a_1)-\frac{2}{a_2-a_1}\int_{a_1}^{a_2} x(s)ds$, it is easy to get (3). ■

And, the following lemma is used in this paper:

Lemma 2. (Finsler lemma (Skelton et al. (1997))) Let a vector $\zeta \in \mathbb{R}^n$, a symmetric matrix $M \in \mathbb{R}^{n\times n}$, and $\Gamma \in \mathbb{R}^{m\times n}$ such that $rank(\Gamma) < n$. Then the following statements are equivalent:

(1) $\zeta^T M\zeta < 0,\ \ \Gamma\zeta = 0,\ \ \zeta \neq 0$
(2) $(\Gamma^{\perp})^T M\Gamma^{\perp} < 0$.

3. MAIN RESULTS

Before stating further results, for the sake of simplicity on matrix representation, $e_i \in \mathbb{R}^{(2m+2)n\times n}$ $(i = 1, 2, \ldots, 2m+2)$ are defined as block entry matrices; e.g., $e_2^T = [0\ I\ \underbrace{0\ \ldots\ 0}_{2m}]$.

Theorem 3. For given a constant delay h and a positive integer m, the time-delay system (2) is stable, if there exist positive definite matrices P, Q_i $(i = 1, \ldots, m)$, R, such that

$$(\Gamma^{\perp})^T\Sigma(\Gamma^{\perp}) < 0, \quad (4)$$

where

$$\Sigma = \Pi_1 P\Pi_2^T + \Pi_2 P\Pi_1^T + he_2Re_2^T - \Pi_3 M(1)\Pi_3^T - \sum_{i=1}^{m-1}\Pi_4(i)M(i+1)\Pi_4^T(i) + \Phi,$$

$$\Gamma = \left[A\ -I\ \underbrace{0\ \ldots\ 0}_{m-1}\ B\ \underbrace{0\ \ldots\ 0}_{m}\right],$$

$$\Pi_1 = [e_1\ e_{3+m}\ e_{4+m}\ e_{5+m}\ \ldots\ e_{2m+2}],$$

$$\Pi_2 = [e_2\ e_1 - e_3\ e_3 - e_4\ e_4 - e_5\ \ldots\ e_{1+m} - e_{2+m}],$$

$$\Pi_3 = [e_1\ e_3\ e_{3+m}],$$

$$\Pi_4(i) = [e_{2+i}\ e_{3+i}\ e_{3+m+i}],$$

$$\Phi = e_1Q_1e_1^T - \sum_{i=1}^{m} e_{2+i}Q_ie_{2+i}^T + \sum_{i=1}^{m-1} e_{2+i}Q_{i+1}e_{2+i}^T,$$

$$M(i) = \frac{1}{h_i - h_{i-1}}\begin{bmatrix} 4R & 2R & -\frac{6}{h_i-h_{i-1}}R \\ \star & 4R & -\frac{6}{h_i-h_{i-1}}R \\ \star & \star & \frac{12}{(h_i-h_{i-1})^2}R \end{bmatrix},$$

$$h_i = \frac{h}{m}i.$$

Proof. Define a Lyapunov functional candidate for system (2) as:

$$V(t) = \eta_1^T(t)P\eta_1(t) + \sum_{i=1}^{m}\int_{t-h_i}^{t-h_{i-1}} x^T(s)Q_ix(s)ds + \int_{-h_m}^{0}\int_{t+\theta}^{t} x^T(s)Rx(s)dsd\theta, \quad (5)$$

where $\eta_1(t) = \left[x^T(t)\ \int_{t-h_1}^{t} x^T(s)ds\ \int_{t-h_2}^{t-h_1} x^T(s)ds\ \ldots\ \int_{t-h_m}^{t-h_{m-1}} x^T(s)ds\right]^T$.

Then, the time-derivative of $V(t)$ along the solution of the system (2) gives

$$\dot{V}(t) = 2\eta_1^T(t)P\eta_2(t) + \sum_{i=1}^{m}\Big(x^T(t-h_{i-1})Q_ix(t-h_{i-1}) - x^T(t-h_i)Q_ix(t-h_i)\Big) + h\dot{x}^T(t)R\dot{x}(t) - \int_{t-h_m}^{t} \dot{x}^T(s)R\dot{x}(s)ds,$$

where $\eta_2(t) = \Big[\dot{x}^T(t)\ \Big(x^T(t) - x^T(t-h_1)\Big)\ \Big(x^T(t-h_1) - x^T(t-h_2)\Big)\ \ldots\ \Big(x^T(t-h_{m-1}) - x^T(t-h_m)\Big)\Big]^T$.

Applying Lemma 1 and defining a vector that $\zeta(t) = \left[x^T(t)\ \dot{x}^T(t)\ \beta_1^T(t)\ \beta_2^T(t)\right]^T$ with $\beta_1(t) = \left[x^T(t-h_1) \dots x^T(t-h_m)\right]^T$ and $\beta_2(t) = \left[\int_{t-h_1}^{t} x^T(s)ds \dots \int_{t-h_m}^{t-h_{m-1}} x^T(s)ds\right]^T$, the following new upper bound of $\dot{V}(t)$ is obtained:

$$\dot{V}(t) \le \zeta^T(t)\Sigma\zeta(t), \tag{6}$$

and according to Lemma 2 with $0 = \Gamma\zeta(t)$, Eq. (6) is equivalent to $(\Gamma^{\perp})^T\Sigma(\Gamma^{\perp})$. Therefore, if (4) is holds, then the system (2) is stable. This completes the proof. ■

Remark 1. It is noted that, when $m = 1$, the Lyapunov functional (5) can be

$$\begin{aligned} V(t) = &\begin{bmatrix} x(t) \\ \int_{t-h}^{t} x(s)ds \end{bmatrix}^T P \begin{bmatrix} x(t) \\ \int_{t-h}^{t} x(s)ds \end{bmatrix} \\ &+ \int_{t-h}^{t} x^T(s)Q_1x(s)ds \\ &+ \int_{-h}^{0}\int_{t+\theta}^{t} x^T(s)Rx(s)dsd\theta. \end{aligned} \tag{7}$$

By letting $P = \begin{bmatrix} P & Q \\ \star & Z \end{bmatrix}$, $Q_1 = S$, Lyapunov functional (7) reduces to the Lyapunov functional of Seuret and Gouaisbaut (2012). Therefore, the proposed method extend the results from Seuret and Gouaisbaut (2012, 2013a).

Remark 2. The considered Lyapunov functional (5) is very simple and fundamentally used in numerous papers. Until now, there are many attempt in constructing Lyapunov functional to reduce conservatism of stability of the systems, such as terms of double and triple integral of states and time derivative of states, terms of fourth integral of states and so on. It should be pointed out that our result may be improved easily, when above commented Lyapunov functional technique is applied to our method.

In Theorem 1, a time-delay is evenly divided into m periods. On the contrary, when uneven dividing bounds are considered, the following theorem is obtained.

Corollary 4. For given a constant delay h, and a positive integer m, positive scalars $\alpha_i < 1$ $(i = 1, \dots, m-1)$ the linear time-delay system (2) is stable, if there exist positive definite matrices P, Q_i $(i = 1, \dots, m)$, R, such that

$$(\Gamma^{\perp})^T\bar{\Sigma}(\Gamma^{\perp}) < 0, \tag{8}$$

where

$$\begin{aligned} \bar{\Sigma} &= \Pi_1 P \Pi_2^T + \Pi_2 P \Pi_1^T + he_2Re_2^T - \Pi_3\bar{M}(1)\Pi_3^T \\ &\quad - \sum_{i=1}^{m-1} \Pi_4(i)\bar{M}(i+1)\Pi_4^T(i) + \Phi, \end{aligned}$$

$$\bar{M}(i) = \frac{1}{\bar{h}_i - \bar{h}_{i-1}} \begin{bmatrix} 4R & 2R & -\frac{6}{\bar{h}_i - \bar{h}_{i-1}}R \\ \star & 4R & -\frac{6}{\bar{h}_i - \bar{h}_{i-1}}R \\ \star & \star & \frac{12}{(\bar{h}_i - \bar{h}_{i-1})^2}R \end{bmatrix},$$

$\bar{h}_i = \alpha_i(h - \bar{h}_{i-1}) + \bar{h}_{i-1}$ $(i = 1, \dots, m-1)$,
$\bar{h}_0 = 0$, $\bar{h}_m = h$,

and other notations are defined in Theorem 1.

Proof. The proof of Corollary 4 is same to Theorem 3 when h_i are replaced to $\bar{h}_i$, so it is omitted.

Remark 3. In Theorem 3, the range of the time delay, h, is divided into m subintervals evenly, i.e. $h_i = \frac{h}{m}i$ $(i = 1, \dots, m)$. On the other hand, by introducing tuning parameters $\alpha_i \in (0,\ 1)$ $(i = 1, \dots, m-1)$, uneven m subintervals of h, i.e. $\bar{h}_i = \alpha_i(h - \bar{h}_{i-1}) + \bar{h}_{i-1}$, $(i = 1, \dots, m-1)$ are considered in Corollary 4. It should be noted that Theorem 3 is a special case of Corollary 4, e.g. when $m = 3$, $\alpha_1 = \frac{1}{3}$, and $\alpha_2 = \frac{1}{2}$, Corollary 4 is same to Theorem 3 with $m = 3$. The advantage of this approach is that the feasible region of stability criterion can be enhanced thanks to adjustable tuning parameters, and it will be shown through numerical examples.

4. NUMERICAL EXAMPLES

In this section, two numerical examples are given to show less conservative results of proposed methods than the existing ones.

4.1 Example 1

Consider the most well known model of time-delay system (2):

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}.$$

This system is known that its maximum analytic delay bound is $h = 6.1721$ and it can be easily computed by delay sweeping techniques. Our results from Theorem 3 and Corollary 4 and recent results are shown in Table 1 in which the maximum bound on delay using Corollary 4 is listed with best case of tuning parameter values. From Table 1, it can be seen that Theorem 3 improves the feasible region of stability criteria compared to all remarkable existing works. Furthermore, Corollary 4 also verifies the effectiveness in improvement of feasible region. It should be noticed that Theorem 3 with $m = 6$ and Corollary 4 with $m = 5$, and many cases of $[\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4]$, give the maximum upper bound of the delay of the system as 6.1721 which is theoretical bound to ensure the stability of the system. In order to show the effectiveness of tuning parameters, by applying Corollary 4, the maximum upper bounds on delay of the system with various tuning parameters in the cases $m = 2$ and $m = 3$ are presented in Table 2 and Fig. 1, respectively.

Table 1. Comparison of maximum bound of h.

Methods	h	Number of variables
He et al. (2007)	4.472	18
Shao (2009)	4.472	13
Kao and Rantzer (2007)	6.1107	33
Ariba et al. (2010)	5.12	36
Sun et al. (2010)	5.02	108
Kim (2011)	4.97	206
Gu et al. (2003)	6.059	25
Ariba and Gouaisbaut (2009)	5.120	33
Seuret and Gouaisbaut (2012)	5.901	16
Seuret and Gouaisbaut (2013b)	6.059	16
Theorem 3 ($m=2$)	6.1577	30
Theorem 3 ($m=3$)	6.1686	48
Theorem 3 ($m=4$)	6.1710	70
Theorem 3 ($m=5$)	6.1718	96
Theorem 3 ($m=6$)	**6.1721**	126
Corollary 4 ($m=2$) $\alpha_1=0.6$	6.1596	30
Corollary 4 ($m=3$) $[\alpha_1\ \alpha_2]=[0.6\ 0.7]$	6.1691	48
Corollary 4 ($m=4$) $[\alpha_1\ \alpha_2\ \alpha_3]=[0.3\ 0.4\ 0.5]$	6.1712	70

Table 2. The maximum bound of h with $m=2$ and various α_1.

α_1	0.1	0.2	0.3	0.4	0.5
h	6.1035	6.1301	6.1402	6.1507	6.1580
α_1	0.6*	0.7	0.8	0.9	
h	6.1596	6.1522	6.1500	6.1050	

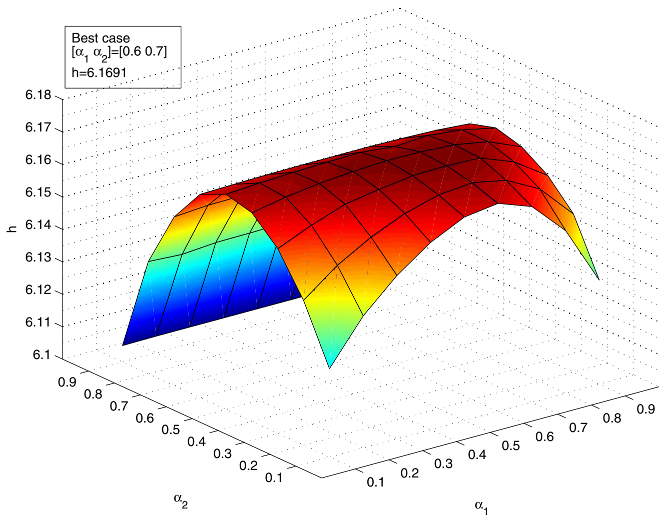

Fig. 1. The allowable maximum h with $m=3$ and various α_1 and α_2.

4.2 Example 2

Let us consider the linear time delay system (2) with the following matrices:

$$A=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix},\quad B=\begin{bmatrix}0 & 0\\ 0 & -1\end{bmatrix}.$$

By Theorem 3 and Corollary 4, the improvement results of this paper are shown in Table 3. It is noticed that the best case of α_i $(i=1,\ldots,m-1)$ of example 2 is evenly divided subinterval case, i.e. $\alpha_i=\frac{1}{m-i+1}$ $(i=1,\ldots,m-1)$, it means Theorem 3 is the best case of Corollary 4. In addition, the results by Corollary 4 with $m=2$ is presented in Table 4 as various α_1 cases.

Table 3. Comparison of maximum bound of h.

Methods	h	Number of variables
Wu et al. (2004)	1.82	27
Park and Ko (2007)	1.99	35
Kim (2011)	2.52	33
Theorem 3 ($m=2$)	3.1349	30
Theorem 3 ($m=3$)	3.1402	48
Theorem 3 ($m=4$)	3.1411	70
Corollary 4 ($m=2$) $\alpha_1=0.5$	3.1349	30
Corollary 4 ($m=3$) $[\alpha_1\ \alpha_2]=[\frac{1}{3}\ 0.5]$	3.1402	48
Corollary 4 ($m=4$) $[\alpha_1\ \alpha_2\ \alpha_3]=[0.25\ \frac{1}{3}\ 0.5]$	3.1411	70

Table 4. The maximum bound of h with $m=2$ and various α_1.

α_1	0.1	0.2	0.3	0.4	0.5*
h	3.0774	3.1050	3.1220	3.1316	3.1349
α_1	0.6	0.7	0.8	0.9	
h	3.1316	3.1220	3.1050	3.0774	

5. CONCLUSIONS

In this paper, the problem of delay-dependent stability analysis of time-delay systems has been discussed. A new lemma has been proposed based on Wirtinger inequality which reduce Jensen's inequality gap and gives less conservatism of stability of the system. In addition, by introducing tuning parameters, a more general criterion has been derived. Two numerical examples have been given to illustrate the less conservatism of the proposed method.

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