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INCLUSION PRINCIPLE AND OVERLAPPING DECOMPOSITIONS OF DISTRIBUTED-TIME-DELAY SYSTEMS

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Abstract: The inclusion principle for distributed-time-delay systems is defined. The properties of input-output map equivalence and preservation of stability between two systems, one of which includes the other, are presented. Controller and observer design, using the inclusion principle, are also discussed. Finally, overlapping decompositions of distributed-time-delay systems is illustrated. The use of overlapping decompositions in controller and observer design is also presented.

Keywords: Large-scale systems; time-delay systems; distributed time-delay; overlapping decompositions; controller design; observer design.

1. INTRODUCTION

Many physical systems, especially large-scale systems, usually involve time-delays. When the amount of the time-delay is small, it may either be ignored or approximated by finite-dimensional dynamics (Brezinski (1996)). When the amount of the time-delay is large, however, it must be explicitly considered (Loiseau et al. (2009)). The controller design problem for a time-delay system is more difficult than for a finite-dimensional system, since a time-delay system is infinite-dimensional (Niculescu (2001)). When the time-delay is distributed, this problem becomes even more challenging (e.g., see Santos et al. (2006)).

When designing controllers for large-scale systems, decomposition techniques are usually needed. Many large-scale systems, however, involve subsystems which are loosely interconnected among themselves, but strongly interconnected through certain dynamics (e.g., see Šiljak (1991)). To obtain meaningful decompositions for such systems, the approach of *overlapping decompositions* has first been introduced by Ikeda and Šiljak (1980). This approach have since been used to design controllers for large-scale systems (e.g., Ikeda et al. (1981); İftar and Özgüner (1987, 1990, 1998); İftar (2004)). The theoretical framework of the overlapping decompositions is the *inclusion principle* (Ikeda et al. (1984)). Although the inclusion principle and overlapping decompositions have been recently considered for time-delay systems with discrete time-delays (e.g., Bakule et al. (2005a,b); Bakule and Rossell (2008); İftar (2008)), to the author's best knowledge, systems with distributed time-delay have not yet been considered within this framework. Therefore, in this work, we first formally define the inclusion principle for distributed-time-delay systems in Section 2. The properties of input-output map equivalence and preservation of stability between two systems, one of which includes the other, are presented in Section 3. Controller and observer design, using the inclusion principle, are discussed in Sections 4 and 5 respectively. Finally, overlapping decompositions and the use of overlapping decompositions in controller and observer design are presented in Section 6.

Throughout the paper, $\mathbf{R}$ and $\mathbf{R}_+$ denote the sets of, respectively, real numbers and non-negative real numbers. For positive integers k and l, $\mathbf{R}^k$ and $\mathbf{R}^{k\times l}$ denote the spaces of, respectively, k-dimensional real vectors and $k\times l$-dimensional real matrices. I_k denotes the $k\times k$-dimensional identity matrix. 0 may denote either the scalar zero, a zero vector, a zero matrix, or a matrix function which is identically zero. For a vector function $x(\cdot)$, $\dot{x}(t)$ is the derivative of $x(t)$ with respect to t.

2. INCLUSION PRINCIPLE

In this section, we extend the inclusion principle to distributed-time-delay systems. Consider linear time-invariant (LTI) retarded distributed-time-delay systems, Σ:

$$\dot{x}(t) = \int_{-\tau}^{0} \left(A(\theta)x(t+\theta) + B(\theta)u(t+\theta)\right) d\theta \qquad (1)$$

$$y(t) = \int_{-\tau}^{0} C(\theta)x(t+\theta)d\theta \qquad (2)$$

and $\hat{\Sigma}$:

$$\dot{\hat{x}}(t) = \int_{-\tau}^{0} \left(\hat{A}(\theta)\hat{x}(t+\theta) + \hat{B}(\theta)\hat{u}(t+\theta)\right) d\theta \qquad (3)$$

$$\hat{y}(t) = \int_{-\tau}^{0} \hat{C}(\theta)\hat{x}(t+\theta)d\theta \qquad (4)$$

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where the initial conditions for Σ and $\hat{\Sigma}$ are respectively given as:

$$x(\theta) = \phi(\theta) \quad \text{and} \quad \hat{x}(\theta) = \hat{\phi}(\theta) \;, \qquad \theta \in [-\tau, 0] \tag{5}$$

for some functions $\phi : [-\tau, 0] \to \mathbf{R}^n$ and $\hat{\phi} : [-\tau, 0] \to \mathbf{R}^{\hat{n}}$. In (1)–(4), $x \in \mathbf{R}^n$, $u \in \mathbf{R}^p$, and $y \in \mathbf{R}^q$ are, respectively, the state, the input, and the output vectors of Σ, and $\hat{x} \in \mathbf{R}^{\hat{n}}$, $\hat{u} \in \mathbf{R}^p$, and $\hat{y} \in \mathbf{R}^q$ are, respectively, the state, the input, and the output vectors of $\hat{\Sigma}$. It is assumed that the dimensions of the input and output vectors of Σ and $\hat{\Sigma}$ are the same, but $\hat{\Sigma}$ has a larger dimensional state vector than that of Σ; i.e., $\hat{n} > n$. In (1)–(4), $A(\cdot) : [-\tau, 0] \to \mathbf{R}^{n\times n}$, $B(\cdot) : [-\tau, 0] \to \mathbf{R}^{n\times p}$, $C(\cdot) : [-\tau, 0] \to \mathbf{R}^{q\times n}$, $\hat{A}(\cdot) : [-\tau, 0] \to \mathbf{R}^{\hat{n}\times\hat{n}}$, $\hat{B}(\cdot) : [-\tau, 0] \to \mathbf{R}^{\hat{n}\times p}$, and $\hat{C}(\cdot) : [-\tau, 0] \to \mathbf{R}^{q\times\hat{n}}$ are bounded matrix functions, except that they may involve Dirac delta terms. By this assumption, Σ and $\hat{\Sigma}$ are allowed to have discrete time-delays besides distributed time-delays. Finally, t denotes the time variable and τ is the maximum time-delay in Σ and $\hat{\Sigma}$.

Definition 1: $\hat{\Sigma}$ *includes* Σ and Σ is *included by* $\hat{\Sigma}$ if there exist a full row-rank matrix $U \in \mathbf{R}^{n\times\hat{n}}$ and a full column-rank matrix $V \in \mathbf{R}^{\hat{n}\times n}$ with $UV = I_n$, such that for all $\phi(\cdot)$ and for all $u(\cdot)$, the choice

$$\hat{\phi}(\theta) = V\phi(\theta) \;, \qquad \theta \in [-\tau, 0] \tag{6}$$

and

$$\hat{u}(t) = u(t) \;, \qquad t \geq -\tau \tag{7}$$

implies

$$x(t) = U\hat{x}(t) \;, \qquad t \geq -\tau \tag{8}$$

and

$$y(t) = \hat{y}(t) \;, \qquad t \geq 0 \;. \tag{9}$$

Two important special cases of inclusion are restriction and aggregation. As we shall see, restriction is useful in controller design and aggregation is useful in observer design.

Definition 2: Σ is said to be a *restriction* of $\hat{\Sigma}$ if there exists a full column-rank matrix $V \in \mathbf{R}^{\hat{n}\times n}$ such that for all $\phi(\cdot)$ and for all $u(\cdot)$, the choice (6) and (7) implies

$$\hat{x}(t) = Vx(t) \;, \qquad t \geq -\tau \tag{10}$$

and (9).

Definition 3: Σ is said to be an *aggregation* of $\hat{\Sigma}$ if there exists a full row-rank matrix $U \in \mathbf{R}^{n\times\hat{n}}$ such that for all $\hat{\phi}(\cdot)$ and for all $\hat{u}(\cdot)$, the choice

$$\phi(\theta) = U\hat{\phi}(\theta) \;, \qquad \theta \in [-\tau, 0] \tag{11}$$

and (7) implies (8) and (9).

The following theorem formally establishes that both restriction and aggregation are special cases of inclusion.

Theorem 1: If Σ is a restriction or an aggregation of $\hat{\Sigma}$, then $\hat{\Sigma}$ includes Σ.

Proof: If Σ is a restriction of $\hat{\Sigma}$, then, since V is of full column-rank, there exists a full row-rank matrix $U \in \mathbf{R}^{n\times\hat{n}}$ with $UV = I_n$. Then (10) implies (8). Thus, $\hat{\Sigma}$ includes Σ.

If Σ is an aggregation of $\hat{\Sigma}$, then, since U is of full row-rank, there exists a full column-rank matrix $V \in \mathbf{R}^{\hat{n}\times n}$ with $UV = I_n$. Then, since (8) and (9) are satisfied for any $\hat{\phi}(\cdot)$, they are satisfied for any $\hat{\phi}(\cdot)$ in the range of V. Thus, $\hat{\Sigma}$ includes Σ. □

Now, without loss of any generality, let us relate the matrix functions of Σ and $\hat{\Sigma}$ as follows:

$$\hat{A}(\theta) = VA(\theta)U + M(\theta) \tag{12}$$

$$\hat{B}(\theta) = VB(\theta) + N(\theta) \tag{13}$$

$$\hat{C}(\theta) = C(\theta)U + L(\theta) \tag{14}$$

for $\theta \in [-\tau, 0]$, where U and V are as in Definitions 1–3 and $M(\cdot) : [-\tau, 0] \to \mathbf{R}^{\hat{n}\times\hat{n}}$, $N(\cdot) : [-\tau, 0] \to \mathbf{R}^{\hat{n}\times p}$, and $L(\cdot) : [-\tau, 0] \to \mathbf{R}^{q\times\hat{n}}$ are bounded matrix functions, except that they may involve Dirac delta terms. Then we have the following results:

Theorem 2: Σ is a restriction of $\hat{\Sigma}$ if and only if

$$M(\theta)V = 0 \;, \qquad N(\theta) = 0 \;, \qquad \text{and} \qquad L(\theta)V = 0 \tag{15}$$

for $\theta \in [-\tau, 0]$.

Proof: First note that (10) for $t \in [-\tau, 0]$ follows from (6). Next, multiply both sides of (1) by V and use (12) and (13). Also use (14) in (2). Compare the resulting equations with (3)–(4) to show that conditions (15) are sufficient for (10) and (9). The necessity of condition $N(\theta) = 0$ follows from the same comparison, since $u(\cdot)$ is arbitrary. To show the necessity of the other two conditions in (15), consider the obtained equations at $t = 0$, in which case $x(t + \theta)$ is replaced by $\phi(\theta)$. The necessity of $M(\theta)V = 0$ and $L(\theta)V = 0$ now follows since $\phi(\cdot)$ is arbitrary. □

Theorem 3: Σ is an aggregation of $\hat{\Sigma}$ if and only if

$$UM(\theta) = 0 \;, \qquad UN(\theta) = 0 \;, \qquad \text{and} \qquad L(\theta) = 0 \tag{16}$$

for $\theta \in [-\tau, 0]$.

Proof: First note that (8) for $t \in [-\tau, 0]$ follows from (11). Next, multiply both sides of (3) by U and use (12) and (13). Also use (14) in (4). Compare the resulting equations with (1)–(2) to show that conditions (16) are sufficient for (8) and (9). The necessity of condition $UN(\theta) = 0$ follows from the same comparison, since $\hat{u}(\cdot)$ is arbitrary. To show the necessity of the other two conditions in (16), consider the obtained equations at $t = 0$, in which case $\hat{x}(t + \theta)$ is replaced by $\hat{\phi}(\theta)$. The necessity of $UM(\theta) = 0$ and $L(\theta) = 0$ now follows since $\hat{\phi}(\cdot)$ is arbitrary. □

3. INPUT-OUTPUT MAPS AND STABILITY

When $\hat{\Sigma}$ includes Σ, some important properties are preserved between the two systems. Here we will consider two such properties: input-output map and stability. Let us first formally define these properties.

Definition 4: Two systems with the same number of inputs and outputs, such as Σ and $\hat{\Sigma}$, are said to *have the same input-output map* if, for any input, they produce the same output in response to the same input when their initial conditions are zero.

Definition 5: A system, such as Σ or $\hat{\Sigma}$, is said to be *bounded-input bounded-output (BIBO) stable* if, in response to any bounded input, it produces a bounded output when its initial condition is zero.

Definition 6: A system, such as Σ or $\hat{\Sigma}$, is said to be (asymptotically) stable if, for any bounded initial condition, its state remains bounded (and asymptotically goes to zero as time goes to infinity) when its input is zero.

Now we can prove the following.

Theorem 4: If $\hat{\Sigma}$ includes Σ, then Σ and $\hat{\Sigma}$ have the same input-output map.

Proof: When both Σ and $\hat{\Sigma}$ have zero intial conditions, then (6) is satisfied. Then, when the two systems have the same input, i.e., when (7) is satisfied, by (9), they produce the same output. □

Remark 1: Note that, Theorem 4 implies that when $\hat{\Sigma}$ includes Σ, then the two systems have the same transfer function matrix, i.e.,

$$\bar{\hat{C}}(s)\left(sI_{\hat{n}} - \bar{\hat{A}}(s)\right)^{-1}\bar{\hat{B}}(s) = \bar{C}(s)\left(sI_n - \bar{A}(s)\right)^{-1}\bar{B}(s)$$

where $\bar{R}(s) := \int_{-\tau}^{0} R(\theta)e^{s\theta}d\theta$, where R stands for either A, B, C, $\hat{A}$, $\hat{B}$, or $\hat{C}$.

Theorem 5: If $\hat{\Sigma}$ includes Σ, then Σ is BIBO stable if and only if $\hat{\Sigma}$ is BIBO stable.

Proof: Follows from Theorem 4. □

Theorem 6: If $\hat{\Sigma}$ includes Σ, then (asymptotic) stability of $\hat{\Sigma}$ implies (asymptotic) stability of Σ.

Proof: When both Σ and $\hat{\Sigma}$ have zero inputs, (7) holds. Furthermore, for any bounded $\phi(\cdot)$, $\hat{\phi}(\cdot)$ given by (6) is also bounded. Then, if $\hat{\Sigma}$ is (asymptotically) stable, $\hat{x}(t)$ is bounded for all $t \geq 0$ (and asymptotically goes to zero as t goes to infinity). Then, by (8), $x(t)$ is also bounded for all $t \geq 0$ (and asymptotically goes to zero as t goes to infinity). Thus, Σ is also (asymptotically) stable. □

Remark 2: Since, by Theorem 1, restriction and aggregation are two special cases of inclusion, Theorems 4–6 also hold when Σ is either a restriction or an aggregation of $\hat{\Sigma}$.

4. CONTROLLER DESIGN

Here we will consider only state feedback controllers. The present analysis can also be extended to output feedback controllers, which we intend to do in a future work. Another alternative, when the whole state is not available for feedback, is to use an observer to observe the state, and, by using the *separation principle* (Fattouh et al. (2000)), use this observation instead of the actual state vector. We will discuss observer design in the next section. Consider the following state feedback controllers for Σ and $\hat{\Sigma}$ respectively:

$$v(t) = \int_{-\sigma}^{0} K(\theta)x(t+\theta)d\theta \tag{17}$$

and

$$\hat{v}(t) = \int_{-\sigma}^{0} \hat{K}(\theta)\hat{x}(t+\theta)d\theta \tag{18}$$

where $v \in \mathbf{R}^p$ and $\hat{v} \in \mathbf{R}^p$ are the outputs of the controllers (17) and (18) respectively (which are to be applied to the inputs of, respectively, Σ and $\hat{\Sigma}$), σ is the maximum time-delay in the controllers, and $K(\cdot) : [-\sigma, 0] \to \mathbf{R}^{p\times n}$ and $\hat{K}(\cdot) : [-\sigma, 0] \to \mathbf{R}^{p\times\hat{n}}$ are bounded matrix functions, except that they may involve Dirac delta terms. Here, it is assumed that $\sigma \leq \tau$. This assumption is justified since, in general, one does not need to employ any time-delays in a controller which are longer than the time-delays of the system to be controlled. Note that, above formulation allows representation of controllers which may have distributed and/or discrete time-delays, as well as delay-free static controllers. A delay-free static controller of the form $v(t) = K_0x(t)$, where $K_0 \in \mathbf{R}^{p\times n}$ can be represented as in (17) by letting $\sigma = 0^+$ and $K(\theta) = K_0\delta(\theta)$, where $\delta(\cdot)$ is the Dirac delta function.

Since the outputs, v and $\hat{v}$, of (17) and (18) are respectively to be applied to the inputs of Σ and $\hat{\Sigma}$, in order to satisfy condition (7) after the application of these controllers, we must have the following property:

Definition 7: The controller (18) for $\hat{\Sigma}$ is said to be *contractible* to the controller (17) for Σ if there exists a full column-rank matrix $V \in \mathbf{R}^{\hat{n}\times n}$ such that for all $\phi(\cdot)$ and for all $u(\cdot)$, the choice (6) and (7) implies

$$v(t) = \hat{v}(t)\ , \quad t \geq 0 \tag{19}$$

Theorem 7: Suppose that $\hat{\Sigma}$ includes Σ. Then, the controller (18) for $\hat{\Sigma}$ is contractible to the controller (17) for Σ if

$$K(\theta)U = \hat{K}(\theta)\ , \quad \theta \in [-\sigma, 0]\ , \tag{20}$$

where U is as in Definition 1.

Proof: Since $\hat{\Sigma}$ includes Σ, when (6) and (7) are satisfied, (8) is satisfied. Substitute (8) into (17) and compare with (18). Then, since $\sigma \leq \tau$, (20) implies (19). □

Since, for most applications (see Section 6), the controller is to be designed for the larger dimensional system $\hat{\Sigma}$ and then to be contracted for application to Σ, it is important that any controller designed for $\hat{\Sigma}$ to be contractible to a controller for Σ. However, in order to obtain a $K(\cdot)$ which satisfies (20), for any $\theta \in [-\sigma, 0]$, any row of $\hat{K}(\theta)$ must be spaned by the rows of U. Since rank$(U) = n < \hat{n}$, however, this means that not every controller of the form (18) is contractible to a controller of the form (17) in general. However, if Σ is a restriction of $\hat{\Sigma}$, then any controller of the form (18) is contractible to a controller of the form (17), as shown by the following result.

Theorem 8: If Σ is a restriction of $\hat{\Sigma}$, then any controller of the form (18) for $\hat{\Sigma}$ is contractible to a controller of the form (17) for Σ with

$$K(\theta) = \hat{K}(\theta)V\ , \quad \theta \in [-\sigma, 0]\ , \tag{21}$$

where V is as in Definition 2.

Proof: Since Σ is a restriction of $\hat{\Sigma}$, (6) and (7) implies (10). Substitute (10) into (18) and compare with (17). Then, since $\sigma \leq \tau$, (21) implies (19). □

Now, let us apply the outputs of the controllers (17) and (18) to the inputs of, respectively, Σ and $\hat{\Sigma}$ as follows:

$$u(t) = v(t) + w(t)\ , \qquad t \geq 0 \tag{22}$$

and

$$\hat{u}(t) = \hat{v}(t) + \hat{w}(t)\ , \qquad t \geq 0 \tag{23}$$

where $w \in \mathbf{R}^p$ and $\hat{w} \in \mathbf{R}^p$ are the external inputs of the respective closed-loop systems. Let us denote these closed-loop systems by Σ_c and $\hat{\Sigma}_c$ respectively. Then, we have the following result.

Theorem 9: Let Σ be a restriction of $\hat{\Sigma}$ and controller (18) be contractible to controller (17). Also suppose that (6) is satisfied not only for $\theta \in [-\tau, 0]$, but for $\theta \in [-\tau - \sigma, 0]$. Then Σ_c is a restriction of $\hat{\Sigma}_c$.

Proof: The condition (10) for $t \in [-\tau - \sigma, 0]$ follows from (6). To establish the same condition for $t \geq 0$, write the equation describing the dynamics of Σ_c by substituting (17) into (22) and this into (1). Similarly, write the equation describing the dynamics of $\hat{\Sigma}_c$ by substituting (18) into (23) and this into (3). Then, multiply both sides of the equation describing the dynamics of Σ_c by V and also use (12), (13), (15), (21), and $\hat{w}(t) = w(t)$, for $t \geq -\tau$ (the equivalent of (7) for the closed-loop systems). Compare the resulting equation with the equation describing the dynamics of $\hat{\Sigma}_c$ to establish (10) for $t \geq 0$. Once (10) is established, (9) can be established by using (14), the last condition in (15), and $UV = I_n$ in (4). □

The above theorem, together with the results in Section 3, leads to the following result.

Corollary 1: Let Σ be a restriction of $\hat{\Sigma}$ and controller (18) be contractible to controller (17). Also suppose that (6) is satisfied not only for $\theta \in [-\tau, 0]$, but for $\theta \in [-\tau - \sigma, 0]$. Then

(i) Σ_c and $\hat{\Sigma}_c$ have the same input-output map (respectively from w to y and from $\hat{w}$ to $\hat{y}$).
(ii) Σ_c is BIBO stable if and only if $\hat{\Sigma}_c$ is BIBO stable.
(iii) If $\hat{\Sigma}_c$ is (asymptotically) stable then Σ_c is (asymptotically) stable.

The above results allow one to design a controller for the larger dimensional system $\hat{\Sigma}$ to satisfy certain properties, such as stability and input-output response, and then contract it (using (21)) to apply on the system Σ, whenever Σ is a restriction of $\hat{\Sigma}$.

5. OBSERVER DESIGN

Consider the observer Ω, described by

$$\dot{z}(t) = \int_{-\sigma}^{0} \left(F(\theta)z(t+\theta) + G(\theta)y(t+\theta) + E(\theta)u(t+\theta)\right) d\theta \tag{24}$$

$$v(t) = \int_{-\sigma}^{0} \left(H(\theta)z(t+\theta) + K(\theta)y(t+\theta)\right) d\theta \tag{25}$$

for Σ and the observer $\hat{\Omega}$, described by

$$\dot{\hat{z}}(t) = \int_{-\sigma}^{0} \left(\hat{F}(\theta)\hat{z}(t+\theta) + \hat{G}(\theta)\hat{y}(t+\theta) + \hat{E}(\theta)\hat{u}(t+\theta)\right) d\theta \tag{26}$$

$$\hat{v}(t) = \int_{-\sigma}^{0} \left(\hat{H}(\theta)\hat{z}(t+\theta) + \hat{K}(\theta)\hat{y}(t+\theta)\right) d\theta \tag{27}$$

for $\hat{\Sigma}$. The initial conditions for Ω and $\hat{\Omega}$ are respectively given as:

$$z(\theta) = \zeta(\theta) \quad \text{and} \quad \hat{z}(\theta) = \hat{\zeta}(\theta)\ , \quad \theta \in [-\sigma, 0] \tag{28}$$

for some functions $\zeta : [-\sigma, 0] \to \mathbf{R}^m$ and $\hat{\zeta} : [-\sigma, 0] \to \mathbf{R}^{\hat{m}}$. In (24)–(27), $F(\cdot) : [-\sigma, 0] \to \mathbf{R}^{m\times m}$, $G(\cdot) : [-\sigma, 0] \to \mathbf{R}^{m\times q}$, $E(\cdot) : [-\sigma, 0] \to \mathbf{R}^{m\times p}$, $H(\cdot) : [-\sigma, 0] \to \mathbf{R}^{n\times m}$, $K(\cdot) : [-\sigma, 0] \to \mathbf{R}^{n\times q}$, $\hat{F}(\cdot) : [-\sigma, 0] \to \mathbf{R}^{\hat{m}\times\hat{m}}$, $\hat{G}(\cdot) : [-\sigma, 0] \to \mathbf{R}^{\hat{m}\times q}$, $\hat{E}(\cdot) : [-\sigma, 0] \to \mathbf{R}^{\hat{m}\times p}$, $\hat{H}(\cdot) : [-\sigma, 0] \to \mathbf{R}^{\hat{n}\times\hat{m}}$, and $\hat{K}(\cdot) : [-\sigma, 0] \to \mathbf{R}^{\hat{n}\times q}$ are bounded matrix functions, except that they may involve Dirac delta terms, σ is the maximum time-delay in Ω and $\hat{\Omega}$, and $z \in \mathbf{R}^m$ and $\hat{z} \in \mathbf{R}^{\hat{m}}$ are the state vectors of Ω and $\hat{\Omega}$ respectively. Finally, $v \in \mathbf{R}^n$ and $\hat{v} \in \mathbf{R}^{\hat{n}}$ are the outputs of Ω and $\hat{\Omega}$ respectively, which are assumed to be *observations* of $x \in \mathbf{R}^n$ and $\hat{x} \in \mathbf{R}^{\hat{n}}$ respectively. It is assumed that $\hat{m} \geq m$ and $\sigma \leq \tau$. The first of these assumptions is justified since the dynamics of Σ consists of a part of the dynamics of $\hat{\Sigma}$ and hence Σ should not require an observer with a higher dimension than that of $\hat{\Sigma}$. The second assumption is justified since to observe the state of a system with maximum time-delay τ, one does not need to employ an observer with a longer time-delay.

As in the case of controller design, the usual application is to design an observer $\hat{\Omega}$ for the larger dimensional system $\hat{\Sigma}$ and then obtain an observer Ω for Σ by *contracting* $\hat{\Omega}$. For this, however, the following must hold.

Definition 8: The observer $\hat{\Omega}$ for $\hat{\Sigma}$ is said to be *contractible* to the observer Ω for Σ if there exist a full row-rank matrix $U \in \mathbf{R}^{n\times\hat{n}}$, a full row-rank matrix $P \in \mathbf{R}^{m\times\hat{m}}$, and a full column-rank matrix $Q \in \mathbf{R}^{\hat{m}\times m}$ with $PQ = I_m$, such that for all $\zeta(\cdot)$, for all $\hat{\phi}(\cdot)$, and for all $\hat{u}(\cdot)$, when (11) and (7) are satisfied, the choice

$$\hat{\zeta}(\theta) = Q\zeta(\theta)\ , \qquad \theta \in [-\sigma, 0]\ , \tag{29}$$

implies

$$z(t) = P\hat{z}(t)\ , \qquad t \geq -\sigma \tag{30}$$

and

$$v(t) = U\hat{v}(t)\ , \qquad t \geq 0 \tag{31}$$

Since $\hat{\Omega}$, designed for $\hat{\Sigma}$, is to be contracted to Ω to observe the state of Σ, it is important that any observer designed for $\hat{\Sigma}$ to be contractible to an observer for Σ. The following theorem shows that this is the case if Σ is an aggregation of $\hat{\Sigma}$.

Theorem 10: If Σ is an aggregation of $\hat{\Sigma}$, then any observer of the form $\hat{\Omega}$ for $\hat{\Sigma}$ is contractible to an observer of the form Ω for Σ with

$$F(\theta) = \hat{F}(\theta) \ , \qquad G(\theta) = \hat{G}(\theta) \ , \qquad E(\theta) = \hat{E}(\theta) \ , \quad (32)$$

$$H(\theta) = U\hat{H}(\theta) \ , \qquad \text{and} \qquad K(\theta) = U\hat{K}(\theta) \ , \quad (33)$$

for $\theta \in [-\sigma, 0]$, where U is as in Definition 3.

Proof: Let $m = \hat{m}$ and $P = Q = I_m$. Note that, (30) for $t \in [-\sigma, 0]$ follows from (29). To establish (30) and (31) for $t \geq 0$, note that, since Σ is an aggregation of $\hat{\Sigma}$, (11) and (7) implies (8) and (9). Use (11), (7)–(9), (29), (32), and (33) in (24)–(25) and compare with (26)–(27) to show (30) and (31). □

Now, suppose that an observer of the form $\hat{\Omega}$, which achieves *desired observation*, is designed for $\hat{\Sigma}$ and contracted to an observer of the form Ω for Σ. The observer Ω must then achieve desired observation for Σ. First, let us formally define what we mean by desired observation.

Definition 9: An observer, such as Ω, designed for a system, such as Σ, is said to achieve *desired observation* with respect to a given norm $\|\cdot\|$ on $\mathbf{R}^n$ and a *bound function* $g(\cdot) : [0, \infty) \to \mathbf{R}_+$ (which may depend on initial conditions and typically $\lim_{t\to\infty} g(t) = 0$), if for any $\phi(\cdot)$, for any $u(\cdot)$ and for any $\zeta(\cdot)$, $\|x(t) - v(t)\| \leq g(t)$, for all $t \geq 0$.

Now we have the following result.

Theorem 11: Let $\|\cdot\|_n$ and $\|\cdot\|_{\hat{n}}$ be norms defined on $\mathbf{R}^n$ and $\mathbf{R}^{\hat{n}}$ respectively. Let $\|\cdot\|_{n,\hat{n}}$ denote the matrix norm induced by $\|\cdot\|_n$ and $\|\cdot\|_{\hat{n}}$. Also let Σ be an aggregation of $\hat{\Sigma}$ and $\hat{\Omega}$ be contractible to Ω, where (11), (7), and (29) are satisfied. Suppose that $\hat{\Omega}$ achieves desired observation with respect to $\|\cdot\|_{\hat{n}}$ and a given $\hat{g}(\cdot) : [0, \infty) \to \mathbf{R}_+$. Then Ω achieves desired observation with respect to $\|\cdot\|_n$ and $g(\cdot) := \gamma\hat{g}(\cdot)$, where $\gamma := \|U\|_{n,\hat{n}}$, where U is as in Definition 3.

Proof: Since (11), (7), and (29) are satisfied, (8) and (31) are also satisfied. Thus, $\|x(t) - v(t)\|_n = \|U(\hat{x}(t) - \hat{v}(t))\|_n \leq \gamma\|\hat{x}(t) - \hat{v}(t)\|_{\hat{n}}$. Then, $\|\hat{x}(t) - \hat{v}(t)\|_{\hat{n}} \leq \hat{g}(t)$, for $t \geq 0$, implies $\|x(t) - v(t)\|_n \leq g(t)$, for $t \geq 0$. □

The above results imply that one can design an observer for the larger dimensional system $\hat{\Sigma}$ to achieve desired observation and then contract it (using (32)–(33)) to an observer for the system Σ, whenever Σ is an aggregation of $\hat{\Sigma}$.

6. OVERLAPPING DECOMPOSITIONS

In practice large-scale systems may have subsystems whose dynamics may overlap in many different ways (see İftar (1993)). The simplest case, which has also been the most investigated case in the literature, is two subsystems which are overlapped through certain dynamics. The state, the input, and the output vectors of the system in this case can be decomposed as follows:

$$x = \begin{bmatrix} x_1 \\ x_c \\ x_2 \end{bmatrix} , \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} , \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} , \qquad (34)$$

where $x_i \in \mathbf{R}^{n_i}$, $u_i \in \mathbf{R}^{p_i}$, and $y_i \in \mathbf{R}^{q_i}$ are, respectively, the state, the input, and the output vectors of the i^{th} subsystem, for $i = 1, 2$, and $x_c \in \mathbf{R}^{n_c}$ is the state vector of the overlapping part. Such a system can then be *expanded* by using either

$$V = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_c} & 0 \\ 0 & I_{n_c} & 0 \\ 0 & 0 & I_{n_2} \end{bmatrix} \text{ and } U = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & \alpha I_{n_c} & \beta I_{n_c} & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \quad (35)$$

or

$$V = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & \alpha I_{n_c} & 0 \\ 0 & \beta I_{n_c} & 0 \\ 0 & 0 & I_{n_2} \end{bmatrix} \text{ and } U = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & I_{n_c} & I_{n_c} & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} , \quad (36)$$

where $\alpha + \beta = 1$ (typically, $\alpha = \beta = \frac{1}{2}$). We note that the first choice is more convenient in the case of controller design and the second choice is more convenient in the case of observer design.

Now, consider a system Σ, described as in (1)–(2) and decomposed as in (34). Let the matrix functions in (1)–(2) be partitioned compatible with (34) as follows:

$$A(\cdot) = \begin{bmatrix} A_{11}(\cdot) & A_{1c}(\cdot) & A_{12}(\cdot) \\ A_{c1}(\cdot) & A_{cc}(\cdot) & A_{c2}(\cdot) \\ A_{21}(\cdot) & A_{2c}(\cdot) & A_{22}(\cdot) \end{bmatrix} , \ B(\cdot) = \begin{bmatrix} B_{11}(\cdot) & B_{12}(\cdot) \\ B_{c1}(\cdot) & B_{c2}(\cdot) \\ B_{21}(\cdot) & B_{22}(\cdot) \end{bmatrix} ,$$

and

$$C(\cdot) = \begin{bmatrix} C_{11}(\cdot) & C_{1c}(\cdot) & C_{12}(\cdot) \\ C_{21}(\cdot) & C_{2c}(\cdot) & C_{22}(\cdot) \end{bmatrix} .$$

If the purpose is to design a controller, then an expansion $\hat{\Sigma}$, described as in (3)–(4), of Σ, such that Σ is a restriction of $\hat{\Sigma}$, can be obtained by choosing V and U as in (35) with $\alpha = \beta = \frac{1}{2}$ and the matrix functions as in (12)–(14), where

$$M(\cdot) = \begin{bmatrix} 0 & \frac{1}{2}A_{1c}(\cdot) & -\frac{1}{2}A_{1c}(\cdot) & 0 \\ 0 & \frac{1}{2}A_{cc}(\cdot) & -\frac{1}{2}A_{cc}(\cdot) & 0 \\ 0 & -\frac{1}{2}A_{cc}(\cdot) & \frac{1}{2}A_{cc}(\cdot) & 0 \\ 0 & -\frac{1}{2}A_{2c}(\cdot) & \frac{1}{2}A_{2c}(\cdot) & 0 \end{bmatrix} , \ N(\cdot) = 0$$

and

$$L(\cdot) = \begin{bmatrix} 0 & \frac{1}{2}C_{1c}(\cdot) & -\frac{1}{2}C_{1c}(\cdot) & 0 \\ 0 & -\frac{1}{2}C_{2c}(\cdot) & \frac{1}{2}C_{2c}(\cdot) & 0 \end{bmatrix} .$$

Note that these choices satisfy the conditions (15); hence, Σ is a restriction of $\hat{\Sigma}$. The state, $\hat{x} \in \mathbf{R}^{\hat{n}}$, of the expanded system $\hat{\Sigma}$ can now be decomposed as

$$\hat{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} , \ \hat{x}_i \in \mathbf{R}^{\hat{n}_i} , \ \hat{n}_i := n_i + n_c , \ i = 1, 2 . \quad (37)$$

Thus, the expanded system $\hat{\Sigma}$ is composed of two disjoint subsystems, $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$, which are *weakly interconnected* (assuming that the only strong interconnections between the original subsystems are through the overlapping part - see Ikeda and Šiljak (1980) for the finite-dimensional case). The state, the input, and the output vectors of $\hat{\Sigma}_i$, for $i = 1, 2$, are, respectively, $\hat{x}_i$, u_i and y_i. Now, suppose that a local controller of the form

$$\hat{v}_i(t) = \int_{-\sigma}^{0} \hat{K}_i(\theta)\hat{x}_i(t+\theta)d\theta \tag{38}$$

is designed for $\hat{\Sigma}_i$, for $i = 1, 2$. A decentralized controller for $\hat{\Sigma}$ can then be described as in (18), where

$$\hat{K}(\theta) = \begin{bmatrix} \hat{K}_1(\theta) & 0 \\ 0 & \hat{K}_2(\theta) \end{bmatrix}, \qquad \theta \in [-\sigma, 0] .$$

This controller can then be contracted to a controller of the form (17), where $K(\cdot)$ is given by (21). Note that this controller has an *overlapping decentralized structure*, where u_1 gets feedback from x_1 and x_c and u_2 gets feedback from x_c and x_2. Assuming that the controller (18) stabilizes the system $\hat{\Sigma}$, by Corollary 1, the contracted controller (17) stabilizes the original system Σ. Furthermore, the two closed-loop systems have the same input-output map.

Now, let us consider observer design for the system Σ, whose state, input, and output vectors are decomposed as in (34). Then, an expansion $\hat{\Sigma}$ of Σ, such that Σ is an aggregation of $\hat{\Sigma}$, can be obtained by choosing V and U as in (36) with $\alpha = \beta = \frac{1}{2}$ and the matrix functions as in (12)–(14), where

$$M(\cdot) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2}A_{c1}(\cdot) & \frac{1}{2}A_{cc}(\cdot) & -\frac{1}{2}A_{cc}(\cdot) & -\frac{1}{2}A_{c2}(\cdot) \\ -\frac{1}{2}A_{c1}(\cdot) & -\frac{1}{2}A_{cc}(\cdot) & \frac{1}{2}A_{cc}(\cdot) & \frac{1}{2}A_{c2}(\cdot) \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$N(\cdot) = \begin{bmatrix} 0 & 0 \\ \frac{1}{2}B_{c1}(\cdot) & -\frac{1}{2}B_{c2}(\cdot) \\ -\frac{1}{2}B_{c1}(\cdot) & \frac{1}{2}B_{c2}(\cdot) \\ 0 & 0 \end{bmatrix}, \qquad \text{and} \qquad L(\cdot) = 0 .$$

These choices satisfy the conditions (16); hence, Σ is an aggregation of $\hat{\Sigma}$. The state, $\hat{x} \in \mathbf{R}^{\hat{n}}$, of the expanded system $\hat{\Sigma}$ can then be decomposed as in (37). Then, as in the case of controller design, the expanded system $\hat{\Sigma}$ is composed of two disjoint subsystems, $\hat{\Sigma}_1$ and $\hat{\Sigma}_2$, which are weakly interconnected. Next, suppose that a local observer $\hat{\Omega}_i$ is designed for $\hat{\Sigma}_i$, for $i = 1, 2$. An observer $\hat{\Omega}$ can then be obtained for the expanded system $\hat{\Sigma}$ by combining these two local observers. This overall observer can then be contracted to an observer Ω for the original system Σ as described in Theorem 10. Assuming that the observer $\hat{\Omega}$ achieves desired observation for $\hat{\Sigma}$, by Theorem 11, the contracted observer Ω achieves desired observation for the original system Σ.

7. CONCLUSION

Inclusion principle and overlapping decompositions has been discussed for LTI retarded distributed-time-delay systems. It has been shown that many concepts which are valid for finite-dimensional systems and for discrete-time-delay systems have extensions in this case. Extensions of the present results to neutral time-delay systems (both in the discrete and distributed time-delay cases) and to nonlinear and/or time-varying time-delay systems are subjects of future research. Another possible direction for future research is to consider overlapping decompositions of input and output spaces besides the state-space.

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