

Converse Lyapunov–Krasovskii theorems for uncertain time-delay systems

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Abstract: In this article, we give a collection of converse Lyapunov–Krasovskii theorems for uncertain time-delay systems. We show that the existence of a weakly-degenerate Lyapunov–Krasovskii functional is necessary and sufficient condition for the global exponential stability of the time-delay systems. This is carried out using the switched system transformation approach.

Keywords: Time-delay system; switched systems; Lyapunov–Krasovskii functional; Converse Lyapunov theorems; exponential stability.

1. INTRODUCTION

The stability of *time-varying delay* systems is a problem of current interest. Two principal approaches in the stability analysis are the Lyapunov–Krasovskii method and Lyapunov–Razumikhin method (see e.g. Hale and Lunel [1993]). A variety of stability criteria, based on these two approaches, have been developed in this context (see e.g. Niculescu [2001], Richard [2003], Fridman [2006], Fridman and Niculescu [2008] and references therein). These criteria are formulated as linear matrix inequalities (LMIs) which yield sufficient conditions for stability. Switched system theory offers a complementary insight in this context. In Hetel et al. [2008] the authors establish a theoretical link between the Lyapunov–Krasovskii approach and the switched system transformation in the context of discrete-time systems with time-varying delays. They prove that applying the multiple Lyapunov functions approach to the switched systems representation is equivalent to using a general, delay dependent, Lyapunov–Krasovskii functional for the initial system. This paper shares the same spirit as Hetel et al. [2008] with the significant difference that it considers the case of continuous-time systems with time-varying delays. An important feature of this setting is that the switched system representation, obtained by standard functional representation, describes an evolution in an infinite dimensional space. Converse Lyapunov theorems for switched systems in Banach and Hilbert spaces were carried out in Hante and Sigalotti [2011]. In this paper, we follow the ideas of Hante and Sigalotti [2011], providing a collection of converse Lyapunov–Krasovskii theorems for uncertain time-delay systems.

Several converse Lyapunov–Krasovskii theorems have been presented in the literature for systems described by retarded functional differential equations, in a general (possibly time-varying but not uncertain) setting (see e.g. Huang

[1989], Gu et al. [2003], Kharitonov and Zhabko [2003], Karafyllis and Jiang [2010], Pepe and Karafyllis [2013]).

More in the spirit of the results presented here are the converse Lyapunov theorems for uncertain nonlinear retarded differential equations obtained in Karafyllis [2006], Karafyllis et al. [2008]. The latter are obtained for a very general class of dynamics (which in particular are allowed to be non-autonomous).

In this work, we consider a linear uncertain time-delay system

$$\dot{x}(t) = \sum_{i=1}^p A_{\sigma(t)}^i x(t - \tau_i(\sigma(t))) \quad t \geq 0 \quad (1)$$

where $x(t) \in \mathbb{R}^n$ represents the system state at time t , the signal $\sigma(\cdot)$ is a piecewise constant function taking values in a (possibly infinite) index set \mathbb{S} , $\tau_i(\cdot)$ represents the time-delay function with $0 \leq \tau_i(\sigma(t)) \leq r$, and A_{σ}^i is a $n \times n$ matrix for every $\sigma \in \mathbb{S}$, $i \in \{1, \dots, p\}$. We are interested in properties that are uniform with respect to $\sigma(\cdot)$ which plays the role of a switched signal. We give a collection of converse Lyapunov–Krasovskii theorems which characterize the uniform exponential stability of the solutions of (1) in terms of the existence of functionals with suitable commensurability conditions and decreasing uniformly along the solutions of (1). One of the novelties of our results is that these functionals may be *weakly-degenerate*, i.e., they may not have a strictly positive norm-dependent lower bound, in contrast with what is known in the literature.

The paper is organized as follows. Section 2 is devoted to the problem framework, where we discuss the well-posedness and the switched system representation of system (1). The statement of the main results is presented in Section 3. The main contribution of this paper is given in Section 4. Section 5 is devoted to the comparison of

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our results with the Lyapunov–Krasovskii theorem given in Hale and Lunel [1993].

2. PROBLEM FRAMEWORK

Suppose $r \geq 0$ is a given real number, \mathbb{R}^n is an n -dimensional linear vector space over the reals with norm $\|\cdot\|$, $C = C([-r, 0], \mathbb{R}^n)$ is the Banach space of continuous functions mapping the interval $[-r, 0]$ into \mathbb{R}^n with the topology of uniform convergence. We denote the norm of an element $\varphi \in C$ by $\|\varphi\|_C = \sup_{-r \leq \theta \leq 0} |\varphi(\theta)|$. Let \mathbb{S} be an index set and consider a family of real matrices parameterized by \mathbb{S} as follows

$$Q := \{A_\sigma \in \mathbb{R}^{n \times n} \mid \sigma \in \mathbb{S}\}.$$

Let us associate with Q the linear uncertain time-delay system

$$\dot{x}(t) = \sum_{i=1}^p A_{\sigma(t)}^i x(t - \tau_i(\sigma(t))) \quad (2)$$

where $x(t) \in \mathbb{R}^n$ represents the system state at time t , the signal $\sigma(\cdot)$ is a piecewise constant function taking values in \mathbb{S} , $\tau(\cdot) = (\tau_1(\cdot), \dots, \tau_p(\cdot))$, with $0 \leq \tau_i(\sigma) \leq r$, and $A_\sigma^i \in Q$ for every $\sigma \in \mathbb{S}$, $i \in \{1, \dots, p\}$. We are interested in properties that are uniform with respect to $\sigma(\cdot)$ which plays the role of a switched signal.

Let the initial condition be

$$x(t_0 + \theta) = \varphi(\theta), \quad \theta \in [-r, 0], \quad (3)$$

where $\varphi(\cdot) \in C$ and $t_0 \geq 0$. If $t \geq t_0$, we adopt the standard notation x_t for the history function defined by

$$x_t : \theta \mapsto x(t + \theta), \quad \theta \in [-r, 0].$$

Of special interest is the case in which Q is a bounded subset of $\mathbb{R}^{n \times n}$, i.e. there exists a positive constant m such that

$$|A_\sigma v| \leq m|v| \quad \forall v \in \mathbb{R}^n, \sigma \in \mathbb{S}. \quad (4)$$

2.1 Switched system representation

A function x is said to be a *solution* of system (2) on $[-r, +\infty)$ with initial condition $\varphi \in C$ if x is absolutely continuous on $[0, +\infty)$, $x_t \in C([-r, 0], \mathbb{R}^n)$ for every $t \geq 0$ and $x(t)$ satisfies system (2), for almost every $t \geq 0$. Under the hypothesis that Q is bounded, it is well-known that Cauchy problem (2)-(3) has a unique solution for each initial condition $\varphi \in C$ (Hale and Lunel [1993]).

Recall that with any $\sigma \in \mathbb{S}$ one can associate a C_0 -semigroup $T_\sigma(t) : C \rightarrow C$ defined by $T_\sigma(t)(x_0) = x(t + \cdot)$ Hale and Lunel [1993]. The evolution operator corresponding to a piecewise constant signal $\sigma(t) = \sum_{k \geq 0} \mathbf{1}_{[t_k, t_{k+1})}(t) \sigma_k$

with $t_0 \geq 0$, $t_k < t_{k+1}$ for $k \geq 0$ is given by

$$T_{\sigma(\cdot)}(t) = T_{\sigma_k}(t - t_k) T_{\sigma_{k-1}}(t_k - t_{k-1}) \dots T_{\sigma_0}(t_1 - t_0)$$

for each $t \in [t_k, t_{k+1})$. In order to define the evolution in this sense one just needs a family of C^0 -semigroups. This is exactly the notion of switched system

$$\begin{aligned} \dot{x}_t &= T_{\sigma(\cdot)}(t)x_0, \\ x_{t_0} &= \varphi \in C, \end{aligned} \quad (5)$$

considered in Hante and Sigalotti [2011] for general Banach spaces.

3. STATEMENT OF THE MAIN RESULTS

The notion of uniform exponential stability is recalled in the following definition.

Definition 1. We say that system (2) is uniformly exponentially stable, with respect to $\|\cdot\|_C$, if there exist constants $a \geq 1$ and $b > 0$ such that for every initial condition $\varphi \in C$ the solution of (2) exists for every $t \in [t_0, +\infty)$ and satisfies

$$\|x_t\|_C \leq a e^{-b(t-t_0)} \|\varphi\|_C, \quad t \geq t_0, \sigma(\cdot)\text{-uniformly.} \quad (6)$$

Here and in the following, σ -uniformly means that the constants a, b do not depend on the choice of the piecewise constant $\sigma : [0, \infty) \rightarrow \mathbb{S}$.

Remark 1. Because of the linear nature of system (2), one can easily check that (2) is uniformly exponentially stable if and only if it is uniformly asymptotically stable. We recall that system (2)-(3) is uniformly asymptotically stable, if for every $R > 0$ there exist $r > 0$ such that, for every solution $x(\cdot)$, $\|\varphi\|_C < r$ implies $\|x_t\|_C < R$, and if, moreover, the solutions of (2) converge to zero uniformly (i.e., there is a constant $\delta > 0$, independent of t_0 and σ , such that for every $\epsilon > 0$, there is a $\zeta(\epsilon)$ such that $\|\varphi\|_C < \delta$ implies $\|x_t\|_C < \epsilon$ for every $t \geq t_0 + \zeta$ and $\sigma \in \mathbb{S}$).

Remark 2. By definition of $T_{\sigma(\cdot)}(t)$, we have that system (2) is uniformly exponentially stable if and only if there exist constants $\alpha > 0$ and $\beta > 0$ such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(C)} \leq \alpha e^{-\beta(t-t_0)}, \quad t \geq t_0, \sigma(\cdot)\text{-uniformly.} \quad (7)$$

From now on, without loss of generality, we will assume $t_0 = 0$.

For a function $V : C \rightarrow [0, \infty)$ we define the generalized derivatives

$$\overline{D}_\sigma V(x) = \limsup_{t \rightarrow 0} \frac{V(T_\sigma(t)x) - V(x)}{t},$$

and

$$\underline{D}_\sigma V(x) = \liminf_{t \rightarrow 0} \frac{V(T_\sigma(t)x) - V(x)}{t},$$

noting the possibility that $\underline{D}_\sigma V(x), \overline{D}_\sigma V(x) = \infty$ for some $x \in C$ and $\sigma \in \mathbb{S}$.

The main result of this paper is the following theorem.

Theorem 1. Consider system (2). Under the assumption that Q is a bounded subset of $\mathbb{R}^{n \times n}$, i.e. that condition (4) holds, the following statements are equivalent:

- (i) System (2) is uniformly exponentially stable, with respect to $\|\cdot\|_C$.
- (ii) There exists a function $V : C \rightarrow [0, \infty)$ such that $\sqrt{V(\cdot)}$ is a norm on C ,

$$\underline{c} \|\varphi\|_C^2 \leq V(\varphi) \leq \overline{c} \|\varphi\|_C^2$$

for some constants $\underline{c}, \overline{c} > 0$ and

$$\overline{D}_\sigma V(\varphi) \leq -\|\varphi\|_C^2, \quad \sigma \in \mathbb{S}, \varphi \in C.$$

- (iii) There exists a continuous function $V : C \rightarrow [0, \infty)$ such that

$$V(\varphi) \leq c \|\varphi\|_C^2$$

for some constant $c > 0$ and

$$\underline{D}_\sigma V(\varphi) \leq -|\varphi(0)|^2, \quad \sigma \in \mathbb{S}, \varphi \in C.$$

Clearly, a Lyapunov function $V(\cdot)$ satisfying condition (iii) does not necessarily satisfy the stronger conditions appearing in condition (ii). Hence, condition (iii) is better suited for proving the global uniform exponential stability of a linear uncertain time-delay system, while condition (ii) provides more information on a linear uncertain time-delay system that is known to be globally uniformly exponentially stable, by tightening the properties satisfied by $V(\cdot)$.

4. SUFFICIENT CONDITIONS FOR STABILITY: WEAKLY-DEGENERATE LYAPUNOV-KRASOVSKII FUNCTIONAL

In this section we establish two equivalence results for the global uniform exponential stability of system (2). The first equivalence is a straightforward application of [Hante and Sigalotti, 2011, Theorem 3] which gives a necessary and sufficient condition for the global uniform exponential stability for switched system of the form (5). The second one is given through a degenerate Lyapunov-Krasovskii functional.

4.1 Uniform exponential boundedness and first converse Lyapunov-Krasovskii theorem

In this section we give a first converse Lyapunov-Krasovskii theorem. Before that, we show that the solutions of (2) are $\sigma(\cdot)$ -uniformly exponentially bounded, i.e. there exist $M, w > 0$ such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(C)} \leq Me^{wt}, \quad t \geq 0, \sigma(\cdot)\text{-uniformly.} \quad (8)$$

This property, which is a necessary condition for the $\sigma(\cdot)$ -uniform exponential stability of the switched system (5), plays a crucial role in the following.

The $\sigma(\cdot)$ -uniform exponential boundedness of the solutions of (2) is given by the following lemma.

Lemma 2. Under the assumption that Q is a bounded subset of $\mathbb{R}^{n \times n}$, the solutions of (2) are $\sigma(\cdot)$ -uniformly exponentially bounded.

Proof. Let $\varphi \in C$. By integrating system (2) and using equation (4), one has for every $t \geq 0$ and every $\mu \in [-r, 0]$

$$\begin{aligned} |x(t + \mu)| &\leq \max\{\|\varphi\|_C, |\varphi(0)| + \int_0^{t+\mu} m \|x_s\|_C ds\} \\ &\leq \|\varphi\|_C + m \int_0^{t+\mu} \|x_s\|_C ds, \end{aligned}$$

that is

$$\|x_t\|_C \leq \|\varphi\|_C + m \int_0^t \|x_s\|_C ds.$$

Thanks to Gronwall's Lemma, we have the following inequality

$$\|x_t\|_C \leq \|\varphi\|_C e^{mt}, \quad (9)$$

which implies that the solution of (2) is $\sigma(\cdot)$ -uniformly exponentially bounded in C . Which ends the proof. \square

Due to the $\sigma(\cdot)$ -uniform exponential boundedness of the solutions of (2), a direct application of [Hante and Sigalotti, 2011, Theorem 3] gives a first necessary and sufficient condition for the global uniform exponential stability of (2). This is given by the following theorem.

Theorem 3. Under the assumption that Q is a bounded subset of $\mathbb{R}^{n \times n}$, we have that system (2) is uniformly exponentially stable, with respect to $\|\cdot\|_C$, if and only if there exists a function $V : C \rightarrow [0, \infty)$ such that $\sqrt{V(\cdot)}$ is a norm on C ,

$$V(\varphi) \leq c \|\varphi\|_C^2,$$

for some constant $c > 0$ and

$$D_\sigma V(\varphi) \leq -\|\varphi\|_C^2, \quad \sigma \in \mathbb{S}, \varphi \in C.$$

Proof. The proof results directly from Lemma 2 and [Hante and Sigalotti, 2011, Theorem 3] together. \square

4.2 Degenerate Lyapunov-Krasovskii functional

In this section we give a second converse Lyapunov-Krasovskii theorem. The crucial step to do this is given by the following lemma proved in Hante and Sigalotti [2011], which extends a result obtained in Triggiani [1994] in the framework of strongly continuous semigroups, to switched system of the form (5).

Lemma 4. Let $(X, \|\cdot\|_X)$ be a Banach space. Assume that

- (i) there exist constants $M \geq 1$ and $w > 0$ such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Me^{wt}, \quad t \geq 0, \sigma(\cdot)\text{-uniformly,}$$
- (ii) there exist a constant $c \geq 0$ and some $p \in [1, +\infty)$ such that

$$\int_0^{+\infty} \|T_{\sigma(\cdot)}(t)x\|_X^p \leq c \|x\|_X^p, \quad x \in X, \sigma(\cdot)\text{-uniformly.}$$

Then there exist constants $K \geq 1$ and $\mu > 0$ such that

$$\|T_{\sigma(\cdot)}(t)\|_{\mathcal{L}(X)} \leq Ke^{-\mu t}, \quad t \geq 0, \sigma(\cdot)\text{-uniformly.}$$

The second equivalence result is given by the following.

Theorem 5. Under the assumption that Q is a bounded subset of $\mathbb{R}^{n \times n}$, we have that system (2) is uniformly exponentially stable, with respect to $\|\cdot\|_C$, if and only if there exists a continuous function $V : C \rightarrow [0, +\infty)$ such that

$$V(\varphi) \leq c \|\varphi\|_C^2, \quad (10)$$

for some constant $c > 0$ and

$$D_\sigma V(\varphi) \leq -|\varphi(0)|^2, \quad \sigma \in \mathbb{S}, \varphi \in C. \quad (11)$$

Proof. Suppose that there is a continuous function $V : C \rightarrow [0, +\infty)$ such that conditions (10)-(11) hold. From Lemma 2, we know that the solutions of system (2) are σ -uniformly exponentially bounded, i.e. there exists constants $M \geq 1$ and $w > 0$ such that equation (8) holds. Then, thanks to Lemma 4, it suffices to prove that there exists a constant $c > 0$ such that

$$\int_0^{+\infty} \|x_t\|_C ds \leq c \|\varphi\|_C, \quad \varphi \in C, \sigma(\cdot)\text{-uniformly,} \quad (12)$$

to conclude the global exponential stability of system (2). For all $\sigma \in \mathbb{S}$, $\varphi \in C$ and for $t \geq 0$, we have

$$V(x_t) - V(x_0) \leq - \int_0^t |x_s(0)| ds, \quad (13)$$

as it follows from (11) (see for instance Hagood and Thomson [2006]). Using the fact that V is positive, one deduces the following inequality

$$\int_0^t |x(s)| ds \leq c \|\varphi\|_C, \quad t \geq 0 \quad (14)$$

from equation (13). In the sequel, we deduce from equation (14) that the property (12) holds. Let $s \geq r$. There exists $\tau_s \in [-r, 0]$ such that $\|x_s\|_C = |x(s + \tau_s)|$. Let $N \geq 1$ be a natural number. There exists $1 \leq j_s \leq N$ such that

$$0 \leq \tau_s + \frac{r}{N} j_s \leq \frac{r}{N}.$$

We have that

$$\begin{aligned} |x(s + \tau_s)| &\leq \left| x\left(s - j_s \frac{r}{N}\right) \right| + \int_{s - \frac{r}{N} j_s}^{s + \tau_s} |\dot{x}(\theta)| d\theta \\ &\leq \left| x\left(s - j_s \frac{r}{N}\right) \right| + m \frac{r}{N} \|x_s\|_{C[-2r, 0]} \\ &\leq \sum_{j=1}^N \left| x\left(s - j \frac{r}{N}\right) \right| + m \frac{r}{N} \|x_s\|_{C[-2r, 0]}. \end{aligned}$$

Remark that we have

$$\begin{aligned} \int_r^t \|x_s\|_{C[-2r, 0]} ds &= \int_0^{t-r} \|x_s\|_{C[-r, r]} ds \\ &\leq \int_0^{t-r} \|x_s\|_{C[-r, 0]} ds + \int_0^{t-r} \|x_s\|_{C[0, r]} ds \\ &\leq \int_0^t \|x_s\|_{C[-r, 0]} ds + \int_r^t \|x_s\|_{C[-r, 0]} ds \\ &\leq 2 \int_0^t \|x_s\|_{C[-r, 0]} ds, \end{aligned}$$

and remark that

$$\int_r^t \sum_{j=1}^N \left| x\left(s - j \frac{r}{N}\right) \right| ds \leq N \int_0^t |x(s)| ds,$$

then

$$\int_r^t \|x_s\|_C ds \leq N \int_0^t |x(s)| ds + 2m \frac{r}{N} \int_0^t \|x_s\|_C ds,$$

which implies together with equation (9) that

$$\left(1 - 2m \frac{r}{N}\right) \int_0^t \|x_s\|_C ds \leq \int_0^r \|x_s\|_C ds + N \int_0^t |x(s)| ds,$$

from which we conclude that for sufficiently large N , we have

$$\int_0^t \|x_s\|_C ds \leq c_1 \int_0^t |x(s)| ds + c_2 \|\varphi\|_C ds, \quad (15)$$

with

$$c_1 = \frac{N}{1 - 2m \frac{r}{N}} \quad \text{and} \quad c_2 = \frac{e^{mr} - 1}{m \left(1 - 2m \frac{r}{N}\right)}.$$

Equations (14) and (15) together imply that

$$\int_0^t \|x_s\|_C ds \leq c_3 \|\varphi\|_C ds, \quad (16)$$

with $c_3 = cc_1 + c_2$. When t tends to $+\infty$ we deduce that

$$\int_0^{+\infty} \|x_t\|_C ds \leq c_3 \|\varphi\|_C ds, \quad (17)$$

which concludes, thanks to Lemma 4, the proof of the global uniform exponential stability of system (2).

Conversely, suppose that system (2) is uniformly exponentially stable with respect to $\|\cdot\|_C$. Then, from Theorem 3, there exists a function $V : C \rightarrow [0, \infty)$ such that $\sqrt{V(\cdot)}$ is a norm on C ,

$$V(\varphi) \leq c \|\varphi\|_C^2$$

for some constant $c > 0$ and

$$\underline{D}_\sigma V(\varphi) \leq -\|\varphi\|_C^2, \quad \sigma \in \mathbb{S}, \varphi \in C. \quad (18)$$

From the fact that $|\varphi(0)| \leq \|\varphi\|_C$, equation (18) implies

$$\underline{D}_\sigma V(\varphi) \leq -|\varphi(0)|^2, \quad \sigma \in \mathbb{S}, \varphi \in C,$$

which ends the proof. \square

4.3 Proof of Theorem 1

Theorem 1 states that the existence of a function $V : C \rightarrow [0, \infty)$ such that,

$$\underline{c} \|\varphi\|_C^2 \leq V(\varphi) \leq \bar{c} \|\varphi\|_C^2$$

for some constants $\underline{c}, \bar{c} \geq 0$ and

$$\overline{D}_\sigma V(\varphi) \leq -\|\varphi\|_C^2, \quad \sigma \in \mathbb{S}, \varphi \in C,$$

is a necessary and sufficient condition for the uniform exponential stability of system (2). It also states that this functional may not have a strictly positive norm-dependent lower bound. Theorem 1 is a direct consequence of our previous results together with [Hante and Sigalotti, 2011, Theorem 6]. In fact, Theorem 5 gives the equivalence (i) \Leftrightarrow (iii). The second equivalence (i) \Leftrightarrow (ii) arises directly from [Hante and Sigalotti, 2011, Theorem 6]. Hence the proof of Theorem 1.

5. DISCUSSION

We compare here the results obtained in the previous section with the Lyapunov–Krasovskii theorem given in Hale and Lunel [1993]. Let us recall that the latter concerns general retarded functional differential equation of the form

$$\dot{x} = f(t, x_t). \quad (19)$$

Theorem 6. Suppose that $f : \mathbb{R} \times C \rightarrow \mathbb{R}^n$ is a continuous function. Suppose that for any bounded set B , f maps $\mathbb{R} \times B$ into a bounded set of \mathbb{R}^n , and $u, v, w : [0, +\infty) \rightarrow [0, +\infty)$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there is a continuous function $V : \mathbb{R} \times C \rightarrow \mathbb{R}$ such that

$$u(|\varphi(0)|) \leq V(t, \varphi) \leq v(\|\varphi\|_C)$$

$$\overline{D}V(t, \varphi) \leq -w(|\varphi(0)|)$$

then the solution $x = 0$ of equation (6) is uniformly stable. If $u(s) \rightarrow +\infty$ as $s \rightarrow +\infty$, the solutions of equation (6) are uniformly bounded. If $w(s) > 0$ for $s > 0$, then the solution $x = 0$ is uniformly asymptotically stable.

Of course, this theorem is not given for switched systems and the uniformity property mentioned is with respect to the initial condition. In Pepe and Karafyllis [2013] the authors show that the existence of a Lyapunov–Krasovskii functional is a necessary and sufficient condition for the uniform global asymptotic stability and the global exponential stability of autonomous systems described by neutral functional differential equations in Hale's form.

One can verify that the proof of Theorem 6 can be straightforwardly modified to consider system (2), and this is reformulated by the following theorem.

Theorem 7. Consider system (2). If there is a continuous function $V : C \rightarrow \mathbb{R}$ such that

$$\underline{c}|\varphi(0)|^2 \leq V(\varphi) \leq \bar{c}\|\varphi\|_C^2 \quad (20)$$

for constants $\underline{c}, \bar{c} > 0$ and

$$\bar{D}_\sigma V(\varphi) \leq -|\varphi(0)|^2, \quad \sigma \in \mathbb{S}, \varphi \in C \quad (21)$$

then system (2) is uniformly exponentially stable, with respect to $\|\cdot\|_C$.

Thanks to our previous results, we can give a converse version for Theorem 7. More precisely, Theorem 1 implies that if system (2) is uniformly exponentially stable, with respect to $\|\cdot\|_C$, then there exists a continuous function $V : C \rightarrow \mathbb{R}$ such that equations (20)-(21) hold.

6. CONCLUSION

In this work we give a collection of converse Lyapunov–Krasovskii theorems for uncertain time-delay systems. These results, which are summarized by Theorem 1, are essentially given by Theorems 3 and 5. By Theorem 3, we show that the existence of a squared norm $V(\cdot)$ on C , with suitable commensurability conditions, is a necessary and sufficient condition for the uniform exponential stability of system (2). Theorem 5 represents our main result, where the assumption that $V(\cdot)$ is a squared norm on C is dropped. One of the novelties of our results is that these functionals may not have a strictly positive norm-dependent lower bound, in contrast with what is known in the literature.

Concerning the differences between conditions (ii) and (iii) appearing in the statement of Theorem 1, we already noticed that a Lyapunov function $V(\cdot)$ satisfying condition (iii) does not necessarily satisfy the stronger condition (ii). Hence, condition (iii) is better suited for proving the global uniform exponential stability of a linear uncertain time-delay system, while condition (ii) provides more information on a linear uncertain time-delay system that is known to be globally uniformly exponentially stable, by tightening the properties satisfied by $V(\cdot)$.

Finally, a comparison of our results with the Lyapunov–Krasovskii theorem given in Hale and Lunel [1993] is discussed in Section 5. These results are carried out using the switched system transformation approach.

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