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Design of Sliding Mode Controllers for Mismatched Uncertain Systems with Unmeasurable States to Achieve Asymptotic Stability ⋆

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Abstract: Based on the Lyapunov stability theorem, a sliding mode controller is proposed in this paper for a class of uncertain multi-input multi-output (MIMO) nonlinear systems to solve regulation problems. The perturbed plant contains partly unmeasurable states and unknown mismatched and matched perturbations. By utilizing the designed auxiliary dynamic equations for state estimation, the proposed sliding mode controller is able to drive the state's trajectory towards a designated sliding surface in finite time and achieve asymptotic stability. Furthermore, the upper bounds of perturbations are not required in the design process due to some adaptive mechanisms are embedded in the auxiliary dynamic equations and controller. A numerical example is also demonstrated for showing the applicability of the proposed design technique.

Keywords: sliding mode control(SMC), Lyapunov stability theorem, mismatched perturbations

1. INTRODUCTION

In many practical applications the state variables of the control systems are not available for measurement. Therefore, design of observers, or utilization of output feedback control technique, has been carried out by many researchers. Chen and Kano [2002] designed a new state observer for a class of linear systems to estimate the true state so that the estimated errors are bounded. Praly [2003], Krishnamurthy *et al*. [2003], Boizot *et al*. [2010], Hammouri *et al*. [2010], Farza *et al*. [2011], Ménard *et al*. [2010] designed the high-gain observers, but all these works did not consider the effects of perturbations. Xiong and Saif [2001], Chen and Chen [2007], Walcott and Zak [1987], Ha *et al*. [2003] discussed the state observation but only for systems with matched perturbations. Jiang and Wu [2002], Kung and Chen [2005] designed perturbation observer and observer-based indirect adaptive fuzzy sliding mode controller respectively for single-input single-output (SISO) nonlinear systems. A higher order sliding mode observer was developed by Floquet and Barbot [2007] for a class of MIMO locally weakly observable, nonlinear systems with unknown inputs. Kalsi *et al*. [2010] also designed sliding-mode observers for systems with unknown inputs, the state estimation error was shown to be uniformly ultimately bounded. Spurgeon [2008] surveyed some observers for systems with mismatched perturbation, but these observers may only be directly applied to systems where the upper bounds of perturbations have to be known in advance. Several different high gain sliding mode observers were proposed by Veluvolu *et al*. [2007], Veluvolu and Soh [2009], Veluvolu *et al*. [2011] for nonlinear systems with mismatched model uncertainties and/or perturbations. However, these observers may only be applied to the systems where the upper bound of disturbance distribution vector has to be known in advance, and the disturbance distribution vector must be in a special form (Veluvolu *et al*. [2007], Veluvolu and Soh [2009]), or must be bounded by a known upper bound (Veluvolu *et al*. [2011]).

As for applying the output feedback control technique, Yan *et al*. [2010] studied a robust stabilization problem for a class of linear time-varying delay systems with time-delayed nonlinear disturbances. Cheng *et al*. [2006] proposed an adaptive output feedback variable structure tracking controller for a class of MIMO dynamic systems with mismatched uncertainties and disturbances. By constructing output feedback stabilizers, Zhai *et al*. [2013] stabilized a class of large-scale uncertain nonlinear systems. Liu *et al*. [2011] studied an adaptive output feedback control for uncertain SISO nonlinear systems with partial unmeasured states. Qian and Du [2012] developed a sampled-data output feedback controller to stabilize a class of nonlinear systems.

In this paper we propose a design methodology of sliding mode controllers for a class of mismatched uncertain nonlinear systems with unmeasurable states in order to achieve asymptotic stability. It is well known that if mismatched perturbations are present in the systems, the famous invariant property of sliding mode control is lost, that is, these mismatched perturbations

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will still affect the dynamics of the system during the sliding mode (Hu *et al.* [2000], Sam *et al.* [2004]). Hence stabilizing the control systems with mismatched perturbations is not an easy task, lots of studies using SMC technique have been carried out (Chan *et al.* [2000], Choi [2003], Tao *et al.* [2003], Chang and Cheng [2007], Cheng and Chang [2008]). However, all these methodologies required that the state variables are measurable. It is also observed that the aforementioned observers and output feedback control techniques can not be directly applied to the systems we considered in this paper. Therefore, in this study we designed an auxiliary dynamic equations so that the proposed sliding mode controllers are able to utilize their estimated states to achieve asymptotic stability. Furthermore, the upper bounds of perturbations are not required to know during the design process due to some adaptive mechanisms are embedded in both the proposed controller and the auxiliary dynamic equations. An example is also illustrated for showing the feasibility of the proposed control technique.

2. SYSTEMS DESCRIPTION AND PROBLEM FORMULATIONS

Consider a class of MIMO nonlinear dynamic systems with mismatched perturbations governed by the following equations

$$\dot{\mathbf{x}}_1 = \mathbf{A}_{11}\mathbf{x}_1 + \mathbf{f}_1(t,\mathbf{x}_2) + \Delta\mathbf{p}_1(t,\mathbf{x}), \tag{1}$$

$$\dot{\mathbf{x}}_2 = \mathbf{f}_2(t,\mathbf{x}) + \mathbf{B}_2\mathbf{u} + \Delta\mathbf{p}_2(t,\mathbf{x}), \tag{2}$$

where $\mathbf{x} = [\mathbf{x}_1^T \ \ \mathbf{x}_2^T]^T$ represents the state of the system, $\mathbf{x}_1 \in R^{n-m}$ is not available for measurement, whereas $\mathbf{x}_2 \in R^m$ is measurable. The constant matrices $\mathbf{A}_{11}$, $\mathbf{B}_2 \in R^{m\times m}$ are known, and the vector $\mathbf{u} \in R^m$ is the control input. The vector $\Delta\mathbf{p}_1(t,\mathbf{x})$ represents the unknown mismatched model uncertainty and/or nonlinearity, and the vector $\Delta\mathbf{p}_2(t,\mathbf{x})$ denotes the unknown matched nonlinearity and/or external disturbance.

In order to control the plant (1) and (2) successfully, the following assumptions are assumed to be valid throughout this paper:

A1. The matrix $\mathbf{A}_{11}$ has stable eigenvalues. The input gain matrix $\mathbf{B}_2$ is invertible.

Remark 1. In this paper the output of the plant can be treated as $\mathbf{y} = \mathbf{x}_2$. Suppose that the number of the output is q, then $q = m$ in this paper. If $\mathbf{A}_{11}$ is not stable and $m < q$, then under certain conditions, it is possible for one to stabilize $\mathbf{A}_{11}$ first by using output feedback method (Edwards and Spurgeon [1998]). Then apply the proposed method to stabilize the whole control system.

A2.(Edwards and Spurgeon [1998]) There exist unknown positive constants c_i $(i = 0,1,2)$ such that the following inequalities

$$\|\Delta\mathbf{p}_1(t,\mathbf{x})\| \leqslant c_2\|\mathbf{x}_2\|, \quad \|\Delta\mathbf{p}_2(t,\mathbf{x})\| \leqslant c_0 + c_1\|\mathbf{x}\|$$

are satisfied in the domain of interest.

Remark 2. Since $\Delta\mathbf{p}_2(t,\mathbf{x})$ is dependent on full state $\mathbf{x}$, its upper bound is also assumed to be dependent on full state $\mathbf{x}$. However, the upper bound of mismatched perturbation $\Delta\mathbf{p}_1(t,\mathbf{x})$ is assumed to be dependent only on the state $\mathbf{x}_2$, this constraint indicates that the stability guaranteed by the proposed control scheme is in local sense. In fact this idea (the upper bound of perturbation depends only on partial state) has been used by many researchers, especially those who designed the robust observers or output feedback control scheme, for example, Edwards and Spurgeon [1998], Ha *et al.* [2003], Kwan [1996], Cheng *et al.* [2006]. In fact one can relax the constraints of the upper bounds of $\Delta\mathbf{p}_1(t,\mathbf{x})$ and $\Delta\mathbf{p}_2(t,\mathbf{x})$ as

$$\|\Delta\mathbf{p}_1(t,\mathbf{x})\| \leqslant \sum_{j=1}^{\gamma_1}\beta_j\|\mathbf{x}_2\|^j, \quad \Delta\mathbf{p}_2(t,\mathbf{x}) \leqslant \sum_{j=0}^{\gamma_2}\lambda_j\|\mathbf{x}\|^j, \tag{3}$$

where β_j and λ_j are unknown positive constants, γ_1 and γ_2 are designed positive constants. Although $\gamma_1 = 1$ and $\gamma_2 = 1$ were assumed in this paper for simplicity, the design of the controllers under the constraints (3) can be proceeded in a similar way.

A3. The upper bound of the known nonlinear vector $\mathbf{f}_1(t,\mathbf{x}_2)$ satisfies the inequality $\|\mathbf{f}_1(t,\mathbf{x}_2)\| \le g(\mathbf{x}_2)$, where $g(\mathbf{x}_2)$ is a vanishing function, and $g(\mathbf{x}_2)$ is bounded if $\mathbf{x}_2$ is bounded. The function $\mathbf{f}_2(t,\mathbf{x})$ is Lipschitz in $\mathbf{x}$ in the domain of interest.

Remark 3. The proposed method can still be directly applied to the case where the number of unmeasurable states is smaller than $n-m$.

The objective of this paper is to design a sliding mode controller for the dynamic equations (1) and (2) under the conditions that state $\mathbf{x}_1$ is unmeasurable and the perturbations $\Delta\mathbf{p}_1$, $\Delta\mathbf{p}_2$ exist, so that the full state variable $\mathbf{x}$ is able to approach zero as $t \to \infty$. The proposed design methodology is presented in the following sections.

3. DESIGN OF THE ROBUST SLIDING MODE CONTROLLERS

Since the state variable $\mathbf{x}_1$ is unmeasurable, we first introduce an auxiliary dynamic equations to estimate the state $\mathbf{x}_1$, so that the estimation error $\mathbf{e}_{\mathbf{x}_1} \triangleq \mathbf{x}_1 - \hat{\mathbf{x}}_1$ is capable of reaching zero as $t \to \infty$ when the proposed controller is employed. These auxiliary dynamic equations are given by

$$\dot{\hat{\mathbf{x}}}_1 = \mathbf{A}_{11}\hat{\mathbf{x}}_1 + \mathbf{f}_1(t,\mathbf{x}_2), \tag{4}$$

$$\dot{\mathbf{h}} = \begin{cases} \mathbf{f}_2(t,\hat{\mathbf{x}}) + \mathbf{B}_2\mathbf{u}, & \text{for } \mathbf{e}_\mathbf{h} = \mathbf{0}, \\ \mathbf{f}_2(t,\hat{\mathbf{x}}) + \mathbf{B}_2\mathbf{u} + \dfrac{\mathbf{e}_\mathbf{h}}{\|\mathbf{e}_\mathbf{h}\|}\big[\hat{\alpha}_0(t) + \hat{\alpha}_1(t)\|\hat{\mathbf{x}}\| + \hat{\alpha}_2(t)\|\mathbf{x}_2\| \\ \quad +\hat{\alpha}_3(t)\|\mathbf{x}_2\|\|\hat{\mathbf{x}}\|\big] + \kappa\mathbf{e}_\mathbf{h}, & \text{for } \mathbf{e}_\mathbf{h} \neq \mathbf{0}, \end{cases} \tag{5}$$

where $\mathbf{e}_\mathbf{h} \triangleq \mathbf{x}_2 - \mathbf{h}$, $\hat{\mathbf{x}} \triangleq [\hat{\mathbf{x}}_1^T \ \ \mathbf{x}_2^T]^T$, and κ is a designed positive constant. The time-varying gains $\hat{\alpha}_i(t)$, $(i \in I,\ 0 \le i \le 3)$ are computed from the following adaptive laws

$$\dot{\hat{\alpha}}_0(t) = \begin{cases} 0, & \text{for } \mathbf{e}_\mathbf{h} = \mathbf{0}, \\ 1, & \text{for } \mathbf{e}_\mathbf{h} \neq \mathbf{0}. \end{cases} \qquad \dot{\hat{\alpha}}_1(t) = \begin{cases} 0, & \text{for } \mathbf{e}_\mathbf{h} = \mathbf{0}, \\ \|\hat{\mathbf{x}}\|, & \text{for } \mathbf{e}_\mathbf{h} \neq \mathbf{0}. \end{cases}$$

$$\dot{\hat{\alpha}}_2(t) = \begin{cases} 0, & \text{for } \mathbf{e}_\mathbf{h} = \mathbf{0}, \\ \|\mathbf{x}_2\|, & \text{for } \mathbf{e}_\mathbf{h} \neq \mathbf{0}. \end{cases} \qquad \dot{\hat{\alpha}}_3(t) = \begin{cases} 0, & \text{for } \mathbf{e}_\mathbf{h} = \mathbf{0}, \\ \|\mathbf{x}_2\|\|\hat{\mathbf{x}}\|, & \text{for } \mathbf{e}_\mathbf{h} \neq \mathbf{0}. \end{cases} \tag{6}$$

According to the previous auxiliary equations, one can easily obtain the error dynamic equations as

$$\dot{\mathbf{e}}_{\mathbf{x}_1} = \mathbf{A}_{11}\mathbf{e}_{\mathbf{x}_1} + \Delta\mathbf{p}_1(t,\mathbf{x}), \tag{7}$$

$$\dot{\mathbf{e}}_\mathbf{h} = \begin{cases} \mathbf{f}_2(t,\mathbf{x}) - \mathbf{f}_2(t,\hat{\mathbf{x}}) + \Delta\mathbf{p}_2(t,\mathbf{x}), & \text{for } \mathbf{e}_\mathbf{h} = \mathbf{0}, \\ \mathbf{f}_2(t,\mathbf{x}) - \mathbf{f}_2(t,\hat{\mathbf{x}}) - \dfrac{\mathbf{e}_\mathbf{h}}{\|\mathbf{e}_\mathbf{h}\|}\big[\hat{\alpha}_0(t) + \hat{\alpha}_1(t)\|\hat{\mathbf{x}}\| - \kappa\mathbf{e}_\mathbf{h} \\ \quad +\hat{\alpha}_2(t)\|\mathbf{x}_2\| + \hat{\alpha}_3(t)\|\mathbf{x}_2\|\|\hat{\mathbf{x}}\|\big] + \Delta\mathbf{p}_2, & \text{for } \mathbf{e}_\mathbf{h} \neq \mathbf{0}, \end{cases} \tag{8}$$

To achieve the objective of regulation of the state variable $\mathbf{x}$, we design the sliding function and the controller respectively as

$$\mathbf{s}(t) = \mathbf{h}(t), \tag{9}$$

$$\mathbf{u}(t) = (\mathbf{B}_2)^{-1}\Big[\mathbf{u}_f(t) + \mathbf{u}_s(t) + \mathbf{u}_{adp}(t)\Big], \tag{10}$$

where $\mathbf{u}_f(t) = -\mathbf{f}_2(t, \hat{\mathbf{x}}) - \kappa \mathbf{e_h}(t)$, and

$$\mathbf{u}_s(t) = \begin{cases} \mathbf{0}, & \text{for } \mathbf{s}(t) = \mathbf{0}, \\ -\rho \dfrac{\mathbf{s}(t)}{\|\mathbf{s}(t)\|}, & \text{for } \mathbf{s}(t) \neq \mathbf{0}, \end{cases}$$

$$\mathbf{u}_{adp}(t) = \begin{cases} \mathbf{0}, \text{ for } \mathbf{e_h}(t) = \mathbf{0}, \\ -\dfrac{\mathbf{e_h}}{\|\mathbf{e_h}\|}\big[\hat{\alpha}_0(t) + \hat{\alpha}_1(t)\|\hat{\mathbf{x}}\| + \hat{\alpha}_2(t)\|\mathbf{x}_2\| \\ \quad +\hat{\alpha}_3(t)\|\mathbf{x}_2\|\|\hat{\mathbf{x}}\|\big], \text{ for } \mathbf{e_h}(t) \neq \mathbf{0}, \end{cases}$$

ρ is a designed positive constant, which determines the time required by the controlled system entering the sliding mode. The adaptive laws $\hat{\alpha}_i(t)$ are given by (6).

Noted that the adaptive mechanisms (6) are intentionally embedded in both controller (10) and auxiliary dynamic equations (5) to overcome the mismatched and/or matched perturbations whose upper bounds are unknown. The effectiveness of the proposed controller as well as the auxiliary dynamic equations is demonstrated in the next section.

4. ROBUSTNESS OF SYSTEM'S STABILITY

In this section we prove that the stability of overall controlled systems is guaranteed under the proposed control scheme. Firstly the following theorem shows that the trajectories of controlled system will be driven into the designated sliding surface $\mathbf{s}(t) = \mathbf{0}$ within a finite time.

Theorem 1: *Consider the systems* (4) *and* (5) *with the assumptions A1 to A3. If the sliding function* (9) *and the controller* (10) *are used, then the state trajectories will be driven towards the sliding hyperplane* $\mathbf{s}(t) = \mathbf{h}(t) = \mathbf{0}$ *within a finite time* t_f.

Proof: A Lyapunov function candidate is chosen as $V_1 = \|\mathbf{s}\|$. Now we consider the following three cases:

(1) $\mathbf{s} \neq \mathbf{0}$, $\mathbf{e_h} \neq \mathbf{0}$: By using (5) and (10), one computes the time derivative of V_1 along the trajectory of (5) as

$$\begin{aligned} \dot{V}_1 &= \frac{\mathbf{s}^T}{\|\mathbf{s}\|}\Big[\mathbf{f}_2(t, \hat{\mathbf{x}}) + \mathbf{B}_2\mathbf{u} + \frac{\mathbf{e_h}}{\|\mathbf{e_h}\|}\big(\hat{\alpha}_0(t) + \hat{\alpha}_1(t)\|\hat{\mathbf{x}}\| \\ &\quad + \hat{\alpha}_2(t)\|\mathbf{x}_2\| + \hat{\alpha}_3(t)\|\mathbf{x}_2\|\|\hat{\mathbf{x}}\|\big) + \kappa\mathbf{e_h}\Big] \\ &= -\frac{\mathbf{s}^T}{\|\mathbf{s}\|}\Big[\frac{\mathbf{s}}{\|\mathbf{s}\|}\rho\Big] = -\rho < 0 \end{aligned}$$

(2) $\mathbf{s} \neq \mathbf{0}$, $\mathbf{e_h} = \mathbf{0}$: The time derivative of V_1 is $\dot{V}_1 = \frac{\mathbf{s}^T}{\|\mathbf{s}\|}\big[\mathbf{f}_2(t, \hat{\mathbf{x}}) + \mathbf{B}_2\mathbf{u}\big] = -\rho < 0$.

(3) $\mathbf{s} = \mathbf{0}$: $\dot{V}_1 = 0$ since $V_1 = 0$.

From the above three cases, it can be seen that no matter what the value of $\mathbf{e_h}$ is, the value of V_1 is bounded for $t \geq t_0$, and its value will decay until $\mathbf{s} = \mathbf{0}$. It can also be verified that the sliding variable $\mathbf{s}$, and hence $\mathbf{h}$, will approach zero within a finite time $t_f \leq t_0 + \frac{V_1(t_0)}{\rho}$. □

The following theorem proves that the error dynamic systems (7) and (8) will be asymptotically stable, and the stability of overall controlled systems is guaranteed if the proposed controller (10) is employed.

Theorem 2: *Consider the dynamic equations* (7), (8) *with assumptions A1, A2, and the proposed controller* (10). *Suppose that* $\|\hat{\mathbf{e}}\| \triangleq \|\mathbf{x} - \hat{\mathbf{x}}\| \leq g_0 + g_1\|\hat{\mathbf{x}}\|$, *where* g_0 *and* g_1 *are unknown constants. Then*
(a) *the state variables* $\mathbf{e_h}$ *and* $\mathbf{x}_2$ *will approach zero as* $t \to \infty$;
(b) *the estimation error* $\mathbf{e}_{\mathbf{x}_1}$ *will approach zero as* $t \to \infty$;
(c) *both the estimated state* $\hat{\mathbf{x}}_1(t)$ *and the state* $\mathbf{x}_1(t)$ *will approach zero after the controlled system enters the sliding mode;*
(d) *there exist finite constants* $\alpha_{i\infty}$ *such that* $\lim_{t\to\infty} \hat{\alpha}_i(t) = \alpha_{i\infty}$ $(i = 0, 1, 2, 3)$; and
(e) *the stability of overall controlled system is guaranteed.*
Proof: Since $\mathbf{A}_{11}$ is a stable matrix, the Lyapunov equation

$$\mathbf{A}_{11}^T\mathbf{P} + \mathbf{P}\mathbf{A}_{11} = -\mathbf{Q} \tag{11}$$

has a unique symmetric and positive definite solution $\mathbf{P}$ for any given positive definite symmetric matrix $\mathbf{Q}$ (Chen [1999]). This also implies that

$$\mathbf{e}_{\mathbf{x}_1}^T\mathbf{P}\mathbf{A}_{11}\mathbf{e}_{\mathbf{x}_1} = \mathbf{e}_{\mathbf{x}_1}^T\mathbf{A}_{11}^T\mathbf{P}\mathbf{e}_{\mathbf{x}_1} = -\frac{1}{2}\mathbf{e}_{\mathbf{x}_1}^T\mathbf{Q}\mathbf{e}_{\mathbf{x}_1}. \tag{12}$$

Since $\|\hat{\mathbf{e}}\| = \|\mathbf{e}_{\mathbf{x}_1}\| \leq g_0 + g_1\|\hat{\mathbf{x}}\|$, from assumption A2 it can be seen that

$$\begin{aligned} \|\Delta\mathbf{p}_2\| &\leq c_0 + c_1\|\hat{\mathbf{e}} + \hat{\mathbf{x}}\| \leq c_0 + c_1(g_0 + g_1\|\hat{\mathbf{x}}\|) + c_1\|\hat{\mathbf{x}}\| \\ &\leq r_0 + r_1\|\hat{\mathbf{x}}\|, \end{aligned} \tag{13}$$

where $r_0 \triangleq c_0 + c_1g_0$, $r_1 \triangleq c_1g_1 + c_1$ are two unknown constants.
(a) Let $\alpha_0 \triangleq r_0 + Lg_0$, $\alpha_1 \triangleq r_1 + Lg_1$, $\alpha_2 \triangleq c_2g_0\|\mathbf{P}\|$, $\alpha_3 \triangleq c_2g_1\|\mathbf{P}\|$ be four unknown constants to be adapted, where L is the Lipschitz constant. Define a Lyapunov function candidate as

$$V = \|\mathbf{e_h}\| + \frac{1}{2}\left(\mathbf{e}_{\mathbf{x}_1}^T\mathbf{P}\mathbf{e}_{\mathbf{x}_1} + \sum_{i=0}^{3}\tilde{\alpha}_i^2(t)\right), \tag{14}$$

where $\tilde{\alpha}_i(t) \triangleq \hat{\alpha}_i(t) - \alpha_i$, $0 \leq i \leq 3$, are the adaptive errors. Note also that $\dot{\tilde{\alpha}}_i(t) = \dot{\hat{\alpha}}_i(t)$. In this part we consider the following three cases:
case 1: $\mathbf{e_h} \neq \mathbf{0}$
By using (11), (12), (13), (6), and assumption A3, the derivative of V along the trajectories of (7) and (8) is

$$\begin{aligned} \dot{V} &= \frac{\mathbf{e_h}^T}{\|\mathbf{e_h}\|}\dot{\mathbf{e}}_\mathbf{h} + \mathbf{e}_{\mathbf{x}_1}^T\mathbf{P}\dot{\mathbf{e}}_{\mathbf{x}_1} + \sum_{i=0}^{3}\tilde{\alpha}\dot{\hat{\alpha}}_i \\ &\leq \mathbf{e}_{\mathbf{x}_1}^T\mathbf{P}\mathbf{A}_{11}\mathbf{e}_{\mathbf{x}_1} + \|\mathbf{e}_{\mathbf{x}_1}^T\|\|\mathbf{P}\|\|\Delta\mathbf{p}_1\| + \|\mathbf{f}_2(t, \mathbf{x}) - \mathbf{f}_2(t, \hat{\mathbf{x}})\| \\ &\quad - \kappa\|\mathbf{e_h}\| - \big(\hat{\alpha}_0 + \hat{\alpha}_1\|\hat{\mathbf{x}}\| + \hat{\alpha}_2\|\mathbf{x}_2\| + \hat{\alpha}_3\|\hat{\mathbf{x}}\|\|\mathbf{x}_2\|\big) \\ &\quad + \|\Delta\mathbf{p}_2\| + \sum_{i=0}^{3}\tilde{\alpha}\dot{\hat{\alpha}}_i \\ &\leq -\frac{1}{2}\mathbf{e}_{\mathbf{x}_1}^T\mathbf{Q}\mathbf{e}_{\mathbf{x}_1} + \|\mathbf{e}_{\mathbf{x}_1}\|\|\mathbf{P}\|c_2\|\mathbf{x}_2\| + L\|\mathbf{x} - \hat{\mathbf{x}}\| - \kappa\|\mathbf{e_h}\| \\ &\quad - \big(\hat{\alpha}_0 + \hat{\alpha}_1\|\hat{\mathbf{x}}\| + \hat{\alpha}_2\|\mathbf{x}_2\| + \hat{\alpha}_3\|\hat{\mathbf{x}}\|\|\mathbf{x}_2\|\big) + r_0 + r_1\|\hat{\mathbf{x}}\| \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=0}^{3}\tilde{\alpha}\dot{\hat{\alpha}}_i \\
&\le -\frac{1}{2}\mathbf{e}_{\mathbf{x}_1}^T\mathbf{Q}\mathbf{e}_{\mathbf{x}_1} - \kappa\|\mathbf{e_h}\| - \big(\hat{\alpha}_0 + \hat{\alpha}_1\|\hat{\mathbf{x}}\| + \hat{\alpha}_2\|\mathbf{x}_2\| \\
&+ \hat{\alpha}_3\|\hat{\mathbf{x}}\|\|\mathbf{x}_2\|\big) + (r_0 + Lg_0) + (r_1 + Lg_1)\|\hat{\mathbf{x}}\| \\
&+ (c_2 g_0\|\mathbf{P}\|)\|\mathbf{x}_2\| + (c_2 g_1\|\mathbf{P}\|)\|\hat{\mathbf{x}}\|\|\mathbf{x}_2\| + (\hat{\alpha}_0 - \alpha_0) \\
&+ (\hat{\alpha}_1 - \alpha_1)\|\hat{\mathbf{x}}\| + (\hat{\alpha}_2 - \alpha_2)\|\mathbf{x}_2\| + (\hat{\alpha}_3 - \alpha_3)\|\hat{\mathbf{x}}\|\|\mathbf{x}_2\| \\
&\le -\frac{1}{2}\mathbf{e}_{\mathbf{x}_1}^T\mathbf{Q}\mathbf{e}_{\mathbf{x}_1} - \kappa\|\mathbf{e_h}\| < 0.
\end{aligned}
$$

Case 2: $\mathbf{e_h} = \mathbf{0}$ and $t < t_f$

In this case, $V = \big(\mathbf{e}_{\mathbf{x}_1}^T\mathbf{P}\mathbf{e}_{\mathbf{x}_1} + \sum_{i=0}^{3}\tilde{\alpha}_i^2(t)\big)/2$ in accordance with (14). Then from (7), (12), and assumption A2

$$
\begin{aligned}
\dot{V} &= \mathbf{e}_{\mathbf{x}_1}^T\mathbf{P}\big[\mathbf{A}_{11}\mathbf{e}_{\mathbf{x}_1} + \Delta\mathbf{p}_1\big] \le -\mathbf{e}_{\mathbf{x}_1}^T\mathbf{Q}\mathbf{e}_{\mathbf{x}_1}/2 + c_2\|\mathbf{e}_{\mathbf{x}_1}\|\|\mathbf{P}\|\|\mathbf{x}_2\| \\
&\le -\lambda_{min}(\mathbf{Q})\|\mathbf{e}_{\mathbf{x}_1}\|^2/2 + c_2\|\mathbf{e}_{\mathbf{x}_1}\|\|\mathbf{P}\|\|\mathbf{x}_2\|. \qquad (15)
\end{aligned}
$$

case 3: $\mathbf{e_h} = \mathbf{0}$ and $t \ge t_f$

According to Theorem 1, $\mathbf{e_h} = \mathbf{x}_2\ \forall\, t \ge t_f$, hence $\mathbf{x}_2 = \mathbf{e_h} = \mathbf{0}$ in this case. From (15) it is seen

$$\dot{V} \le -\lambda_{min}(\mathbf{Q})\|\mathbf{e}_{\mathbf{x}_1}\|^2/2.$$

The above analysis all implies that $\mathbf{e_h}$ will approach zero as $t \to \infty$. From Theorem 1 it is known that the sliding variable $\mathbf{s} = \mathbf{h}$ will approach zero within a finite time t_f, the equation $\mathbf{e_h} = \mathbf{x}_2 - \mathbf{h}$ indicates that the state variable $\mathbf{x}_2$ will also be asymptotically stable.

(b) It is easy to see that in case 1 and case 3, $\mathbf{e}_{\mathbf{x}_1}$ will approach zero as $t \to \infty$. In case 2 if $\|\mathbf{e}_{\mathbf{x}_1}\| > \frac{2c_2\|\mathbf{P}\|\|\mathbf{x}_2\|}{\lambda_{min}(\mathbf{Q})}$ then $\dot{V} < 0$. Although the magnitude of $\mathbf{e}_{\mathbf{x}_1}$ may increase if $\|\mathbf{e}_{\mathbf{x}_1}\| < \frac{2c_2\|\mathbf{P}\|\|\mathbf{x}_2\|}{\lambda_{min}(\mathbf{Q})}$, this will only last at most finite time t_f. Since $\mathbf{x}_2$ is asymptotically stable as proved in (a), the trajectory of $\mathbf{e}_{\mathbf{x}_1}$ will still reach zero as $t \to \infty$.

(c) For analyzing the stability of (4), another Lyapunov function candidate is chosen as $V_2 = \hat{\mathbf{x}}_1^T\mathbf{P}\hat{\mathbf{x}}_1/2$. The time derivative of V_2 along the trajectory of (4) is

$$
\begin{aligned}
\dot{V}_2 &= \hat{\mathbf{x}}_1^T\mathbf{P}\dot{\hat{\mathbf{x}}}_1 = \hat{\mathbf{x}}_1^T\mathbf{P}\big[\mathbf{A}_{11}\hat{\mathbf{x}}_1 + \mathbf{f}_1(t,\mathbf{x}_2)\big] \\
&\le -\hat{\mathbf{x}}_1^T\mathbf{Q}\hat{\mathbf{x}}_1/2 + \|\hat{\mathbf{x}}_1^T\|\|\mathbf{P}\|\|\mathbf{f}_1(t,\mathbf{x}_2)\| \\
&\le -\lambda_{min}(\mathbf{Q})\|\hat{\mathbf{x}}_1\|^2/2 + \|\hat{\mathbf{x}}_1\|\|\mathbf{P}\|g(\mathbf{x}_2), \qquad (16)
\end{aligned}
$$

It is easy to verify that if $\|\hat{\mathbf{x}}_1\| > \frac{2\|\mathbf{P}\|g(\mathbf{x}_2)}{\lambda_{min}(\mathbf{Q})}$ then $\dot{V}_2 < 0$. Since $\mathbf{x}_2$ is bounded and is asymptotically stable for $t \ge t_f$ as proved in part (a) ($g(\mathbf{x}_2)$ is vanishing function), $\hat{\mathbf{x}}_1$ must be bounded and will reach zero as $t \to \infty$. According to the equation $\mathbf{e}_{\mathbf{x}_1} = \mathbf{x}_1 - \hat{\mathbf{x}}_1$ and part (b), $\mathbf{x}_1$ will also be asymptotically stable for $t \ge t_f$ since $\mathbf{e}_{\mathbf{x}_1}$ and $\hat{\mathbf{x}}_1$ are asymptotically stable.

(d) From part (a) it is shown in case 1 and case 3 that $\dot{V} \le 0$, hence V is bounded in these two cases. In case 2 $\dot{V}$ may be greater than zero if $\|\mathbf{e}_{\mathbf{x}_1}\| < \frac{2c_2\|\mathbf{P}\|\|\mathbf{x}_2\|}{\lambda_{min}(\mathbf{Q})}$ and $t < t_f$. Since t_f is a finite number and V is continuous, V is still bounded in this case. Hence $\hat{\alpha}_i(t) \in L_\infty$, $0 \le i \le 3$, for all time. From (6), it is known that $\hat{\alpha}_i(t)$ are monotonically increasing functions, and these functions are all bounded, therefore, according to the Proposition 2.14 in Tao [2003], one can conclude that there exist finite constants $\alpha_{i\infty}$, $0 \le i \le 3$, such that $\lim_{t\to\infty}\hat{\alpha}_i(t) = \alpha_{i\infty}$.

(e) From (a) to (d), it is seen that not only the state variable $\mathbf{x}(t) = [\mathbf{x}_1^T(t)\ \ \mathbf{x}_2^T(t)]^T$ is bounded, but also it is asymptotically stable when the controlled system is in the sliding mode. Both $\hat{\mathbf{x}}_1$ and $\mathbf{h}$ are asymptotically stable too, and all the adaptive gains $\hat{\alpha}_i(t)$, $0 \le i \le 3$, are bounded and will reach constants $\alpha_{i\infty}$, respectively. Therefore, the control input function (10) will be bounded, and one can conclude that the stability of overall controlled system is guaranteed. □

Theorem 2 clearly shows that the proposed controller as well as the auxiliary dynamic equations are effective in dealing with systems which contain mismatched and/or matched perturbations since the overall controlled system is able to achieve asymptotic stability. From part (b) and (c) of Theorem 2, it is also seen that the stability of $\mathbf{x}_2$ is very crucial to the success of the proposed control scheme. The auxiliary dynamic equations may not be employed alone to estimate state variable $\mathbf{x}_1$ since they need the controller to drive the sliding variable $\mathbf{s} = \mathbf{h}$ to zero within a finite time. Then the asymptotic stability of the state $\mathbf{x}_2$ is guaranteed. Another key factor for the success of the proposed control scheme is the finite time t_f. Since V may be unbounded if $\|\mathbf{e}_{\mathbf{x}_1}\| < \frac{2c_2\|\mathbf{P}\|\|\mathbf{x}_2\|}{\lambda_{min}(\mathbf{Q})}$ and t_f is not finite.

It is proved in the previous theorem that the trajectory of $\mathbf{e_h}$ will reach zero as $t \to \infty$, and all the adaptive rules $\alpha_i(t)$ are bounded. However, these adaptive rules given by (6) in general might not be acceptable since in most practical applications $\mathbf{e_h}$ will not be exactly equal to zero due to the computing accuracy and noise in the control system. It also means that these adaptive rules may increase as time increases. A simple remedy of this problem is to utilize the dead-zone technique (Slotine and Li [1991]), *i.e.*,

$$
\dot{\hat{\alpha}}_0(t) = \begin{cases} 1, & \text{for } \|\mathbf{e_h}\| > \epsilon, \\ 0, & \text{otherwise,} \end{cases} \qquad
\dot{\hat{\alpha}}_1(t) = \begin{cases} \|\hat{\mathbf{x}}\|, & \text{for } \|\mathbf{e_h}\| > \epsilon, \\ 0, & \text{otherwise,} \end{cases}
$$

$$
\dot{\hat{\alpha}}_2(t) = \begin{cases} \|\mathbf{x}_2\|, & \text{for } \|\mathbf{e_h}\| > \epsilon, \\ 0, & \text{otherwise,} \end{cases} \qquad
\dot{\hat{\alpha}}_3(t) = \begin{cases} \|\mathbf{x}_2\|\|\hat{\mathbf{x}}\|, & \text{for } \mathbf{e_h} \ne \mathbf{0} \\ 0, & \text{otherwise,} \end{cases} \qquad (17)
$$

where ϵ is a designed small positive constant. Note also that increasing the value of κ will drive the trajectory of $\mathbf{e_h}$ toward zero faster.

5. EXAMPLE AND SIMULATION

Consider the perturbed dynamic equations in the form of (1) and (2) with $\mathbf{x}_1 \triangleq [x_1\ \ x_2\ \ x_3]^T$, $\mathbf{x}_2 \triangleq [x_4\ \ x_5]^T$, and

$$
\mathbf{A}_{11} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 3 \\ 0 & -4 & -2 \end{bmatrix}, \quad
\mathbf{f}_1 = \begin{bmatrix} x_4x_5 \\ 3x_5\sin(2t) \\ 2x_5\cos(2x_4) \end{bmatrix},
$$

$$
\mathbf{f}_2 = \begin{bmatrix} 2x_2x_4\cos(2t) \\ 3x_1x_3 \end{bmatrix}, \quad
\mathbf{B}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.
$$

Note that x_1, x_2, and x_3 are unmeasurable states, whereas x_4 and x_5 are measurable. For demonstrating the robustness of the proposed control scheme and computer simulation, it is assumed that the mismatched and matched lumped perturbations $\Delta\mathbf{p}_1$, $\Delta\mathbf{p}_2$ are

$$
\Delta\mathbf{p}_1 = \begin{bmatrix} 0.2x_4\cos(x_1x_2) \\ 0.5x_5\sin(2t) \\ -0.3x_4\sin(x_1x_3x_5) \end{bmatrix}, \quad
\Delta\mathbf{p}_2 = \begin{bmatrix} 0.2x_1x_4\cos(2t) \\ 0.5x_2x_3 + 0.3\sin t \end{bmatrix}.
$$

The objective of this example is to use the proposed control scheme so that all the states $x_1(t)$ to $x_5(t)$ can be driven to zero as $t \to \infty$. The sliding surface function, auxiliary dynamic equations, and the controller are designed in accordance with (9), (4), (5), and (10) respectively. The adaptive rules (17) are employed with $\epsilon = 0.001$. The design parameters are set to be $\rho = 10$, $\kappa = 1$.

The simulation results with calculation step size $1m$ sec and initial conditions $\mathbf{h}(0) = [-5 \quad 5]^T$ and $\mathbf{x}(0) = [2 \quad 5 \quad -4 \quad 3 \quad -1]^T$ are shown from Fig. 1 to Fig. 7. It is clearly shown that all the state variables are asymptotically stable in Fig. 1 and Fig. 2. Note that the measurable states x_4 and x_5 are forced to be asymptotically stable after $t_f \approx 0.7$ sec, which is the time when sliding variables reach zero. The estimation error $\mathbf{e}_{\mathbf{x}_1}$ shown in Fig. 3 and trajectory of $\mathbf{e}_{\mathbf{h}}$ in Fig. 4 all approach zero as $t \to \infty$. Fig. 5 depicts the sliding variables all reaching zero in finite time t_f. The control input functions $u_1(t)$ and $u_2(t)$ displayed in Fig. 6 are all bounded. Fig. 7 clearly illustrates that each adaptive gain $\hat{\alpha}_i(t)$ approaches a constant respectively.

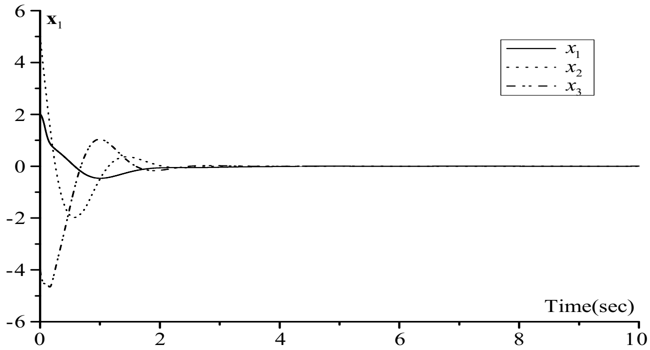

Fig. 1. The state variables x_1, x_2 and x_3.

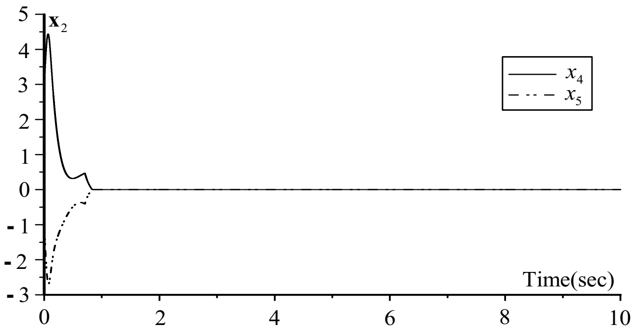

Fig. 2. The state variables x_4 and x_5.

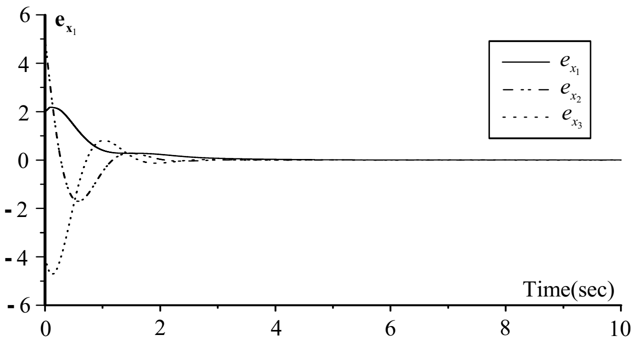

Fig. 3. The estimation errors e_{x_1}, e_{x_2}, and e_{x_3}.

6. CONCLUSIONS

In this study a methodology of designing sliding mode controllers is successfully proposed for a class of mismatched perturbed MIMO nonlinear systems with unmeasurable states. Although the designed auxiliary dynamic equations need the aid of proposed controller to estimate the unmeasurable states, the proposed control scheme is capable of achieving asymptotic stability. For future study, relaxing the constraint in assumption A1 is worth considering.

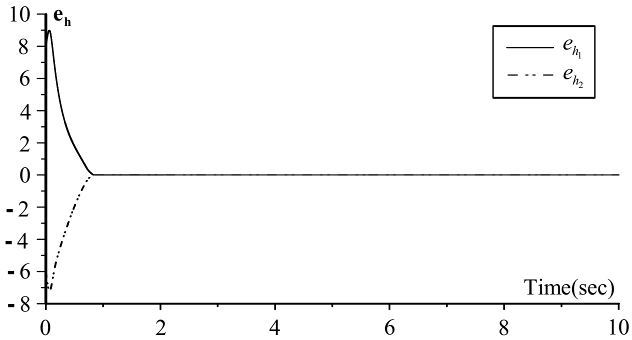

Fig. 4. The estimation errors e_{x_4} and e_{x_5}.

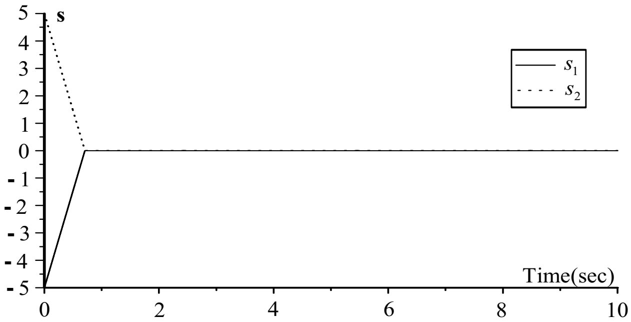

Fig. 5. The sliding functions s_1 and s_2.

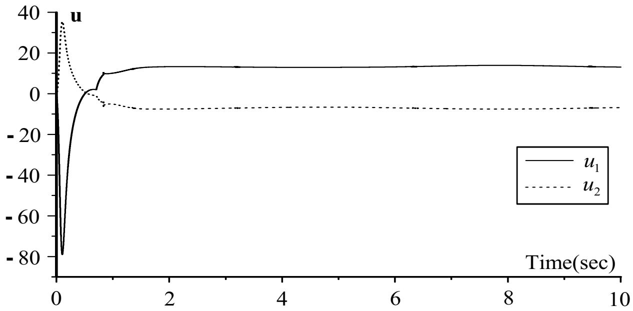

Fig. 6. The control input functions u_1 and u_2.

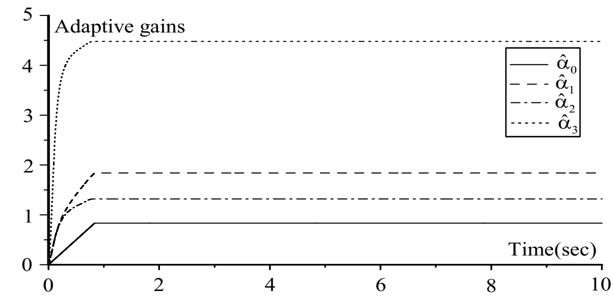

Fig. 7. The adaptive gains $\hat{\alpha}_i$, $i = 0, 1, 2, 3$.

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