Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Some Results on Optimal Control for a Partially Observed Linear Stochastic System with an Exponential Quadratic Cost

T.E. Duncan * **B. Pasik-Duncan** **

* *Department of Mathematics, University of Kansas, Lawrence, KS 66045 USA, (e-mail: duncan@math.ku.edu)*
** *Department of Mathematics, University of Kansas, Lawrence, KS 66045 USA, (e-mail: bozenna@math.ku.edu)*

Abstract: A control problem for a partially observed linear stochastic system with an exponential quadratic cost functional is formulated and explicitly solved. It is assumed given that the estimation of the state is described by the solution of the information filter which is known. This solution is a sufficient statistic for the unknown state based on the observations. In this paper an optimal control is determined explicitly in a simple, direct manner from this sufficient statistic. This approach does not use either the solution of a Hamilton-Jacobi-Bellman equation or a stochastic maximum principle with backward stochastic differential equations. This control problem is often called a linear partially observed risk sensitive control problem.

Keywords: linear exponential quadratic Gaussian control, stochastic control, partially observed stochastic systems, explicit optimal controls, risk sensitive control, information filter.

1. INTRODUCTION

The control of a completely observed linear system with a Brownian motion and an exponential quadratic cost functional was initially solved by Jacobson (Jacobson [1973]) by exhibiting a smooth solution to the associated Hamilton-Jacobi-Bellman equation. A particularly interesting feature of the solution is that the solution of a Riccati equation determines the optimal feedback control. This Riccati equation only differs from the well known Riccati equation for the linear quadratic Gaussian control problem by an additional term. The corresponding partially observed problem with an exponential quadratic cost was completely solved by Bensoussan and van Schuppen (Bensoussan et al [1985]) after some solutions of special cases by some others (e.g. Kumar et al [1981], Speyer et al [1974]). However the prior solutions of this partially observed problem are complicated and the complications obscure the important features of the solution. In this paper the estimation equation result for this problem that is often called an information filter that minimizes the exponential of a quadratic functional is assumed given and it is shown how to use the method in Duncan [2013] to obtain an optimal control in a simple, direct way. This approach can suggest a duality property between estimation and control that is analogous to the well known duality between estimation and control for linear systems with a quadratic cost.

An economic interpretation of the exponential quadratic cost functional with a parameter μ motivated the study of this control problem. As μ tends to zero the solution of the linear-quadratic control problem is recovered. This linear exponential quadratic control problem is also directly related to a two-person zero sum stochastic differential game with a quadratic payoff. Furthermore this control problem is related to an H^∞ deterministic control problem (e.g. Glover et al [1988]). Thus the linear exponential quadratic Gaussian control problems with either complete or partial observations are important to understand for their own interpretation and for their relation to other control problems.

For the control problem the cost functional has a parameter denoted μ that can be considered as an investor's propensity for risk. If $\mu > 0$ then an investor is said to be risk averse and if $\mu < 0$ then an investor is said to be risk seeking. Thus control problems of this type are often called risk sensitive control.

2. PRELIMINARIES

Initially the system and the observation equations are described. The equation for the system process X is given by

$$dX(t) = (AX(t) + CU(t))dt + FdB(t) \tag{1}$$
$$X(0) = X_0$$

where X_0 is a constant vector in $\mathbb{R}^n$, $X(t) \in \mathbb{R}^n, U(t) \in \mathbb{R}^m, A \in L(\mathbb{R}^n, \mathbb{R}^n), C \in L(\mathbb{R}^m, \mathbb{R}^n), F \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $(B(t), t \geq 0)$ is an $\mathbb{R}^n$-valued standard Brownian motion. The process B is defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

* Research supported in part by NSF grant DMS 1108884, AFOSR grant FA9550-12-1-0384, ARO grant W911NF-10-1-0248.

The observation process $(Y(t), t \in [0,T])$ satisfies the following stochastic equation

$$\begin{aligned} dY(t) &= HX(t)dt + GdV(t) \quad (2) \\ Y(0) &= 0 \end{aligned}$$

where $Y(t) \in L(\mathbb{R}^p)$, $H \in L(\mathbb{R}^n, \mathbb{R}^p)$, $G \in L(\mathbb{R}^p, \mathbb{R}^p)$ is invertible and $(V(t), t \geq 0)$ is an $\mathbb{R}^p$-valued standard Brownian motion that is also defined on $(\Omega, \mathcal{F}, \mathbb{P})$. It is assumed that the processes B and V are independent. Let $(\mathcal{G}(t), t \in [0,T])$ be the natural filtration for the process $(Y(t), t \in [0,T])$ on $(\Omega, \mathcal{F}, \mathbb{P})$. The family of admissible controls, $\mathcal{U}$, is defined as

$\mathcal{U} = \{U : U \text{ is an } \mathbb{R}^m\text{-valued } (\mathcal{G}(t), t \in [0,T]) \text{ progressively measurable process such that } U \in L^2([0,T]) \text{ a.s.}\}$

The cost, $J(\cdot)$, is an exponential quadratic functional of the state and the control that is given as follows

$$\begin{aligned} J(U) = \mu \mathbb{E} \exp[\frac{\mu}{2}\int_0^T (&< QX(s), X(s) > \quad (3) \\ &+ < RU(s), U(s) >)ds + \frac{\mu}{2} < MX(T), X(T) >] \end{aligned}$$

where the dependence of J on μ and X_0 is suppressed for notational convenience, $Q \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $R \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ and $M \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ are symmetric linear transformations, such that $Q > 0$, $R > 0$, $M \geq 0$ and μ is fixed. For the verification of an optimal control in this paper it is assumed that $M = 0$. Some remarks are made later about this restriction and how to eliminate it.

The following two assumptions are used subsequently.

(A1) The parameter μ in (3) is chosen so that $(H^T H - \mu Q + FF^T) > 0$

(A2) The parameter μ in (3) is chosen so that $inf_{t\in[0,T]}(CR^{-1}C^T - \mu P(t)H^T(GG^T)^{-1}HP(t)) > 0$

The appropriate estimation equation, often called the information filter (e.g. Elliott et al [1994]), is given by

$$\begin{aligned} dZ(t) &= (A - P(t)H^T H + \mu P(t)Q)Z(t)dt \quad (4) \\ &\quad + CU(t)dt + P(t)H^T dY(t) \\ Z(0) &= X(0) \end{aligned}$$

and $(P(t), t \in [0,T])$ is the unique, positive symmetric solution of the following Riccati equation

$$\begin{aligned} \frac{dP}{dt} &= AP + PA^T \quad (5) \\ &\quad - P(H^T H - \mu Q + FF^T)P \\ P(0) &= 0 \end{aligned}$$

The process $(\int_0^t PH^T(dY - HZds), \mathcal{G}(t), t \in [0,T])$ is a Brownian motion with incremental covariance GG^T which is verified by the Riccati equation (5) and the absolute continuity result in Duncan [1968]. It follows from the results for the information filter (e.g. Elliott et al [1994], Moore et al [1997]) that for progressively measurable observation actions on the exponential quadratic cost, that it suffices to consider the process $(Z(t), t \in [0,T])$ because this process is the minimizing solution of the best estimate for the exponential of the quadratic form in X formed using Q. Thus the control for (1) is a function of the process Z. This estimate Z is given as follows

$$\begin{aligned} Z(\cdot) = arg\ min_{h\in\mathcal{H}} \mathbb{E}[\mu exp(\frac{\mu}{2}\int_0^t &< Q(X(s) - h(s)), \\ &X(s) - h(s) > ds|\mathcal{G}(t)] \quad (6) \end{aligned}$$

where $\mathcal{H}$ is the family of square integrable $\mathcal{G}(\cdot)$ progressively measurable processes on $[0,T]$ An optimal control is explicitly determined in the next section.

3. OPTIMAL CONTROLS

In this section an optimal control is obtained for the exponential quadratic cost using the family of admissible controls, $\mathcal{U}$. While this optimal control is known (Bensoussan et al [1985]) the proof given here is elementary and direct and provides an explanation for the additional term in the Riccati equation for the optimal feedback control as compared to the well known Riccati equation for the linear quadratic Gaussian control problem. The optimal control is given in the following theorem.

Theorem 1. For the control problem given by the state equation (1), the observation equation (2), and the cost functional (3) there is an optimal control, U^*, from the family of admissible controls, $\mathcal{U}$, that is given by

$$U^*(t) = -R^{-1}C^T S(t)Z(t) \quad (7)$$

where $(S(t), t \in [0,T])$ is the unique positive, symmetric solution of the following Riccati equation

$$\begin{aligned} -\frac{dS}{dt} &= S(A + \mu PQ) + (A^T + \mu QP)S + Q \quad (8) \\ &\quad - (S(CR^{-1}C^T - \mu PH^T(GG^T)^{-1}HP)S \\ S(T) &= 0 \end{aligned}$$

Proof. The proof uses a refinement of a technique from the solution of the completely observable linear exponential quadratic Gaussian control problem (Duncan [2013]). Apply the Ito formula to the process, $(\frac{1}{2} < S(t)Z(t), Z(t) >, t \in [0,T])$ to obtain

$$\frac{1}{2} < S(T)Z(T), Z(T) > -\frac{1}{2} < S(0)X(0), X(0) > \quad (9)$$
$$= \frac{1}{2}\int_0^T (2(< S(t)(A + \mu P(t)Q)Z(t), Z(t) >$$
$$+ < S(t)CU(t), Z(t) >)dt$$
$$+ < S(t)P(t)H^T(dY(t) - HZ(t)dt), Z(t) >)$$
$$+ 2tr(S(t)P(t)H^T(GG^T)^{-1}HP(t))dt - < (S(t)((A +$$
$$+ \mu P(t)Q) + (A^T + \mu QP(t))S(t) + Q)Z(t), Z(t) > dt$$
$$+ < S(t)(CR^{-1}C^T - \mu P(t)H^T(GG^T)^{-1}$$
$$\times HP(t))S(t)Z(t), Z(t) > dt)$$
$$= \frac{1}{2}\int_0^T (< (S(t)CR^{-1}C^T S(t) - Q)Z(t), Z(t) >$$
$$+ 2 < C^T S(t)Z(t), U(t) >)dt + \int_0^T < S(t)P(t)H^T(dY(t)$$
$$- HZ(t)dt), Z(t) > -\mu < S(t)P(t)H^T(GG^T)^{-1}$$
$$\times HP(t)S(t)Z(t), Z(t) > dt + tr(S(t)P(t)H^T(GG^T)^{-1}$$
$$\times HP(t))dt$$

Let $L(U)$ be the quadratic functional that appears in the exponential of the cost functional (3) replacing X by Z as

$$L(U) = \frac{\mu}{2}\int_0^T (< QZ(t), Z(t) > + < RU, U >)dt \quad (10)$$

Using (9) the quadratic functional in the exponential of the cost functional can be expressed as

$$L(U) - \frac{\mu}{2} < S(0)X(0), X(0) > \quad (11)$$
$$= \frac{\mu}{2}[\int_0^T ((< RU, U > + < S(CR^{-1}C^T)Z, Z >$$
$$+ < S(t)CU(t), Z(t) >)dt$$
$$+ 2 < SPH^T(dY - HZdt), Z >$$
$$- \mu < SPH^T(GG^T)^{-1}HPSZ, Z >$$
$$+ \int_0^T tr(SPH^T(GG^T)^{-1}HP))dt]$$
$$= \frac{\mu}{2}\int_0^T |R^{-\frac{1}{2}}(RU + C^T SZ)|^2 dt$$
$$+ \mu\int_0^T < SPH^T(dY(t) - HZdt), Z >$$
$$- \frac{\mu^2}{2}\int_0^T < SPH^T(GG^T)^{-1}HPSZ, Z > dt$$
$$+ \frac{\mu}{2}\int_0^T tr(SPH^T(GG^T)^{-1}HP)dt$$

Thus

$$\mathbb{E}exp[L(U^*)] \quad (12)$$
$$= \mathbb{E}exp[\frac{\mu}{2} < S(0), X(0), X(0) >$$
$$+ \frac{\mu}{2}\int_0^T |R^{-\frac{1}{2}}(RU + C^T SZ)|^2 dt$$
$$+ \mu\int_0^T < 2 < PH^T(dY - HZdt), Z >$$
$$- \frac{\mu^2}{2}\int_0^T < SPH^T(GG^T)^{-1}SZ, Z > dt$$
$$+ \frac{\mu}{2}\int_0^T tr(SPH^T(GG^T)^{-1}HP)dt]$$

The minimizer of this expression, (12), using the admissible controls from $\mathcal{U}$ is U^* given by (7) because this expression for U^* can be expressed as

$$\mathbb{E}exp[L(U)] = \tilde{\mathbb{E}}exp[< S(0)X(0), X(0) > \quad (13)$$
$$+ \frac{\mu}{2}\int_0^T tr(SPH^T(GG^T)^{-1}HPS)dt]$$

where $\tilde{\mathbb{E}}$ is the expectation for $\tilde{\mathbb{P}}$ given by

$$d\tilde{\mathbb{P}} = exp[\mu\int_0^T < PH^T(dY - HZdt), Z > \quad (14)$$
$$- \frac{\mu^2}{2}\int_0^T < SPH^T(GG^T)^{-1}SZ, Z > dt]d\mathbb{P}$$

Recall from the observation equation (2) and the likelihood function result Duncan [1968] that $(\int_0^t (dY - HZdt), t \geq 0)$ is a Brownian motion with the incremental covariance GG^T. The fact that the exponential in (14) is a Radon-Nikodym derivative, that is, it integrates to one, follows from the strong dichotomy for the absolute continuity of Gaussian measures. The optimality claim is verified as follows by considering a control $\tilde{U}$ given by

$$\tilde{U}(t) = U^*(t) + U_1(t) \quad (15)$$

where $U_1(t) = \alpha 1_{[t_0,t_1]}$ and α is $\mathcal{G}(t_0)$ measurable and bounded. For the control $\tilde{U}$ the exponential corresponding to the exponential in (14) is also a Radon-Nikodym derivative so it follows immediately from (12) that

$$\mathbb{E}[L(U^*)] \leq \mathbb{E}exp[L(\tilde{U})] \quad (16)$$

Thus the functional in (12) is minimized by choosing the (optimal) control

$$U^*(t) = -R^{-1}C^T S(t)Z(t) \quad (17)$$

This completes the proof.

If $M \neq 0$ then an additional absolute continuity result has to be applied to a Gaussian random vector at time T. It can be shown that the optimal cost for the problem (1), (2) and (3) is obtained from (13) and the error of estimation that is given by

$$\mu exp[\frac{\mu}{2}\int_0^T tr(P(t)Q)dt] \quad (18)$$

Specifically the optimal cost $J(U^*)$ is

$$J(U^*) = \mu exp[\frac{\mu}{2}(< S(0)X(0), X(0) > + \int_0^T tr((PQ + SPH^T(GG^T)^{-1}HP)dt)] \quad (19)$$

A number of authors have verified the equation (9) for the information filter, so combining this filtering result with the above optimal control result provides a complete solution to the partially observed linear quadratic Gaussian control problem.

4. CONCLUDING REMARKS

The verification method for the optimal control in this paper demonstrates that the additional quadratic term in the Riccati equation (8) as compared to the Riccati equation for the linear quadratic Gaussian control problem arises as the integrand of the increasing process associated with the stochastic integral term in the exponential of the Radon-Nikodym derivative (14) that is used to verify optimality of the control U^* given by (7). This approach provides a possibility of extension to other Gaussian processes, such as the family of fractional Brownian motions. The methods given in Duncan [2006] for prediction should be very applicable to the problems with an arbitrary fractional Brownian motion.

REFERENCES

A. Bensoussan and J. H. van Schuppen, Optimal control of partially observable stochastic systems with an exponential-of-integral performance index. *SIAM J. Control Optim.*, 23: 599-613 1985.

T. E. Duncan, Evaluation of likelihood functions, *Information and Control*, 13, 62-74, 1968.

T. E. Duncan, Prediction for some processes related to a fractional Brownian motion. *Stat. Prob. Lett.*, 76: 128-134, 2006.

T. E. Duncan, Linear exponential quadratic control. *IEEE Trans. Autom. Control*, 58: 2910-2911, 2013..

K. Glover and J. C. Doyle, State space formulae for all controllers that satisfy an H_∞ norm bound and relation to risk sensitivity. *Syst. Contr. Lett.*, 11: 167-172, 1988

D. H. Jacobson, Optimal stochastic linear systems with exponential performance criteria and their relation to deterministic differential games. *IEEE Trans. Autom. control*, AC-19: 124-131, 1973.

P. R. Kumar and J. van Schuppen, On the optimal control of stochastic systems with an exponential-of-integral performance index. *J. Math. Anal. Appl.* 80: 312-332, 1981.

J. L. Speyer, J. Deyst, and D. H. Jacobson, Optimization of stochastic linear systems with additive measurement and process noise using exponential performance criteria. *IEEE Trans. Autom. Control*, AC-19: 358-366, 1974.

R. J. Elliott, L. Aggoun, and J. B. Moore, *Hidden Markov Models*, Springer, 1994.

J. B. Moore, R. J. Elliott and S. Dey, Risk sensitive generalizations of minimum variance estimation and control, *J. Math. Syst., Estimation, Contr.*7, 15 pp. (electronic), 1997.